Mathematical Sciences Letters
An International Journal

http://dx.doi.org/10.12785/msl/030205

The Detour Polynomials and Detour Index of an m-Ring of $2n$-Cycles, $m,n \geq 2$

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Received: 17 Nov. 2013, Revised: 19 Feb. 2014, Accepted: 20 Feb. 2014
Published online: 1 May 2014

Abstract: A graph polynomial based on the detour distance, called detour polynomial, is obtained for a compound graph $H_m^*(C_{2n})$, which consists of a ring of m copies of an even cycle $C_{2n}, m, n \geq 2$. The detour diameter and the minimum detour distance are also determined in this paper.

Keywords: Detour distance, detour diameter, detour polynomial, detour index.

1 Introduction

For the definitions of graph concepts and notations, see the books [4] and [8]. For the definitions of detour distance concepts, see [3], [5], [6] and [7]. The **detour distance** $D(u,v)$ between two distinct vertices u and v in a connected graph G is the maximum of the lengths of $u-v$ paths in G. A $u-v$ path of length $D(u,v)$ is called $u-v$ **detour**. The **detour diameter of** $\boldsymbol{G}$, denoted by $Diam(G)$ (or $diam_D(G)$, is defined by

$Diam(G) = max\{D(u,v) : u,v \in V(G)\}$.

The **detour index** $\boldsymbol{dd(G)}$ of a connected graph G is defined by:

$$dd(G) = \sum_{u,v} D(u,v), \tag{1}$$

where the summation is taken over all unordered pairs of vertices u and v of G.

The **detour distance of a vertex** $\boldsymbol{u}$, denoted by $d_D(u)$ (or $D(u)$) is defined by

$$d_D(u) = \sum_{v \in V(G)} D(u,v). \tag{2}$$

It is clear that

$$dd(G) = \frac{1}{2} \sum_{u \in V(G)} d_D(u). \tag{3}$$

detour index has recently received some attention in the chemical literature [9] and [10], because $dd(G)$ certainly carries interesting information for cyclic compounds.

The **detour polynomial** of a connected graph G [1] denoted by $D(G;x)$, is defined by

$$D(G;x) = \sum_{u,v \in V(G)} x^{D(u,v)}, \tag{4}$$

where the summation is taken over all unordered pairs of distinct vertices u and v of G. It is clear that

$$dd(G) = \frac{d}{dx} D(G;x)|_{x=1}. \tag{5}$$

Moreover, one can easily see that

$$D(G;x) = \sum_{k=1}^{\delta_D} C_D(G,k), \tag{6}$$

where $\delta_D = Diam(G)$ and $C_D(G,k)$ is the number of unordered pairs of distinct vertices u and v such that $D(u,v) = k$.

The **detour polynomial of a vertex** v of G is defined as

$$D(v,G;x) = \sum_{\substack{u \in V(G) \\ u \neq v}} x^{D(v,u)}. \tag{7}$$

It is clear that

$$D(G;x) = \frac{1}{2} \sum_{v \in V(G)} D(v,G;x). \tag{8}$$

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Definition 1: Let G be a connected graph of order ≥ 3, and let u and v be two distinct vertices of G. For $m \geq 2$, we define a compound graph $H_m^*(G)$ as follows: Let $G^{(1)}$, $G^{(2)}$, , $G^{(m)}$ be m disjoint copies of G, and denote the vertices u and v in the i^{th} copy $G^{(i)}$ by $u^{(i)}$ and $v^{(i)}$, respectively. The graph $H_m^*(G)$ is constructed from the union of $G^{(1)}$, $G^{(2)}$, , $G^{(m)}$ with the edges $v^{(m)}u^{(1)}$ and $v^{(i)}u^{(i+1)}$, $i = 1,2,\ldots,m-1$.

In this paper, we take G as an even cycle $C_{2n}, n \geq 2$, and vertices v and u as diametrical vertices in C_{2n}, that is $d_{C_{2n}}(u,v) = n$. The graph $H_m^*(C_{2n})$ is an m-ring of $2n$-cycles. For $m = 3$, $H_3^*(C_{2n})$, is shown in Fig.1.

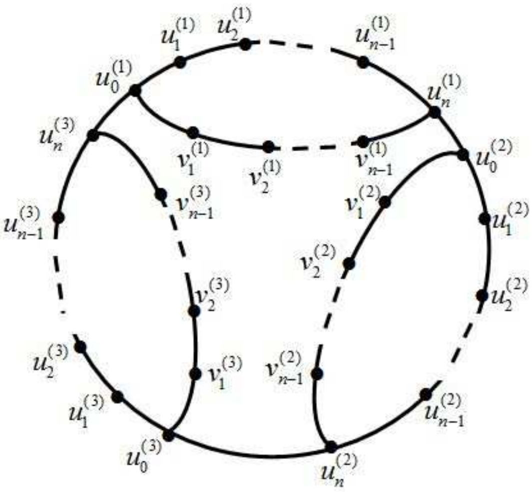

Fig. 1: The graph $H_3^*(C_{2n})$, $n \geq 2$.

For $n = 3$, $H_m^*(C_6)$ is the graph of polyhex armchair nanotube with exactly one row and m hexagons; and for $n = 2$, $H_m^*(C_4)$ is the graph of $TUC_4C_8(R)$ nanotube with one row [2]. The detour polynomial for $H_m^*(C_{2n})$ is obtained, in this paper, from which the detour index $dd(H_m^*(C_{2n}))$ can be computed. The detour diameter and the minimum detour for $H_m^*(C_{2n})$ are also determined in this paper.

