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Existence of Positive Solutions to a Family of Fractional Two Point Boundary Value Problems

*Christina A. Hollon and Jeffrey T. Neugebauer**

Department of Mathematics and Statistics, Eastern Kentucky University, Richmond, Kentucky 40475 USA

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Abstract: In this paper we will consider an αth order fractional boundary value problem, $n-1<\alpha\le n$, $n\in\mathbb{N}$, with boundary conditions that include a fractional derivative at 1. We will develop properties of the Green's Function for this boundary value problem and use these properties along with the Contraction Mapping Principle, and the Schuader's, Krasnosel'skii's, and Leggett-Williams fixed point theorems to prove the existence of positive solutions under different conditions.

Keywords: Fractional boundary value problem, positive solution, fixed points.

1 Introduction

Let $n\ge 2$ denote an integer, and let α and β be positive reals such that $n-1<\alpha\le n$ and $0\le j\le\beta\le n-1$, for some $j\in\{0,\ 1,\dots,\ n-2\}$. We will consider the boundary value problem for the fractional differential equation given by

$$D^{\alpha}_{0+}u+a(t)f(u,u',\dots,u^{(j)})=0,\quad 0<t<1, \tag{1}$$

satisfying the boundary conditions

$$u^{(i)}(0)=0,\ i=0,\ 1,\ \dots,\ n-2,\quad D^{\beta}_{0+}u(1)=0, \tag{2}$$

where D^{α}_{0+} and D^{β}_{0+} are the standard Riemann-Liouville fractional derivatives. We make the following assumptions on the functions f and a:

(H1) $f:[0,\infty)^{j+1}\to[0,\infty)$ is continuous, and
(H2) $a:[0,1]\to[0,\infty)$ with $a\in L^{\infty}[0,1]$ and $|a|_{\infty}=M$.

The topic of fractional calculus, once thought to be inapplicable to real world situations, is now being studied in many branches of science due to the ability of fractional differential equations to model certain situations better than differential equations of integer order. Today, there are an increasing number of papers relating to differential equations of arbitrary order being published. The use of fixed point theory and cone-theoretic techniques to show the existence of solutions to difference equations, ordinary differential equations, and singular boundary value problems is abundant, (see [1,2,3]) but still far less work has been done to develop the existence of solutions to fractional, or arbitrary order differential equations, as in [4,5,6,7].

In this paper, we shall develop properties of the Green's function of (1), (2), constructed in [8], necessary to prove the existence of positive solutions under different conditions using the Contraction Mapping Principle and the Schuader Fixed Point Theorem. We will then restrict $n-2\le\beta<n-1$ and $j=n-2$ and prove the existence of positive solutions to the resulting boundary value problem when certain conditions are met, using Krasnosel'skii's and the Leggett-Williams fixed point theorems.

In the following section, we provide the fundamental definitions of fractional calculus. In the third section, we develop properties of the Green's function necessary to apply the Contraction Mapping Principle and Schuader Fixed

* Corresponding author e-mail: jeffrey.neugebauer@eku.edu

Point Theorem, which we do in the following two sections. In the sixth section, we restrict our problem and apply Krasnosel'skii's Fixed Point Theorem, and in the subsequent section we apply the Leggett-Williams Fixed Point Theorem.

2 Preliminary Definitions

Definition 1. *Let $\nu > 0$. The Riemann-Liouville fractional integral of a function u of order ν, denoted $I_{0^+}^{\nu} u$, is defined as*

$$I_{0^+}^{\nu} u(t) = \frac{1}{\Gamma(\nu)} \int_0^t (t-s)^{\nu-1} u(s)\, \mathrm{d}s, \tag{3}$$

provided that the right-hand side exists.

Definition 2. *Let n denote a positive integer, and assume that the positive real α satisfies $n-1 < \alpha \le n$. The Riemann-Liouville fractional derivative of order α of the function $u : [0,1] \to \mathbb{R}$, denoted $D_{0^+}^{\alpha} u$, is defined as*

$$\begin{aligned} D_{0^+}^{\alpha} u(t) &= \frac{1}{\Gamma(n-\alpha)} \frac{\partial^n}{\partial t^n} \int_0^t (t-s)^{n-\alpha-1} u(s)\, \mathrm{d}s, \\ &= D^n I_{0^+}^{n-\alpha} u(t), \end{aligned} \tag{4}$$

provided the right-hand side exists.

