Progress in Fractional Differentiation and Applications
An International Journal

http://dx.doi.org/10.18576/pfda/020204

Some New Results on the New Conformable Fractional Calculus with Application Using D'Alambert Approach

Olaniyi S. Iyiola[1] *and Eze R. Nwaeze*[2,*]

[1] University of Wisconsin-Milwaukee, Wisconsin, USA
[2] Auburn University, Auburn, AL, 36849, USA

Received: 2 Jan. 2016, Revised: 20 Mar. 2016, Accepted: 28 Mar. 2016
Published online: 1 Apr. 2016

Abstract: In this paper, we propose and prove some new results on the recently proposed conformable fractional derivatives and fractional integral, [Khalil, R., et al., A new definition of fractional derivative, J. Comput. Appl. Math. 264, (2014)]. The simple nature of this definition allows for many extensions of some classical theorems in calculus for which the applications are indispensable in the fractional differential models that the existing definitions do not permit. The extended mean value theorem and the Racetrack type principle are proven for the class of functions which are α-differentiable in the context of conformable fractional derivatives and fractional integral. We also apply the D'Alambert approach to the conformable fractional differential equation of the form: $T_\alpha T_\alpha y + pT_\alpha y + qy = 0$, where p and q are $\alpha-$differentiable functions as application.

Keywords: $\alpha-$differentiable functions, conformable fractional differential equation, conformable fractional derivatives and integrals, Racetrack type principle.

1 Introduction and Preliminaries

The concept of derivative is traditionally associated to integers where the order of derivative is considered to be integer. In 1695 L'Hospital, in his letter to Leibnitz, asked, "what does it mean by $\dfrac{d^n f}{dx^n}$ when $n = 0.5$?". In the bid to answer L'Hospital's question, many researchers tried to put a definition on a fractional derivative. Various types of fractional derivatives were introduced - Riemann-Liouville, Caputo, Hadamard, Erdélyi-Kober, Grünwald-Letnikov, Marchaud and Riesz are just a few to name [1,2,3,4,5,6]. Most of the fractional derivative are defined through fractional integrals [6]. Due to the same reason, those fractional derivatives inherit some non-local behaviors, which lead them to many interesting applications including memory effects and future dependence [7]. In recent time, there are many applications of the fractional derivatives cutting across many fields such as found in control theory of dynamical systems, nanotechnology, viscoelasticity, anomalous transport and anomalous diffusion, financial modeling, random walk see [8,9,10,11,12,13,14, 15,16,17]. These recent discoveries of the applications of fractional calculus have drawn the attention of many researchers in other to gain more insight into the field. Existence and uniqueness of solutions, asymptotic behaviour, analytical and numerical solutions of some of the fractional differential equations both linear and nonlinear see [18,19,20,21,22].

We present the two most popular definitions in the sense that they are mostly used for mathematical modeling in many applications.

Definition 1.[5] *The Riemann-Liouville's (RL) fractional integral operator of order* $\alpha \in [n-1, n)$, *of a function* $f \in L^1(a,b)$ *is given as*

$$I^\alpha w(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x w(\tau)(x-\tau)^{n-\alpha-1} d\tau \tag{1}$$

* Corresponding author e-mail: ern0002@auburn.edu

with Γ as the Gamma function and $I^0 w(x) = w(x)$.
The Riemann-Liouville's (RL) fractional derivative operator is then given as

$$D^{\alpha} w(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^x w(\tau)(x-\tau)^{n-\alpha-1} d\tau. \tag{2}$$

Definition 2.[5] *The Caputo fractional derivative operator of order $\alpha \in [n-1, n)$, of a function $f \in L^1(a,b)$ is given as*

$$^{c}D^{\alpha} w(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x w^n(\tau)(x-\tau)^{n-\alpha-1} d\tau. \tag{3}$$

Now, all definitions including (2) and (3) above satisfy the property that the fractional derivative is linear. This is the only property inherited from the first derivative by all of the definitions. However, there are some inconsistencies of many existing definitions that limit the extent of applications in so many fields. Properties such as the derivative of constant should be zero,the product rule, quotient rule, chain rule, Rolle's theorem, mean value theorem and composition rule and so on are lacking in almost all fractional derivatives.

These inconsistencies and many more have posed a lot of problems in real life applications and have limited how far these fractional calculus could be explored. To overcome some of these and other difficulties, Khalil et al. in [23] came up with an interesting idea that extends the familiar limit definition of the derivatives of a function called conformable fractional derivative. The simple nature of this definition allows for many extensions of some classical theorems in calculus for which the applications are indispensable in the fractional differential models that the existing definitions do not permit.