2 The Detour Diameter of $H_m^*(C_{2n})$

Let W_j be the vertex set of the j^{th} copy of C_{2n}, for $j = 1,2,\ldots,m$. The set W_j is partitioned into:

$$U_j = \left\{u_1^{(j)}, u_2^{(j)}, \ldots, u_{n-1}^{(j)},\right\}, V_j = \left\{v_1^{(j)}, v_2^{(j)}, \ldots, v_{n-1}^{(j)},\right\}$$

and $\left\{u_0^{(j)}, v_n^{(j)}\right\}$ as shown in Fig. 2.

It is clear that:

$V(H_m^*(C_{2n})) = \bigcup_{j=1}^m W_j$, $p(H_m^*(C_{2n})) = 2mn$, $q(H_m^*(C_{2n})) = (2n+1)m$.

Also, one can see that $H_m^*(C_{2n})$ is a 2-connected graph having circumference $m(n+1)$, and every vertex of it is contained in a cycle of length $m(n+1)$. Therefore, the

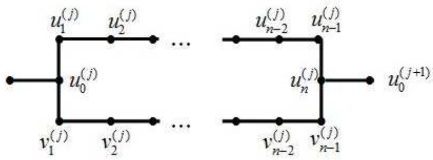

Fig. 2: The j^{th} copy of C_{2n} in $H_m^*(C_{2n})$.

(standard) eccentricity of each vertex is $\left\lfloor \frac{m(n+1}{2} \right\rfloor$. Thus, see Figure 3 (*a* and *b*),

$$rad(H_m^*(C_{2n})) = diamH_m^*(C_{2n}) = \left\lfloor \frac{m(n+1}{2} \right\rfloor. \quad (9)$$

The next proposition determines the detour diameter of $H_m^*(C_{2n})$.

Proposition 1: For $m, n \geq 2$, we have:

$$Diam(H_m^*(C_{2n})) = m(n+1) + 2n - 3. \quad (10)$$

Proof: Let u and v be any two distinct vertices of $H_m^*(C_{2n})$. We consider two main cases:

(I) If u and v are in the same copy, say $C_{2n}^{(i)}$, then let $u = u_i^{(j)}$ and $v = v_k^{(j)}$, and assume, without loss of generality, that $0 \leq j < k \leq n$. From Fig. 3, we notice that

$$D(u_i^{(j)}, u_k^{(j)}) = m(n+1) - (k-i) < m(n+1).$$

Similarly, if $u = v_j^i$ and $v = v_k^{(j)}$, with $1 \leq i < k \leq n-1$, then $D(v_i^{(j)}, v_k^{(j)}) < m(n+1)$.

Moreover, if $u = u_i^{(j)}$, $0 \leq i \leq n$, and $v = v_k^{(j)}$, $1 \leq k \leq n-1$ then

$$\begin{aligned} D(u_i^{(j)}, v_k^{(j)}) &= (m-1)(n+1) + k + 1 + (n-i) \\ &= m(n+1) + k - i \leq m(n+1) + (n-1). \end{aligned}$$

Hence, if u and v are in the same copy of $H_m^*(C_{2n})$, then

$$D(u,v) \leq m(n+1) + 2n - 3. \quad (11)$$

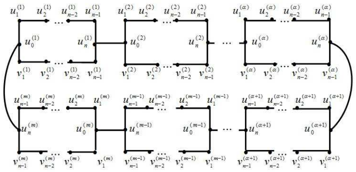

Fig. 3: (a) The graph $H_m^*(C_{2n})$ for even m, $m = 2\alpha$.

(II) If u and v are in different copies of C_{2n}, then we may assume, without loss of generality, that $u = u_i^{(j)}$ and $v = u_k^{(l)}$. This is because the detour from $u_i^{(j)}$ to any vertex

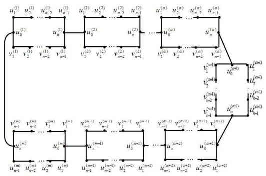

Fig. 4: (b) The graph $H_m^*(C_{2n})$ for odd m, $m = 2\alpha+1$.

w in the l^{th} copy is the same as from $v_i^{(j)}$ to w, for $i = 1,2,\ldots,n-1$. Let $l < j$. Then

$D(u_i^{(j)},u_k^{(l)}) = max\{A_1,A_2\}$

where

$A_1 = (j-l-1)(n+1)+1+D_{C_{2n}}(u_k^{(l)},u_n^{(l)})$
$+D_{C_{2n}}(u_0^{(j)},u_i^{(j)})$

$A_2 = (m+l-j-1)(n+1)+1+D_{C_{2n}}(u_k^{(l)},u_0^{(l)})$
$+D_{C_{2n}}(u_n^{(j)},u_i^{(j)})$.