3 The Green's Function

The Green's Function for the boundary value problem (1), (2) is given by (see [8])

$$G(\beta;t,s) = \begin{cases} \dfrac{t^{\alpha-1}(1-s)^{\alpha-1-\beta}}{\Gamma(\alpha)} - \dfrac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}, & \text{if } 0 \le s \le t < 1, \\[2ex] \dfrac{t^{\alpha-1}(1-s)^{\alpha-1-\beta}}{\Gamma(\alpha)}, & \text{if } 0 \le t \le s < 1. \end{cases} \tag{5}$$

Thus, u is a solution of (1), (2) if and only if

$$u(t) = \int_0^1 G(\beta;t,s) a(s) f(u(s), u'(s), \dots, u^{(j)}(s))\, \mathrm{d}s, \quad 0 \le t \le 1.$$

We will develop properties of (5) to prove the existence of positive solutions to (1), (2).

Lemma 1. *Let β be a positive real and $j \in \{0,1,\dots,n-2\}$ be an integer, satisfying $0 \le j \le \beta \le n-1$. The kernel, $G(\beta;t,s)$, satisfies the following properties:*

$$\frac{\partial^i}{\partial t^i} G(\beta;t,s) \ge 0, \ (t,s) \in [0,1] \times [0,1), \quad i = 0,1,\dots,j, \tag{6}$$

$$\max_{0 \le t \le 1} \int_0^1 \frac{\partial^i}{\partial t^i} G(\beta;t,s)\, \mathrm{d}s = \frac{(\alpha-i)t_i^{\alpha-i-1} - (\alpha-\beta)t_i^{\alpha-1}}{\Gamma(\alpha-i)(\alpha-\beta)(\alpha-i)} := \overline{G}_i, \tag{7}$$

where

$$t_i = \min\left\{ \frac{(\alpha-1-i)}{(\alpha-\beta)},\ 1 \right\}.$$

Proof. Define, for $0 \le s \le t < 1$, the function g_1 by

$$g_1(\beta;t,s) = \frac{t^{\alpha-1}(1-s)^{\alpha-1-\beta} - (t-s)^{\alpha-1}}{\Gamma(\alpha)},$$

and define, for $0 \le t \le s < 1$, the function g_2 by

$$g_2(\beta;t,s) = \frac{t^{\alpha-1}(1-s)^{\alpha-1-\beta}}{\Gamma(\alpha)},\ 0 \le t \le s < 1.$$

In order to prove (6), let s and t be positive reals such that $0 \le s \le t < 1$, and let $i \in \{0,\ 1, \dots,\ j\}$. Then

$$\begin{aligned}
\frac{\partial^i}{\partial t^i} G(\beta;t,s) &= \frac{\partial^i}{\partial t^i} g_1(\beta;t,s) \\
&= \frac{\partial^i}{\partial t^i} \frac{t^{\alpha-1}(1-s)^{\alpha-1-\beta} - (t-s)^{\alpha-1}}{\Gamma(\alpha)} \\
&= \frac{1}{\Gamma(\alpha)} \frac{\partial^i}{\partial t^i} [t^{\alpha-1}(1-s)^{\alpha-1-\beta} - (t-s)^{\alpha-1}] \\
&= \frac{1}{\Gamma(\alpha)} \left[\frac{(1-s)^{\alpha-1-\beta}\Gamma(\alpha)t^{\alpha-1-i}}{\Gamma(\alpha-i)} - \frac{\Gamma(\alpha)(t-s)^{\alpha-1-i}}{\Gamma(\alpha-i)} \right] \\
&= \frac{1}{\Gamma(\alpha-i)} [(1-s)^{\alpha-1-\beta} t^{\alpha-1-i} - (t-s)^{\alpha-1-i}] \\
&\ge \frac{1}{\Gamma(\alpha-i)} [(1-s)^{\alpha-1-i} t^{\alpha-1-i} - (t-s)^{\alpha-1-i}] \\
&= \frac{1}{\Gamma(\alpha-i)} [(t-ts)^{\alpha-1-i} - (t-s)^{\alpha-1-i}].
\end{aligned}$$

But $ts < s$, and hence $\frac{1}{\Gamma(\alpha-i)}[(t-ts)^{\alpha-1-i} - (t-s)^{\alpha-1-i}] > 0$, implying that $\frac{\partial^i}{\partial t^i} g_1(\beta;t,s) \ge 0$.