In this paper, the extended mean value theorem and the Racetrack type principle are proven for the class of functions which are α-differentiable in the context of conformable fractional derivatives and fractional integral. In Section 2 we review some concepts of conformable fractional derivatives equation and fractional integral of the form: $T_\alpha T_\alpha y + pT_\alpha y + qy = 0$, where p and q are $\alpha-$differentiable functions as application.

2 Conformable Fractional Derivative

Definition 3.[23] *Given a function $f : [0,\infty) \to \mathbb{R}$. Then the* **conformable fractional derivative** *of f of* **order** *α is defined by*

$$T_\alpha(f)(t) = \lim_{\varepsilon \to 0} \frac{f(t+\varepsilon t^{1-\alpha}) - f(t)}{\varepsilon}, \tag{4}$$

for all $t > 0$, $\alpha \in (0,1)$. If f is $\alpha-$differentiable in some $(0,a)$, $a > 0$, and $\lim_{t\to 0^+} T_\alpha(f)(t)$ exists, then define $T_\alpha(f)(0) = \lim_{t\to 0^+} f^{(\alpha)}(t)$.

In the sequel, we shall also adopt the notation used in [23]. That is, we will, sometimes write $f^{(\alpha)}(t)$ for $T_\alpha(f)(t)$, to denote the conformable fractional derivatives of f of order α. In addition, if the conformable fractional derivative of f of order α exists, then we simply say that f is $\alpha-$differentiable. It is easy to see that, from the definition, if two functions are $\alpha-$differentiable, so is their sum and difference. See a similar definition in [24].

As a consequence of the above definition, the authors in [23], showed that the $\alpha-$derivative in (4), obeys the product rule, quotient rule, linearity property, and zero derivative for constant functions. Also they proved results similar to the Rolle's Theorem and the Mean Value Theorem in classical calculus. Specifically, they proved the following theorems:

Theorem 1(Rolle's Theorem for Conformable Fractional Differentiable Functions). *Let $a > 0$ and $f : [a,b] \to \mathbb{R}$ be a given function that satisfies*

- *f is continuous on $[a,b]$,*
- *f is $\alpha-$differentiable for some $\alpha \in (0,1)$,*
- *$f(a) = f(b)$.*

Then, there exist $c \in (a,b)$, such that $f^{(\alpha)}(c) = 0$.

Theorem 2(Mean Value Theorem for Conformable Fractional Differentiable Functions). *Let $a > 0$ and $f : [a,b] \to \mathbb{R}$ be a given function that satisfies*

- *f is continuous on $[a,b]$,*
- *f is $\alpha-$differentiable for some $\alpha \in (0,1)$.*

Then, there exist $c \in (a,b)$, *such that*

$$f^{(\alpha)}(c) = \frac{f(b)-f(a)}{\frac{b^\alpha}{\alpha}-\frac{a^\alpha}{\alpha}}.$$

They also gave the following definition for the $\alpha-$fractional integral of a function f starting from $a \geq 0$.

Definition 4. $I_\alpha^a(f)(t) = I_1^a(t^{\alpha-1}f) = \int_a^t \frac{f(x)}{x^{1-\alpha}}dx$, *where the integral is the usual Riemann improper integral, and* $\alpha \in (0,1)$.

With the above definition, it was shown that

Theorem 3. $T_\alpha I_\alpha^a(f)(t) = f(t)$, *for* $t \geq a$, *where* f *is any continuous function in the domain of* I_α.

3 Main Results

We begin by proving the Extended Mean Value Theorem for Conformable Fractional Differentiable Functions.

Theorem 4(Extended Mean Value Theorem for Conformable Fractional Differentiable Functions). *Let* $a > 0$ *and* $f,g : [a,b] \to \mathbb{R}$ *be functions that satisfy*

– f,g *is continuous on* $[a,b]$,
– f,g *is* $\alpha-$*differentiable for some* $\alpha \in (0,1)$.

Then, there exist $c \in (a,b)$, *such that*

$$\frac{f^{(\alpha)}(c)}{g^{(\alpha)}(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}.$$

*Remark.*Observe that Theorem 2 is a special case of this Theorem 4 for $g(x) = \frac{x^\alpha}{\alpha}$.