Since,

$D_{C_{2n}}(u_k^{(l)},u_n^{(l)}), D_{C_{2n}}(u_0^{(j)},u_i^{(j)}), D_{C_{2n}}(u_k^{(l)},u_0^{(l)}),$
$D_{C_{2n}}(u_n^{(j)},u_i^{(j)}) \le n-1,$

and $j-l \le m-1$, $m-(j-l) \le m-1$, then

$$D(u_i^{(l)},u_k^{(l)}) \le (m-2)(n+1)+1+2(2n-1) = m(n+1)+2n-3.$$

Therefore, for any two distinct vertices u and v, in both cases, of $H_m^*(C_{2n})$, we have

$$D(u,v) \le m(n+1)+2n-3.$$

Moreover, one may easily see, from Fig.3(a) and (b), that $D(u_1^{(1)},u_{n-1}^{(2)}) = m(n+1)+2n-3$, which completes the proof.■

The next proposition determines the **minimum detour** of $H_m^*(C_{2n})$, that is

$$D_{min}(H_m^*(C_{2n}) = min\{D(u,v) : u \ne v \wedge u,v \in V(H_m^*(C_{2n}))\}.$$

Proposition 2: For $m,n \ge 2$,

$$D_{min}(H_m^*(C_{2n})) = \begin{cases} \frac{1}{2}m(n+1) & \text{for even } m, \\ \frac{1}{2}(mn+m+n-1) & \text{for odd } m. \end{cases}$$

Proof: Let u and v be any two distinct vertices of $H_m^*(C_{2n})$. If both u and v are in one copy of C_{2n}, which, from Fig.3(a and b), $D(u,v) \ge m(n+1)-n$, which implies that, $D(u,v) \ge \frac{1}{2}m(n+1)$ and $\frac{1}{2}(mn+m+n-1)$. Now, assume that u and v are in different copies of C_{2n}. Then u and v are in a common cycle of length $m(n+1)$.

(1) If m is even, then from Fig. 3(a), $D(u,v) \ge \frac{1}{2}m(n+1)$. One may easily see from Fig.3 (a), that,

$D(u_0^{(1)},u_0^{(\alpha+1)}) = \frac{1}{2}m(n+1)$, in which $\alpha = \frac{m}{2}$.

Therefore, for even m, we have

$D_{min}(H_m^*(C_{2n})) = \frac{1}{2}m(n+1)$.

(2) If m is odd, then from Fig. 3(b),

$D(u,v) \ge \frac{1}{2}[m(n+1)+n-1]$.

Moreover, one may easily see from Fig.3(a), that, $D(u_0^{(1)},u_n^{(\alpha+1)}) = \alpha(n+1)+n$, where $\alpha = \frac{m-1}{2}$.

Therefore, for odd m, we have: $D_{min}(H_m^*(C_{2n})) = \frac{1}{2}(m-1)(n+1)+n = \frac{1}{2}(mn+m+n-1)$, which completes the proof of the proposition.■

3 The Detour Polynomial of $H_m^*(C_{2n})$

From the definition of $H_m^*(C_{2n})$, we notice that, $D(H_m^*(C_{2n})) = \frac{1}{2}\sum_{j=1}^{m} D(W_j,H_m^*(C_{2n});x)$, and for $i,j = 1,2,\ldots,m$, $D(W_j,H_m^*(C_{2n});x) = D(W_i,H_m^*(C_{2n});x)$, in which $D(W_j,H_m^*(C_{2n});x) = \sum_{w\in W_j}(w,H_m^*(C_{2n});x)$.

Therefore,

$$D(H_m^*(C_{2n});x) = \frac{1}{2}mD(W_1,H_m^*(C_{2n});x). \quad (12)$$

Moreover, one may see from Fig. 2 with $j=1$, that

$D(U_1,H_m^*(C_{2n});x) = D(V_1,H_m^*(C_{2n});x),$

and

$D(u_0^{(1)},H_m^*(C_{2n});x) = D(u_n^{(1)},H_m^*(C_{2n});x).$

Thus, from the partition of W_1 and substituting in (12), we get

$$D(H_m^*(C_{2n});x) = m\left[D(U_1,H_m^*(C_{2n});x) + D(u_0^{(1)},H_m^*(C_{2n});x)\right]. \quad (13)$$

To find $D(U_1,H_m^*(C_{2n});x)$, we consider two cases of n, namely: n is even, $n = 2\beta$; and n is odd, $n = 2\beta+1$, and we partition U_1 according to that as shown in Fig.4.

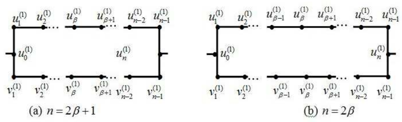

Fig. 5: The first copy of C_{2n}.

In addition to that, we have two cases for m, namely: m is even, say $m = 2\alpha$, and m is odd, say $m = 2\alpha+1$. Therefore, to find $D(H_m^*(C_{2n});x)$, we consider four main cases.

3.1 $D(H_m^*(C_{2n});x)$ *for odd* n *and even* m

It is clear from Fig.4(a), that for $i = 1,2,\ldots,\beta$, $D(u_i^{(1)},H_m^*(C_{2n});x) = D(u_{n-i}^{(1)},H_m^*(C_{2n});x)$. Thus, from (13), we get

$$D(H_m^*(C_{2n});x) = m(D(u_0^{(1)},H_m^*(C_{2n});x) + 2\sum_{i=1}^{\beta} D(u_i^{(1)},H_m^*(C_{2n});x)). \quad (14)$$

The next proposition gives us $D(u_0^{(1)}, H_m^*(C_{2n});x)$.

Proposition 3: For even m and odd n, $m \geq 4$, $n \geq 3$, we have:

$$D(u_0^{(1)}, H_m^*(C_{2n});x) = (1+2A(x))x^{m(n+1)-n} + (1+x^n+2x^nA(x))\sum_{j=2}^{\alpha} x^{(m-j)(n+1)+1} + (x^{-1}+x^{-n-1}+2-2x^{n-1}+2A(x))\sum_{j=\alpha+1}^{m} x^{(n+1)j}, \quad (15)$$

where $A(x) = \sum_{k=1}^{n-1} x^k$.