Next, let $0 \le t \le s < 1$ and $i \in \{0,\ 1, \dots,\ j\}$. Then

$$\begin{aligned}
\frac{\partial^i}{\partial t^i} G(\beta;t,s) &= \frac{\partial^i}{\partial t^i} g_2(\beta;t,s) \\
&= \frac{\partial^i}{\partial t^i} \frac{t^{\alpha-1}(1-s)^{\alpha-1-\beta}}{\Gamma(\alpha)} \\
&= \frac{1}{\Gamma(\alpha)} \frac{(1-s)^{\alpha-1-\beta}\Gamma(\alpha)t^{\alpha-1-i}}{\Gamma(\alpha-i)} \\
&= \frac{1}{\Gamma(\alpha-i)} (1-s)^{\alpha-1-\beta} t^{\alpha-1-i} \\
&\ge 0,
\end{aligned}$$

and hence $\frac{\partial^i}{\partial t^i} G(\beta;t,s) \ge 0$, when $i \in \{0,\ 1, \dots,\ j\}$. This proves (6).

Now,

$$\begin{aligned}
\int_0^t \frac{\partial^i}{\partial t^i} g_1(\beta;t,s)\,\mathrm{d}s &= \int_0^t \frac{\partial^i}{\partial t^i} \left(\frac{t^{\alpha-1}(1-s)^{\alpha-1-\beta} - (t-s)^{\alpha-1}}{\Gamma(\alpha)} \right) \mathrm{d}s \\
&= \frac{t^{\alpha-1-i}(\alpha-i) - t^{\alpha-1-i}(1-t)^{\alpha-\beta}(\alpha-i) - t^{\alpha-i}(\alpha-\beta)}{\Gamma(\alpha-i)(\alpha-\beta)(\alpha-i)},
\end{aligned}$$

and

$$\begin{aligned}
\int_t^1 \frac{\partial^i}{\partial t^i} g_2(\beta;t,s)\,\mathrm{d}s &= \int_t^1 \frac{\partial^i}{\partial t^i} \left(\frac{t^{\alpha-1}(1-s)^{\alpha-1-\beta}}{\Gamma(\alpha)} \right) \mathrm{d}s \\
&= \frac{t^{\alpha-1-i}(1-t)^{\alpha-\beta}}{\Gamma(\alpha-i)(\alpha-\beta)}.
\end{aligned}$$

Hence

$$\int_0^1 \frac{\partial^i}{\partial t^i} G(\beta;t,s)\,\mathrm{d}s = \frac{t^{\alpha-i-1}(\alpha-i) - t^{\alpha-i}(\alpha-\beta)}{\Gamma(\alpha-i)(\alpha-\beta)(\alpha-i)}.$$

Now, by properties of the first derivative of any function, $\max_{t\in[0,1]} \int_0^1 \frac{\partial^i}{\partial t^i} G(\beta;t,s)\,\mathrm{d}s$ occurs when

$$\begin{aligned}\frac{\partial}{\partial t}\left[\int_0^1 \frac{\partial^i}{\partial t^i} G(\beta;t,s)\,\mathrm{d}s\right] &= \frac{\partial}{\partial t}\frac{(\alpha-i)t^{\alpha-i-1} - (\alpha-\beta)t^{\alpha-i}}{\Gamma(\alpha-i)(\alpha-\beta)(\alpha-i)}\\ &= (\alpha-1-i)(\alpha-i)t^{\alpha-2-i} - (\alpha-i)(\alpha-\beta)t^{\alpha-1-i}\\ &= 0,\end{aligned}$$

which occurs when $t = \frac{\alpha-1-i}{\alpha-\beta}$. Note that if $\frac{\alpha-1-i}{\alpha-\beta} > 1$, then by (6), the maximum occurs when $t=1$. It follows that

$$\max_{0\le t\le 1} \int_0^1 \frac{\partial^i}{\partial t^i} G(\beta;t,s)\,\mathrm{d}s = \frac{(\alpha-i)t_i^{\alpha-i-1} - (\alpha-\beta)t_i^{\alpha-i}}{\Gamma(\alpha-i)(\alpha-\beta)(\alpha-i)},$$

where $t_i = \min\left\{\frac{\alpha-i-1}{\alpha-\beta},\ 1\right\}$, which proves (7).