Proof. Consider the function

$$F(x) = f(x) - f(a) + \left(\frac{f(b)-f(a)}{g(b)-g(a)}\right)(g(x)-g(a)).$$

Since F is continuous on $[a,b]$, $\alpha-$differentiable on (a,b), and $F(a) = 0 = F(b)$, then by Theorem 1, there exist a $c \in (a,b)$ such that $F^{(\alpha)}(c) = 0$ for some $\alpha \in (0,1)$. Using the linearity of T_α and the fact that the $\alpha-$derivative of a constant is zero, our result follows.

Theorem 5. *Let* $a > 0$ *and* $f : [a,b] \to \mathbb{R}$ *be a given function that satisfies*

– f *is continuous on* $[a,b]$,
– f *is* $\alpha-$*differentiable for some* $\alpha \in (0,1)$.

If $f^{(\alpha)}(x) = 0$ *for all* $x \in (a,b)$, *then* f *is a constant on* $[a,b]$.

Proof. Suppose $f^{(\alpha)}(x) = 0$ for all $x \in (a,b)$. Let x_1, x_2 be in $[a,b]$ with $x_1 < x_2$. So, the closed interval $[x_1,x_2]$ is contained in $[a,b]$, and the open interval (x_1,x_2) is contained in (a,b).
Hence, f is continuous on $[x_1,x_2]$ and $\alpha-$differentiable on (x_1,x_2). So, by Theorem 2, there exist c between x_1 and x_2 with

$$\frac{f(x_2)-f(x_1)}{\frac{x_2^\alpha}{\alpha}-\frac{x_1^\alpha}{\alpha}} = f^{(\alpha)}(c) = 0.$$

Therefore, $f(x_2) - f(x_1) = 0$ and $f(x_2) = f(x_1)$.
Since x_1 and x_2 are arbitrary numbers in $[a,b]$ with $x_1 < x_2$, then f is a constant on $[a,b]$.

Corollary 1. *Let* $a > 0$ *and* $F,G : [a,b] \to \mathbb{R}$ *be functions such that for all* $\alpha \in (0,1)$, $F^{(\alpha)}(x) = G^{(\alpha)}(x)$ *for all* $x \in (a,b)$. *Then there exist a constant* C *such that* $F(x) = G(x) + C$.

Proof. Simply apply the above theorem to $H(x) = F(x) - G(x)$.

Theorem 6. *Let $a > 0$ and $f : [a,b] \to \mathbb{R}$ be a given function that satisfies*

– *f is continuous on $[a,b]$,*
– *f is $\alpha-$differentiable for some $\alpha \in (0,1)$.*

Then we have the following:

1. *If $f^{(\alpha)}(x) > 0$ for all $x \in (a,b)$, then f is increasing on $[a,b]$.*
2. *If $f^{(\alpha)}(x) < 0$ for all $x \in (a,b)$, then f is decreasing on $[a,b]$.*

Proof. Following similar line of argument as given in the proof of Theorem 5, there exist c between x_1 and x_2 with

$$\frac{f(x_2) - f(x_1)}{\frac{x_2^\alpha}{\alpha} - \frac{x_1^\alpha}{\alpha}} = f^{(\alpha)}(c).$$

1. If $f^{(\alpha)}(c) > 0$, then $f(x_2) > f(x_1)$ for $x_1 < x_2$.
Therefore, f is increasing on $[a,b]$ since x_1 and x_2 are arbitrary numbers of $[a,b]$.
2. If $f^{(\alpha)}(c) < 0$, then $f(x_2) < f(x_1)$ for $x_1 < x_2$.
Therefore, f is decreasing on $[a,b]$ since x_1 and x_2 are arbitrary numbers of $[a,b]$.

Theorem 7(Racetrack Type Principle). *Let $f,g : [a,b] \to \mathbb{R}$ be given functions satisfying*

– *f and g are continuous on $[a,b]$,*
– *f and g are $\alpha-$differentiable for some $\alpha \in (0,1)$.*
– *$f^{(\alpha)}(x) \le g^{(\alpha)}(x)$ for all $x \in (a,b)$.*

Then we have the following:

1. *If $f(a) = g(a)$, then $f(x) \le g(x)$ for all $x \in [a,b]$.*
2. *If $f(b) = g(b)$, then $f(x) \ge g(x)$ for all $x \in [a,b]$.*

Proof. Consider $h(x) = g(x) - f(x)$. Then h is continuous on $[a,b]$ and $\alpha-$differentiable for some $\alpha \in (0,1)$. Also, using the linearity of T_α and the fact that $f^{(\alpha)}(x) \le g^{(\alpha)}(x)$ for all $x \in (a,b)$, $\alpha \in (0,1)$, we obtain

$$h^{(\alpha)}(x) \ge 0, \quad \text{for all} \quad x \in (a,b). \tag{5}$$

So, by Theorem 6, h is increasing (or nondecreasing).
Hence, for any $a \le x \le b$, we have $h(a) \le h(x)$.
Since $h(a) = g(a) - f(a) = 0$ by the assumption, the result follows.