Proof:For $j = 1,2,\ldots,m$, we define the polynomial $F_j(x) = \sum_{w\in W_j} x^{D(u_0^{(1)},w)}$.
Then

$$D(u_0^{(1)}, H_m^*(C_{2n});x) = \sum_{j=1}^{m} F_j(x)$$

. For values of j, we consider three cases.
(1) For $j = 1$, we have, from Fig.3(a): $D(u_0^{(1)}, u_k^{(1)}) = D(u_0^{(1)}, v_k^{(1)}) = m(n+1)-k$, for $k = 1,2,\ldots,n-1$, and $D(u_0^{(1)}, u_n^{(1)}) = m(n+1)-n$. Therefore,

$$F_1(x) = x^{m(n+1)-n} + 2\sum_{k=1}^{n-1} x^{m(n+1)-k} = x^{m(n+1)-n}(1+2\sum_{k=1}^{n-1} x^{n-k}) = x^{m(n+1)-n}(1+2\sum_{k=1}^{n-1} x^{k}). \quad (16)$$

(2) For $2 \leq j \leq \alpha$, we have, using Fig.3(a):
$D(u_0^{(1)}, u_0^{(j)}) = (m+1-j)(n+1)$,
$D(u_0^{(1)}, u_k^{(j)}) = D(u_0^{(1)}, v_k^{(j)}) = (m+1-j)(n+1)+k$,
for $k = 1,2,\ldots,n-1$, and
$D(u_0^{(1)}, u_n^{(j)}) = (m-j)(n+1)+1$.

Therefore, for $j = 2,3,\ldots,\alpha$, we have:

$$\sum_{k=1}^{n-1} F_j(x) = x^{(m+1-j)(n+1)+1} + 2\sum_{k=1}^{n-1} x^{(m+1-j)(n+1)+k} = x^{(m-j)(n+1)+1}(1+x^n+2x^n\sum_{k=1}^{n-1} x^k). \quad (17)$$

(3) For $\alpha+1 \leq j \leq m$, we have:
$D(u_0^{(1)}, u_0^{(j)}) = (j-1)(n+1)$,
$D(u_0^{(1)}, u_n^{(j)}) = j(n+1)-1$,
and, for $k = 1,2,\ldots,n-1$,

$$D(u_0^{(1)}, u_k^{(j)}) = D(u_0^{(1)}, v_k^{(j)}) = (j-1)(n+1) + D_{C_{2n}}(u_0^{(j)}, u_k^{(j)}) = (j-1)(n+1)+2n-k = j(n+1)+n-k-1.$$

Therefore, for $j = \alpha+1, \alpha+2, \ldots, m$,

$$F_j(x) = x^{(j-1)(n+1)} + x^{j(n+1)-1} + 2\sum_{k=1}^{n-1} x^{j(n+1)+n-k-1} = x^{(n+1)j}\left[x^{-1}+x^{-n-1}+2\sum_{k=0}^{n-2} x^k\right. \quad (18)$$

Hence, summing (16), (17) and (18) from $j = 1$ to $j = m$, we get (15).■
Remark (1): The formula (15) holds also for even $n \geq 2$.

Remark (2): For $m = 2$, and any value of n, $n \geq 2$, we have:

$$D(u_0^{(1)}, H_2^*(C_{2n});x) = x^{n+2}(1+2A(x)) + x^{2n+1} + x^{n+1} + 2x^{2n+2} - 2x^{3n+1} + 2A(x)x^{2n+2}. \quad (19)$$

The polynomials $D(u_i^{(1)}, H_m^*(C_{2n});x)$, for $i = 1,2,\ldots,\beta$, odd $n \geq 3$ and even $m \geq 4$, will be obtained in the next proposition.

Proposition 4: For even $m \geq 4$, odd $n \geq 3$, and for $i = 1,2,\ldots\beta\ \left(= \frac{n-1}{2}\right)$, we have:

$$D(u_i^{(1)}, H_m^*(C_{2n});x) = x^{t-i} - x^t + x^{t-i}A(x) + x^i\sum_{k=1}^{n} x^{t-k} + (1+x^n+2x^nA(x))x^{2n+1-i}\sum_{j=2}^{\alpha} x^{(n+1)(m-j)} + x^{(n+1)\alpha+n}(x^{n-i}+x^i+2\sum_{k=1}^{n-1} x^{n+|i-k|} + (x^{i-1}+x^{i-n-1}+2x^{i-1}A(x))\sum_{j=2}^{m} x^{(n+1)j}, \quad (20)$$

where $t = m(n+1)$.

Proof: We define the polynomial, for $j = 1,2,\ldots,m$,

$$F_j(u_i^{(1)};x) = \sum_{\substack{w\in W_j \\ w\neq u_i^{(1)}}} x^{D(u_i^{(1)},w)},$$

for $i = 1,2,\ldots,\beta$.
Then

$$D(u_i^{(1)}, H_m^*(C_{2n});x) = \sum_{j=1}^{m} F_j(u_i^{(1)};x). \quad (21)$$

We have three cases for the values of j, namely, $j=1$, $2\le j\le\alpha$, and $\alpha+1\le j\le m$.

Case (1): $j=1$.