Throughout this paper, we will make use of the Banach space

$$\mathscr{B} = \{u \in C^{(j)}[0,1] : u(0) = u'(0) = \cdots = u^{(j-1)}(0) = 0\},$$

endowed with the norm

$$\|u\| = \max_{0\le t\le 1} |u^{(j)}(t)| = |u^{(j)}|_0,$$

and the operator $T : \mathscr{B} \to \mathscr{B}$ by

$$(Tu)(t) = \int_0^1 G(t,s) f(s,u(s),\ldots,u^{(j)}(s))\,\mathrm{d}s \tag{8}$$

for all $t\in[0,1]$ and $u\in\mathscr{B}$. First, notice if $\hat{u}$ is a fixed point of T, $\hat{u}$ solves (1), (2). Also, notice that, for $i = 1,\ 2,\ldots,j$,

$$\begin{aligned}|u^{(j-i)}(t)| &= |u^{(j-i)}(t) - u^{(j-i)}(0)|\\ &= \left|\int_0^t u^{(j+1-i)}(s)\,\mathrm{d}s\right|\\ &\le \int_0^t \left|u^{(j+1-i)}(s)\right|\,\mathrm{d}s\\ &\le |u^{(j+1-i)}(t)|\\ &\le |u^{(j+1-i)}|_0.\end{aligned}$$

Therefore, $|u|_0 \le |u'|_0 \le \cdots \le |u^{(j-1)}|_0 \le |u^{(j)}|_0 = \|u\|$.

4 Contraction Mapping Principle

The theory behind the use of the Contraction Mapping Principle in proving the existence of fixed points for differential equations has been studied in papers such as [9]. These authors make use of the Contraction Mapping Principle to show the existence of solutions to differential equations of integer order in partially ordered and ordered metric spaces. The existence and uniqueness of solutions to a nonlinear fractional Cauchy problem in a special Banach space is developed in [10]. We will develop a theorem and proof for the existence and uniqueness of solutions of problem (1), (2).

Theorem 1. *Assume (H2) is satisfied and f satisfies a Lipschitz condition $|f(y_0,y_1,\ldots,y_j) - f(z_0,z_1,\ldots,z_j)| \le k|y_j - z_j|$ on $[0,\infty)^{j+1}$. Then, if $Mk\overline{G}_j < 1$, (1), (2) has a unique solution.*

*Proof.*Notice that

$$|(Tu)^{(j)}(t)-(Tv)^{(j)}(t)| \leq Mk\|u-v\| \int_0^1 \frac{\partial^j G(\beta;t,s)}{\partial t^j}\,\mathrm{d}s \\ \leq Mk\overline{G}_j\|u-v\|,$$

and, consequently,

$$\|Tu-Tv\| \leq Mk\overline{G}_j\|u-v\|.$$

Hence, since $Mk\overline{G}_j < 1$, T is a contraction mapping on $\mathscr{B}$, and thus T has a unique fixed point $\hat{u}$, which is the unique solution of (1), (2).

5 Schauder Fixed Point Theorem

The Schauder Fixed Point Theorem has been utilized in the study and proof of existence of solutions to fractional order differential equations and systems of fractional order differential equations as well, see [11,12]. We will use the Schauder fixed point theorem to show the existence of positive solutions of (1), (2).

Theorem 2(Schauder Fixed Point Theorem [13]). *If $\mathscr{M}$ is a closed, bounded, convex subset of a Banach space $\mathscr{B}$ and $T:\mathscr{M}\to\mathscr{M}$ is completely continuous, then T has a fixed point in $\mathscr{M}$.*

Lemma 2.*The operator T is completely continuous on $\mathscr{M}$, where for fixed $N>0$, the set $\mathscr{M}$ is defined to be $\mathscr{M}=\{u\in\mathscr{B}:\|u\|<N\}$.*

The proof is a standard application of the Arzelà-Ascoli theorem.