Similarly, for the part 2 of Theorem 7, since for any $a \le x \le b$, we have $h(x) \le h(b)$ and $h(b) = f(b) - g(b) = 0$, the result follows.

Theorem 8. *Let $0 < a < b$ and $f : [a,b] \to \mathbb{R}$ be continuous function. Then for $\alpha \in (0,1)$,*

$$\left| I_\alpha^a(f)(t) \right| \le I_\alpha^a(|f|)(t).$$

Proof. The result follows directly since

$$\begin{aligned} \left| I_\alpha^a(f)(t) \right| &= \left| \int_a^t \frac{f(x)}{x^{1-\alpha}} dx \right| \\ &\le \int_a^t \left| \frac{f(x)}{x^{1-\alpha}} \right| dx \\ &= \int_a^t \frac{|f(x)|}{x^{1-\alpha}} dx \\ &= I_\alpha^a(|f|)(t). \end{aligned}$$

Corollary 2. *Let $f : [a,b] \to \mathbb{R}$ be continuous function such that*

$$M = \sup_{[a,b]} |f|.$$

Then for any $t \in [a,b]$, $\alpha \in (0,1)$,

$$\left| I_\alpha^a(f)(t) \right| \le M \left(\frac{t^\alpha}{\alpha} - \frac{a^\alpha}{\alpha} \right).$$

Proof. From Theorem 8, we have that for any $t \in [a,b]$, $\alpha \in (0,1)$,

$$\begin{aligned}\left|I_\alpha^a(f)(t)\right| &\leq I_\alpha^a(|f|)(t)\\ &= \int_a^t \frac{|f(x)|}{x^{1-\alpha}}dx\\ &\leq M\int_a^t x^{\alpha-1}dx\\ &= M\left(\frac{t^\alpha}{\alpha}-\frac{a^\alpha}{\alpha}\right).\end{aligned}$$

We now give an example to illustrate Theorem 6.

Example 1. Let $f:[0.5,3]\to\mathbb{R}$ be defined by $f(x)=x^3-3x+2$. Find where f is increasing and decreasing.

Solution: We first compute $f^{(\alpha)}(x)$ for any $\alpha\in(0,1)$. By definition, we have

$$f^{(\alpha)}(x)=3x^{1-\alpha}(x^2-1).$$

So, $f^{(\alpha)}(x)=0$ if and only if $x=-1,0$ or 1.
All numbers less than 0 will not be considered since they do not lie in the domain under consideration.
To this end, we will consider all positive numbers less than one (in particular, $x\in[0.5,1)$) and all numbers greater or equal to one (in particular, $x\in[1,3]$).

– For $x\in[0.5,1)$, $x-1<0$ and $x+1>0$. This implies that for all $\alpha\in(0,1)$, $f^{(\alpha)}(x)<0$ for all $x\in[0.5,1)$. So, f is decreasing on $[0.5,1)$.
– For $x\in[1,3]$, $x-1\geq 0$ and $x+1>0$. This implies that for all $\alpha\in(0,1)$, $f^{(\alpha)}(x)>0$ for all $x\in[1,3]$. So, f is increasing on $[1,3]$.

4 Application: D'Alambert Approach

In this section, we seek to find two solutions of the following conformable fractional differential equation of the form

$$T_\alpha T_\alpha y+p(x)T_\alpha y+q(x)y=0, \tag{6}$$

where p and q are $\alpha-$differentiable functions of x.