From Figs. 4 and 3(a), we notice that,
$D(u_i^{(1)},u_k^{(1)})=t-|i-k|$,
for $k=0,1,...,i-1,i+1,...,n$,
$D(u_i^{(1)},v_k^{(1)})=t+|i-k|$, for $k=1,2,...,n-1$.
Therefor

$$\begin{aligned}F_1(u_i^{(1)};x)&=\sum_{k=0}^{n}x^{t-|i-k|}+\sum_{k=1}^{n-1}x^{t+|i-k|}-x^t\\&=\sum_{k=0}^{i-1}x^{t+k-i}+\sum_{k=i+1}^{n}x^{t+i-k}+\sum_{k=1}^{i}x^{t+i-k}\\&+\sum_{k=i+1}^{n-1}x^{t+k-i}=\sum_{k=0}^{n-1}x^{t+k-i}+\sum_{k=1}^{n}x^{t+i-k}-x^t,\\&\text{for } i=1,2,...,\beta.\end{aligned}\tag{22}$$

Case(2): $2\le j\le\alpha$.
From Figs. 4 and 3(a), we notice that for $i=1,2,...,\beta$ and $k=1,2,...,n-1$,

$$\begin{aligned}D(u_i^{(1)},u_k^{(j)})&=D(u_i^{(1)},v_k^{(j)})\\&=(2n-i)+(m-j)(n+1)+1+(n+k)\\&=(m-j)(n+1)+3n+k+1-i,\end{aligned}$$

$$D(u_i^{(1)},u_0^{(j)})=(m-j)(n+1)+3n+1-i,$$

$$D(u_i^{(1)},u_n^{(j)})=(m-j)(n+1)+2n+1-i.$$

Therefore, for $j=2,3,...,\alpha$, we have

$$F_j(u_i^{(1)};x)=x^{(m-j)(n+1)+2n+1-i}(1+x^n+2x^nA(x)).\tag{23}$$

Case(3): $1+\alpha\le j\le m$.
From Figs. 4 and 3(a), we notice that for $i=1,2,...,\beta$, $k=1,2,...,n-1$,
and $j=\alpha+1,\alpha+2,...,m$, that:

$$\begin{aligned}D(u_i^{(1)},u_k^{(j)})&=max\{S_1,S_2\}\\&=3n+1+|i-k|\\&+max\{(j-2)(n+1),(m-j)(n+1)\}.\end{aligned}$$

where $S_1=(i+n)+(j-2)(n+1)+1+(2n-k)$ and $S_2=(2n-i)+(m-j)(n+1)+1+(n+k)$.
For $j=\alpha+1$, we get

$$D(u_i^{(1)},u_k^{(\alpha+1)})=\alpha(n+1)+2n+|i-k|.\tag{24}$$

For $j=\alpha+2,\alpha+3,...,m$, we have $m-j\le 2\alpha-(\alpha+2)=\alpha-2<j-2$, and, it is clear that $|i-k|<n$, therefore, for $j=\alpha+2,\alpha+3,...,m$

$$\begin{aligned}D(u_i^{(1)},u_k^{(j)})&=(j-2)(n+1)+3n+1+i-k\\&=(n+1)j+n+i-k-1.\end{aligned}\tag{25}$$

Moreover, from the symmetry of C_{2n}, we have:

$$D(u_i^{(1)},v_k^{(\alpha+1)})=(n+1)\alpha+2n+|i-k|,\tag{26}$$

$$D(u_i^{(1)},v_k^{(j)})=(n+1)j+n+i-k-1,\tag{27}$$

for $j=\alpha+2,\alpha+3,...,m$.
Also, from Fig. 3(a), we see that

$$D(u_i^{(1)},u_0^{(j)})=max\{S_3,S_4\}$$

$$=\begin{cases}\alpha(n+1)+2n-i, & \text{for } j=\alpha+1\\(n+1)(j-2)+n+1-i, & \text{for } j=\alpha+2,\alpha+3,...,m,\end{cases}\tag{28}$$

where $S_3=(n+1)(j-2)+n+1+i$, $S_4=(m-j)(n+1)+3n+1-i$,
and

$$\begin{aligned}D(u_i^{(1)},u_n^{(j)})&=max\{S_5,S_6\}\\&=max\{S_7,S_8\}.\end{aligned}$$

where
$S_5=(j-2)(n+1)+1+(n+i)+n$,
$S_6=(2n-i)+(m-j)(n+1)+1$,
$S_7=(n+1)j+i-1$,
and
$S_8=(n+1)(m-j)+2n+1-i$.
Since, $\alpha+1\le j\le 2\alpha(=m)$, then, one can check that:

$$D(u_i^{(1)},u_n^{(j)})=(n+1)j+i-1,\tag{29}$$

for $j=\alpha+1,\alpha+2,...,m$.