Theorem 3.*Let N be fixed and define $\mathscr{M}=\{u\in\mathscr{B}:\|u\|<N\}$ and $\|u\|=|u^{(j)}|_0$. Then (1), (2) has a solution in $\mathscr{M}$.*

*Proof.*By definition, $\mathscr{M}$ is bounded.

To see that $\mathscr{M}$ is closed, let $\{h_i\}_{i=1}^{\infty}\subseteq\mathscr{M}$, and let $h_0\in\mathscr{B}$ be such that $\|h_i-h_0\|\to 0$ as $i\to\infty$. Then $h_i^{(j)}\to h_0^{(j)}$ on $[0,1]$. Thus, since $h_i\in\mathscr{M}$ for all i, $|h_i^{(j)}|\leq N$ for all i, and $|h_0^{(j)}(x)|\leq N$ on $[0,1]$. So, $\|h_0\|\leq N$, and $h_0\in\mathscr{M}$. Hence, $\mathscr{M}$ is closed.

Let $h,\ g\in\mathscr{M}$, and, for real λ with $0\leq\lambda\leq 1$, consider $\lambda h+(1+\lambda)g$. Well, since $h,\ g\in\mathscr{M}$, we have

$$|\lambda h^{(j)}(x)+(1-\lambda)g^{(j)}(x)| \leq |\lambda h^{(j)}(x)|+|(1-\lambda)g^{(j)}(x)| \\ = \lambda|h^{(j)}(x)|+(1-\lambda)|g^{(j)}(x)| \\ \leq \lambda N+(1-\lambda)N \\ = N.$$

Hence $\lambda h+(1-\lambda)g\in\mathscr{M}$ for all $h,\ g\in\mathscr{M}$, and $\mathscr{M}$ is convex.

From Lemma 2, T is completely continuous on $\mathscr{M}$. Hence, the assumptions of the Schauder Fixed Point Theorem are met, and thus T has a fixed point in $\mathscr{M}$ which is a solution of (1), (2).

6 Krasnosel'skii's Fixed Point Theorem

In this section, we will use Krasnosel'skii's well-known fixed point theorem for operators acting on a cone in a Banach space. In order to apply Krasnosel'skii's fixed point theorem, we need $n-2\leq\beta<n-1$ and $j=n-2$. So the specified boundary value problem is

$$D_{0^+}^{\alpha}u+a(t)f(u,u',\ldots,u^{(n-2)})=0,\quad 0<t<1, \tag{9}$$

satisfying the boundary conditions

$$u^{(i)}(0)=0,\ i=0,\ 1,\ \ldots,\ n-2,\quad D_{0^+}^{\beta}u(1)=0, \tag{10}$$

where $D_{0^+}^{\alpha}$ and $D_{0^+}^{\beta}$ are the standard Riemann-Liouville fractional derivatives. The Banach space and norm used in the forthcoming analysis is the same as the Banach space and norm used above, with $j=n-2$ as well. Some authors have used Krasnosel'skii's fixed point theorem to show the existence of solutions of ordinary differential equations, difference equations, and dynamic equations on time scales; however, few papers have been published that were devoted to the study of boundary value problems of fractional order as in [4,5,6], where the authors develop proofs for the existence of positive solutions to the nonlinear fractional boundary value problems

$$D^{\alpha}u+a(t)f(u)=0, \quad 0<t<1, \quad 1<\alpha\leq 2,$$

and

$$D^{\alpha}u+a(t)f(u)=0, \quad 0<t<1, \quad 3<\alpha\leq 4,$$

satisfying boundary conditions

$$u(0)=0=u'(1)=0,$$

and

$$u(0)=0=u'(0)=u''(0)=u'(1)=0,$$

respectively, which are two specific cases of problem (9), (10). We seek to show the existence of positive solutions of (9), (10) for arbitrary positive integer n and positive real α, $n-1<\alpha\leq n$.