Approach: We start by assuming that (6) has a solution, say, y_1. We wish to find the second solution y_2 such that

$$y_2=vy_1,$$

where v is an $\alpha-$differentiable function of x.
For this,

$$\begin{aligned}T_\alpha y_2 &= T_\alpha(vy_1)\\ &= vT_\alpha y_1+y_1T_\alpha v\end{aligned}$$

and

$$\begin{aligned}T_\alpha T_\alpha y_2 &= T_\alpha(vT_\alpha y_1+y_1T_\alpha v)\\ &= vT_\alpha T_\alpha y_1+T_\alpha y_1T_\alpha v+y_1T_\alpha T_\alpha v+T_\alpha y_1T_\alpha v.\end{aligned}$$

But y_2 a solution of (6) if and only if

$$\begin{aligned}0 &= T_\alpha T_\alpha y_2+pT_\alpha y_2+qy_2\\ &= vT_\alpha T_\alpha y_1+T_\alpha y_1T_\alpha v+y_1T_\alpha T_\alpha v+T_\alpha y_1T_\alpha v+pvT_\alpha y_1+py_1T_\alpha v+qvy_1.\end{aligned} \tag{7}$$

Since y_1 is assumed to be a solution of (6), we have that

$$T_\alpha T_\alpha y_1 + pT_\alpha y_1 + qy_1 = 0.$$

With this, the equation (7) boils down to

$$2T_\alpha y_1 T_\alpha v + y_1 T_\alpha T_\alpha v + py_1 T_\alpha v = 0. \tag{8}$$

Now, if we let $w = T_\alpha v$, then (8) becomes

$$T_\alpha w + \left(p + \frac{2}{y_1} T_\alpha y_1\right) w = 0. \tag{9}$$

The problem becomes:
Find w that satisfies (9).
To do this, we simply multiply both sides of Equation (9) by

$$\exp\left[I_\alpha\left(p + \frac{2T_\alpha y_1}{y_1}\right)\right] = y_1^2 e^{I_\alpha p}$$

and use the product rule property of T_α to obtain (here we only employ the properties of T_α and I_α, as discussed in previous section and [23]) :

$$T_\alpha\left(w y_1^2 e^{I_\alpha p}\right) = 0. \tag{10}$$

This implies that

$$w = \frac{ce^{-I_\alpha p}}{y_1^2}, \tag{11}$$

where c is an arbitrary constant. Therefore,

$$v = I_\alpha\left(\frac{ce^{-I_\alpha p}}{y_1^2}\right). \tag{12}$$

Hence, our second solution, y_2, for Equation (6) is given by

$$y_2 = y_1 I_\alpha\left(\frac{ce^{-I_\alpha p}}{y_1^2}\right). \tag{13}$$

We state the above approach as follows:

Theorem 9. *Given a conformable fractional differential equation of the form*

$$T_\alpha T_\alpha y + p(x) T_\alpha y + q(x) y = 0, \tag{14}$$

where p and q are $\alpha-$differentiable functions of x. Suppose y_1 is a solution of (14), then another solution, y_2, is given by

$$y_2 = y_1 I_\alpha\left(\frac{e^{-I_\alpha p}}{y_1^2}\right). \tag{15}$$

Example 2. Consider the differential equation

$$T_{2/3} T_{2/3} y - \sqrt[3]{x} T_{2/3} y = 0.$$

Clearly, $y_1 = 1$ is a solution of such equation. Here $p(x) = -\sqrt[3]{x}$. Using formula (13) and the definition of $I_{2/3}$ we obtain another solution of the form

$$y_2 = I_{2/3}\left(e^{I_{2/3}\sqrt[3]{x}}\right) = I_{2/3}(e^x).$$

Example 3. Consider the differential equation

$$T_{1/2}T_{1/2}y+\frac{\sqrt{x}}{2}T_{1/2}y-y=0.$$

It is easy to see that $y_1=x$ is a solution of the given equation. Here $p(x)=\frac{\sqrt{x}}{2}$. Using formula (13) and the definition of $I_{1/2}$ we obtain another solution of the form

$$y_2=xI_{1/2}\left(\frac{e^{-I_{1/2}(\sqrt{x}/2)}}{x^2}\right)=xI_{1/2}\left(\frac{e^{-x/2}}{x^2}\right).$$

5 Conclusion

The extended mean value theorem and the Racetrack type principle are proposed and proven in this paper for the class of functions which are α-differentiable in the context of conformable fractional derivatives and fractional integral introduced recently in [23]. We apply the D'Alambert approach to the conformable fractional differential equation of the form: $T_\alpha T_\alpha y+pT_\alpha y+qy=0$, where p and q are $\alpha-$differentiable functions as application. The D'Alambert approach is made possible here to solve this class of fractional differential equations simply because of the nature of this new definition of derivatives. This, of course, is necessary since many real life models give such differential equations.