Finally, from (24), (26), (28) and (29), we get:

$$\begin{aligned}F_{\alpha+1}(u_i^{(1)};x)&=2\sum_{k=1}^{n-1}x^{\alpha(n+1)+2n+|i-k|}+x^{\alpha(n+1)+2n-i}\\&+x^{(\alpha+1)(n+1)+i-1}\\&=x^{\alpha(n+1)+n}(2\sum_{k=1}^{n-1}x^{n+|i-k|}+x^{n-i}+x^i).\end{aligned}\tag{30}$$

From (25), (27), (28) and (29), we get for $j=\alpha+2,\alpha+3,...,m$, that:

$$\begin{aligned}F_j(u_i^{(1)};x)&=2\sum_{k=1}^{n-1}x^{(n+1)j+n+i-k-1}+x^{(n+1)(j-2)+n+1+i}\\&+x^{((n+1)j+i-1}\\&=x^{(n+1)j}(2x^{i-1}A(x)+x^{i-n-1}+x^{i-1}).\end{aligned}\tag{31}$$

Hence, from (21), (22), (23), (30) and (31), we get, for $i = 1, 2, ..., \beta$:

$$\begin{aligned} D(u_i^{(1)}, H_m^*(C_{2n}); x) &= F_1(u_i^{(1)}; x) + \sum_{j=2}^{\alpha} F_j(u_i^{(1)}; x) \\ &+ F_{\alpha+1}(u_i^{(1)}; x) + \sum_{j=\alpha+2}^{m} F_j(u_i^{(1)}; x) \\ &= x^{t-i} \sum_{k=0}^{n-1} x^k + x^i \sum_{k=1}^{n} x^{t-k} - x^t \\ &+ \sum_{j=2}^{\alpha} \left[x^{(m-j)(n+1)+2n+1-i} (1 + x^n + 2x^n A(x)) \right] \\ &+ x^{\alpha(n+1)+n} (2 \sum_{k=1}^{n-1} x^{n+|i-k|} + x^{n-i} + x^i) \\ &+ \sum_{j=\alpha+2}^{m} x^{(n+1)j} (2x^{i-1} A(x) + x^{i-n-1} + x^{i-1}). \end{aligned} \tag{32}$$

Simplifying (32), we get (20). Hence the proof is completed.■

Substituting (15) and (32) in (14), we get $D(H_m^*(C_{2n}); x)$ for even $m(\geq 3)$ and odd $n(\geq 3)$.

Remark (3): For $m = 2$ and odd $n \geq 3$, we have:

$$\begin{aligned} D(u_i^{(1)}, H_2^*(C_{2n}); x) &= x^{t-i} - x^t + x^{t-i} A(x) \\ + x^i \sum_{k=1}^{n} x^{t-k} &+ x^{2k+1} (x^{n-i} + x^i + 2 \sum_{k=1}^{n-1} x^{n+|i-k|}), \end{aligned} \tag{33}$$

in which $t = 2n + 2$.
Thus

$$\begin{aligned} D(H_2^*(C_{2n}); x) &= 2D(u_0^{(1)}, H_2^*(C_{2n}); x) \\ &+ 4 \sum_{i=1}^{\beta} D(u_i^{(1)}, H_2^*(C_{2n}); x). \end{aligned} \tag{34}$$

As we have mentioned before that, the graph of a polyhex armchair nanotube with exactly one row and m hexagons is $H_m^*(C_6)$, then it is useful to give its detour polynomial.

Corollary 5: For even m, $m \geq 4$, we have:

$$\begin{aligned} D(H_m^*(C_6); x) &= m(2x^{4m+1} + 2x^{4m} + 6x^{4m-1} \\ + 4x^{4m-2} &+ x^{4m-3} + 4x^{2m+7} + 4x^{2m+6} + 4x^{2m+5} \\ + 4x^{2m+4} &+ x^{2m+3} + x^{2m2} + (4x^{11} + 4x^{10} + 2x^9 \\ &+ 4x^6 + 2x^5 + x^4 + x) \sum_{j=2}^{\alpha} x^{4(m-j)} \\ + (4x^2 + 6x &+ 4 + x^{-1} + 2x^{-3} + x^{-4}) \sum_{j=\alpha+2}^{m} x^{4j}). \end{aligned} \tag{35}$$

Proof: Substituting $n = 3$ in (15) and (20), we get $D(u_0^{(1)}, H_m^*(C_6); x)$ and $D(u_1^{(1)}, H_m^*(C_6); x)$. Then, using (14), we obtain (35).■

Remark (4): For $m = 2$, we put $n = 3$ in (19) and (33) to obtain

$$\begin{aligned} D(H_2^*(C_6); x) &= 2(4x^{11} + 4x^{10} + 6x^9 + 6x^8 + 7x^7 \\ &+ 4x^6 + x^5 + x^4). \end{aligned}$$

Remark (5): Taking the derivative of $D(H_m^*(C_6); x)$ at $x = 1$, we get the detour index:

$$dd(H_m^*(C_6); x) = 2m \left[27m^2 + 30m - 34 \right],$$

which is the result obtained by A. R. Ashrafi, et al. [2] using detour matrix.

3.2 $D(H_m^*(C_{2n}); x)$ *for odd* n *and odd* m

The formula (14) holds for odd n and m. By a method similar to that used in proving Propositions 3 and 4, one can easily established the following propositions:

Proposition 6: For odd $m(= 2\alpha + 1, \alpha \geq 2)$, and odd $n(= 2\beta + 1)$, we have:

$$\begin{aligned} D(u_0^{(1)}, H_m^*(C_{2n}); x) &= x^{t-n} (1 + 2A(x)) \\ + x^{(n+1)(\alpha+1)} &(x^{-1} + 1 + 2x^{n-1} - 2x^{\beta} \\ + 4 \sum_{k=\beta}^{n-2} x^k) &+ (1 + x^n + 2x^n A(x)) \\ (x \sum_{j=2}^{\alpha} x^{(n+1)(m-j)} &+ \sum_{j=\alpha+2}^{m} x^{(n+1)(j-1)}). \end{aligned} \tag{36}$$