References

[1] U. N. Katugampola, New approach to a generalized fractional integral,*Appl. Math. Comput.* **218**, 860-865 (1945).
[2] U. N. Katugampola, New approach to a generalized fractional derivatives, *B. Math. Anal. App.* **6**, 1-15 (2014).
[3] A. A. Kilbas, H. M. Srivastava and J. J. Truijillo, *Theory and applications of fractional differential equations, Elsevier B. V. , Amsterdam, Netherlands,* 2006.
[4] K. B. Oldham and J. Spanier, *The fractional calculus,* Academic Press, New York, 1974.
[5] I. Podlubny, *Fractional Differential Equations,* Academic Press, San Diego, California, U.S.A., 1999.
[6] S. G. Samko, A. A. Lilbas and O. I. Marichev, *Fractional integrals and derivatives: theory and applications,* Gordon and Breach, Yverdon et. Alibi, 1993.
[7] J. A. Machado, And I say to myself: what a fractional world!, *Frac. Calc. Appl.* **14**, 635-654 (2011).
[8] D. Baleanu, Z. B Guvenc and J. T. Machado, Editors. *New Trends in Nanotechnology and Fractional Calculus Applications,* Springer, 2010.
[9] R. Caponetto, G. Dongola, L. Fortuna and I. Petras, *Fractional order systems: modeling and control applications,* volume 72 of World Scientific Series on Nonlinear Science, World Scientific, 2010.
[10] R. Hilfer and L. Anton, Fractional master equations and fractal time random walks, *Phys. Rev. E,* **51**, R848–R851 (1995).
[11] R. Klages, G. Radons and I. Sokolov, Editors, *Anomalous transport: foundations and applications,* Wiley, 2008.
[12] F. Mainardi, *Fractional calculus and waves in linear viscoelasticity,* Imperial College Press, 2010.
[13] C. A. Monje, Y. Chen, B. M. Vinagre, D. Xue and V. Feliu, *Fractional-order systems and controls,* Advances in Industrial Control, Springer, 2010.
[14] E. Scalas, R. Gorenflo, F. Mainardi and M. Meerschaert, *Speculative option valuation and the fractional diffusion equation,* In J. Sabatier and J. T. Machado, editors, *Proceedings of the IFAC Workshop on Fractional Differentiation and its Applications, (FDA 04), Bordeaux, 2004,* (2004).
[15] Y. Zhang, D. A. Benson, M. M. Meerschaert, E. M. LaBolle and H. P. Scheffler, Random walk approximation of fractional-order multiscaling anomalous diffusion, *Phys. Rev. E,* **74**, 026706–026715 (2006).
[16] C Pinto, ARM Carvalho, New findings on the dynamics of HIV and TB coinfection models, *Applied Mathematics and Computation* **242**, 36-46 (2014).
[17] CMA Pinto, ARM Carvalho, Fractional Modeling of Typical Stages in HIV Epidemics with Drug-Resistance, *Progr. Fract. Differ. Appl.* **1**, No. 2, 111-122 (2015).
[18] T. S. Aleroev, M. Kirane and S. A. Malik, Determination of a source term for a time fractional diffusion equation with an integral type over-determining condition, *Electr. J. Differ. Equ.* **270**, 1–16 (2013).
[19] K. M. Furati, O. S. Iyiola and M. Kirane, An inverse problem for a generalized fractional diffusion, *Appl. Math. Comput.* **249**, 24-31 (2014).
[20] O. S. Iyiola and G. O. Ojo, On the analytical solution of Fornberg-Whitham equation with the new fractional derivative, *Pramana J. Phys.,* **85**, 567-575 (2015).
[21] O. S. Iyiola and F. D. Zaman, A fractional diffusion equation model for cancer tumor, *AIP Advances,* **4**, 107121 (2014); doi: 10.1063/1.4898331.

[22] O. S. Iyiola, Exact and approximate solutions of fractional diffusion equations with fractional reaction terms, *Progr. Fract. Differ. Appl.* **2**, 21-30 (2016).
[23] R. Khalil, M. A. Horani, A. Yousef and M. Sababheh, A new definition of fractional derivativel, *J. Comput. Appl. Math.* **264**, 65-70 (2014).
[24] T. Abdeljawad, On conformable fractional calculus, *J. Comput. Appl. Math.* **279**, 57–66 (2015).