Proposition 7: For odd $m(\geq 5)$, odd n, and $i = 1, 2, ..., \beta$, we have:

$$\begin{aligned} D(u_i^{(1)}, H_m^*(C_{2n}); x) &= x^{t-i} - x^t + x^{t-i} A(x) \\ + x^t \sum_{k=1}^{n} x^{i-k} &+ x^{(n+1)(\alpha+1)+i} (x^{n-2i-1} + x^n + 2x^n A(x)) \\ + (1 + x^n + 2x^n + A(x)) &(x^{2n+1-i} \sum_{j=2}^{\alpha+1} x^{(n+1)(m-j)} \\ &+ x^i \sum_{j=\alpha+3}^{m} x^{(n+1)(j-1)}). \end{aligned} \tag{37}$$

Remark (6): The formulas (36) and (37) hold for $m = 3$ providing that $\sum_{j=\alpha+3}^{m} x^{(n+1)(j-1)}$ and $\sum_{j=2}^{\alpha+1} x^{(n+1)(m-j)}$ are omitted.

Finally, substituting (36) and (37) in (14), we get $D(H_m^*(C_{2n}); x)$ for odd n and m.

Corollary 8: For odd $m, m \geq 5$, we have:

$$D(H_m^*(C_6);x) = m((1+4x+6x^2+2x^3+2x^4)x^{4m-3} +(1+x+2x^2+4x^3+2x^6+4x^7+4x^8)x^{2m+1} +(1+x^3+2x^4+2x^5)((1+2x)\sum_{j=\alpha+2}^{m} x^{4(j-1))} +(x+2x^6)\sum_{j=2}^{\alpha} x^{4(m-j)})). \quad (38)$$

Proof: Substituting $n = 3$ in (3.25) and (3.26), and using (3.3), we get (3.27).■

Remark (7): The formula (3.27) holds for $m = 3$, providing that the summation $\sum_{j=2}^{\alpha} x^{(m-j)}$ is omitted.

Remark (8): From Corollary 8 and Remark (7), we get the detour index of $H_m^*(C_6)$ for odd $m(\geq 3)$:

$$dd(H_m^*(C_6)) = m(54m^2+60m-67),$$

which is the same result given by Ashrafi, et. al [2].

3.3 $D(H_m^*(C_{2n}))$ *for even* n *and* m

Let $n = 2\beta, m = 2\alpha$. From Fig.4(b), we notice that W_1 is partitioned into:
$U_1 = \left\{u_1^{(1)}, u_2^{(1)}, \dots, u_{\beta-1}^{(1)}\right\}$, $U'_1 = \left\{u_{\beta+1}^{(1)}, u_{\beta+2}^{(1)}, \dots, u_{n-1}^{(1)}\right\}$, and $\left\{u_\beta^{(1)}, u_0^{(1)}, u_n^{(1)}\right\}$. Thus, from Figs. 3($a$) and 4($b$), we deduced that:

$$D(W_1, H_m^*(C_{2n});x) = 2D(u_0^{(1)}, H_m^*(C_{2n});x) +2D(u_\beta^{(1)}, H_m^*(C_{2n});x) +4D(U, H_m^*(C_{2n});x).$$

Therefore,

$$D(H_m^*(C_{2n});x) = m(D(u_0^{(1)}, H_m^*(C_{2n});x) +D(u_\beta^{(1)}, H_m^*(C_{2n});x) +2\sum_{i=1}^{\beta-1} D(u_i^{(1)}, H_m^*(C_{2n});x)). \quad (39)$$

One may notice that $D(u_0^{(1)}, H_m^*(C_{2n});x)$ is that given in (15) for $m \geq 4$, and in (19) for $m = 2$. Moreover, $D(u_i^{(1)}, H_m^*(C_{2n});x)$, $i = 1,2,\dots,\beta-1$, is that given in (20) for $m, n \geq 4$ and in (33) for $m = 2$. Therefore, we need to find $D(u_\beta^{(1)}, H_m^*(C_{2n});x)$.

Proposition 9: For even $m, n \geq 4$, we have:

$$D(u_\beta^{(1)}, H_m^*(C_{2n});x) = 2x^{t-\beta} - x^t + 2x^\beta \sum_{k=1}^{n-1} x^{t-k} +x^\beta(1+x^n+2x^nA(x)) (x^{n+1}\sum_{j=2}^{\alpha} x^{(m-j)(n+1)} + \sum_{j=\alpha+2}^{m} x^{(j-1)(n+1)}) +2(1+x^\beta+2\sum_{k=\beta+1}^{n-1} x^k)x^{(n+1)\alpha+3\beta}). \quad (40)$$

Proof: The detour distance is obtained from $u_\beta^{(1)}$ to every other vertex of $H_m^*(C_{2n})$ depicted in Fig.3(a). Then, the polynomial $D(u_\beta^{(1)}, H_m^*(C_{2n});x)$ is obtained as given in (40).■

Remark (9): For $m = 2$ and $n \geq 4$, we have:

$$D(u_\beta^{(1)}, H_m^*(C_{2n});x) = 2x^{2n+2-\beta} - x^{2n+2} + 2x^\beta \sum_{k=1}^{n-1} x^{2n+2-k} +2(1+x^\beta+2\sum_{k=\beta+1}^{n-1} x^k)x^{n+1+3\beta}. \quad (41)$$

Remark (10): For $m \geq 4$ and $n = 2$, we have:

$$D(u_1^{(1)}, H_m^*(C_4);x) = 2x^{3m-1} - x^{3m} + x(1+x^2+3x^3) (x^3\sum_{j=2}^{\alpha} x^{3(m-j)} + \sum_{j=\alpha+2}^{m} x^{3(j-1)}) +2(1+x)x^{3(\alpha+1)}. \quad (42)$$

Thus for even m and n, $D(H_m^*(C_{2n});x)$ is obtained from (39) by substituting $D(u_0^{(1)}, H_m^*(C_{2n});x)$, given in (15), $D(u_i^{(1)}, H_m^*(C_{2n});x)$ given (20) for $i = 1,2,\dots,\beta-1$ and $D(u_\beta^{(1)}0, H_m^*(C_{2n});x)$ given in (40), and taking care of the special cases where $n = 2$ or $m = 2$, as given in the Remarks. As we have mentioned before $H_m^*(C_4)$ is the graph of $TUC_4C_8(R)$ nanotube with one row, and it is useful to find its detour polynomial.

Corollary 10: Fro even $m \geq 4$, we have:

$$D(H_m^*(C_{2n});x) = m(x^{3m}+4x^{3m-1}+x^{3m-2}+2x^{3\alpha+4}+4x^{3\alpha+3} +x^{3\alpha+2}+x^{3\alpha}+(2x+3+x^{-1}+x^{-2} +x^{-3})\sum_{j=\alpha+2}^{m} x^{3j}+(2x^7+x^6+3x^4+x^3 +x)\sum_{j=2}^{\alpha} x^{3(m-j)}). \quad (43)$$

Proof: For $m = 2$, we get from (39):

$$D(H_m^*(C_4);x) = m\left[D(u_0^{(1)}, H_m^*(C_4);x) + D(u_1^{(1)}, H_m^*(C_4);x)\right],$$

where $D(u_0^{(1)}, H_m^*(C_4);x)$ is obtained from (15) for $n = 2$, and $D(u_i^{(1)}, H_m^*(C_4);x)$ is given in (42).■

Remark (11): For $m = 2$, we have:

$$D(H_2^*(C_4);x) = 2(2x^7 + 5x^6 + 5x^5 + x^4 + x^3),$$

which can be obtained form (43) by omitting both summations, and putting $m = 2$. Finally, from Corollary 10 and Remark (11), we get the detour index of $H_m^*(C_4)$ for even $m(\geq 2)$

$$dd(H_m^*(C_4)) = 2m(9m^2 + 5m - 8),$$

which is the same formula obtained by Ashrafi, et. al [2].

3.4 $D(H_m^*(C_4);x)$ *for even* n *and odd* m

Let $n = 2\beta$ and $m = 2\alpha + 1$, then from Figs.3(b) and 4(b), we notice that formula (39) holds for this case, in which, for $n \geq 4$ and $m \geq 5$,

$$D(u_0^{(1)}, H_m^*(C_{2n});x) = x^{t-n}(1 + 2A(x)) + (1 + x^{-1} + 2x^{n-1} + 4\sum_{k=1}^{n-1} x^k)x^{(n+1)(\alpha+1)} + (1 + x^n + 2x^n A(x)) (x\sum_{j=2}^{\alpha} x^{(n+1)(m-j)} + \sum_{j=\alpha+2}^{m} x^{(n+1)(j-1)}, \tag{44}$$

$$D(u_\beta^{(1)}, H_m^*(C_{2n});x) = 2x^{t-\beta} - x^t + 2\sum_{k=1}^{n-1} x^{t+\beta-k} + 2(1 + x^n + 2x^n A(x))x^\beta (x^{(\alpha+1)(n+1)} + \sum_{j=\alpha+2}^{2\alpha} x^{(n+1)j}), \tag{45}$$

and $D(u_i^{(1)}, H_m^*(C_{2n});x)$ is given in (37) for $i = 1, 2, ..., \beta - 1$.

Remark (12): For $m = 3$, and even $n, n \geq 4$, $D(H_3^*(C_{2n});x)$ can be obtained from (37), (44) and (45) providing that all summations of the form $\sum_{j=a}^{b}$ for $a > b$, are omitted.

Corollary 11: For odd $m, m \geq 5$, we have

$$D(H_m^*(C_4);x) = m(x^{3m-2} + 4x^{3m-1} + x^{3m} + x^{3\alpha+2} + x^{3\alpha+3} + 4x^{4\alpha+4} + 2x^{3\alpha+6} + 4x^{3\alpha+7} + (1 + x^2 + 2x^3)(x\sum_{j=2}^{\alpha} x^{3(m-j)} + \sum_{j=\alpha+2}^{m} x^{3(j-1)} + 2x\sum_{j=\alpha+2}^{2\alpha} x^{3j})). \tag{46}$$

Proof: It follows from the formulas (45) and (46) by substituting $n = 2$ and omitting the summation $\sum_{k=\beta}^{n-2} x^k$.■

Remark (13): For $m = 3$, and $n = 2$, we get from (46):

$$D(H_3^*(C_4);x) = 3(4x^{10+5x^9+5x^8+5x^7+2x^6+x^5}).$$

Finally, from Corollary 11 and Remark(13), we get the detour index of $H_m^*(C_4)$ for odd $m, m \geq 3$:

$$dd(H_3^*(C_4);x) = m(18m^2 + 10m - 15),$$

which is the same formula given in [2] using detour matrix.

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