Theorem 4(Krasnosel'skii's Fixed Point Theorem [14]). *Let $\mathscr{B}$ be a Banach space, and let $\mathscr{K}\subset\mathscr{B}$ be a cone in $\mathscr{B}$. Assume that Ω_1, Ω_2 are open sets with $0\in\Omega_1$, and $\overline{\Omega}_1\subset\Omega_2$. Let $T:\mathscr{K}\cap(\overline{\Omega}_2\backslash\Omega_1)\to\mathscr{K}$ be a completely continuous operator such that either*

(i)$\|Tu\|\leq\|u\|$, $u\in\mathscr{K}\cap\partial\Omega_1$, and $\|Tu\|\geq\|u\|$, $u\in\mathscr{K}\cap\partial\Omega_2$, or
(ii)$\|Tu\|\geq\|u\|$, $u\in\mathscr{K}\cap\partial\Omega_1$, and $\|Tu\|\leq\|u\|$, $u\in\mathscr{K}\cap\partial\Omega_2$.

Then T has a fixed point in $\mathscr{K}\cap(\overline{\Omega}_2\backslash\Omega_1)$.

We will need the following additional properties of the Green's function.

Lemma 3.*Let γ and s be fixed nonnegative reals, with $0\leq\gamma\leq s<1$, and let β be a positive real such that $n-2<\beta\leq n-1$. The kernel, $G(\beta;t,s)$, satisfies the following properties:*

$$\begin{aligned}\overline{G}_{n-2}&=\max_{t\in[0,1]}\int_0^1\frac{\partial^{n-2}}{\partial t^{n-2}}G(\beta;t,s)\,\mathrm{d}s\\&=\frac{(\alpha-n+1)^{\alpha-n+1}(\alpha-n+2)-(\alpha-n+1)^{\alpha-n+2}}{(\alpha-\beta)^{\alpha-n+2}\Gamma(\alpha-n+2)(\alpha-n+2)},\end{aligned}\tag{11}$$

where $\overline{G}_{n-2}$ is the specific case of $\overline{G}_i$ as defined in Lemma 1 where $i=n-2$, and

$$\min_{\gamma\leq t\leq 1}\frac{\partial^{n-2}}{\partial t^{n-2}}G(\beta;t,s)\geq[1-(1-\gamma)^{\beta-n+2}]\gamma^{\alpha-n+1}s\frac{\partial^{n-2}}{\partial t^{n-2}}G(\beta;s,s).\tag{12}$$

*Proof.*Let $i=n-2$. Notice that $t_{n-2}=\dfrac{\alpha-n+1}{\alpha-\beta}$ since $\alpha-\beta\geq\alpha-n+1$, implying that $\dfrac{\alpha-n+1}{\alpha-\beta}\leq 1$. Hence,

$$\begin{aligned}\max_{0\leq t\leq 1}\int_0^1\frac{\partial^{n-2}}{\partial t^{n-2}}G(\beta;t,s)\,\mathrm{d}s&=\frac{\dfrac{(\alpha-n+1)^{\alpha+1-n}}{(\alpha-\beta)^{\alpha+1-n}}(\alpha-n+2)-\dfrac{(\alpha-n+1)^{\alpha-n+2}}{(\alpha-\beta)^{\alpha-n+2}}}{(\alpha-\beta)^{\alpha-n+2}\Gamma(\alpha-n+2)(\alpha-n+2)}\\&=\frac{(\alpha-n+1)^{\alpha-n+1}(\alpha-n+2)-(\alpha-n+1)^{\alpha-n+2}}{(\alpha-\beta)^{\alpha-n+2}\Gamma(\alpha-n+2)(\alpha-n+2)},\end{aligned}$$

which proves (11).

To prove (12), note that

$$\frac{\partial^{n-1}}{\partial t^{n-1}}g_1(t,s)=\frac{(1-s)^{\alpha-1-\beta}t^{\alpha-n}-(t-s)^{\alpha-n}}{\Gamma(\alpha-n+1)}.$$

Now,

$$\begin{aligned}(1-s)^{\alpha-1-\beta}t^{\alpha-n}-(t-s)^{\alpha-n}&=(1-s)^{\alpha-\beta-1}t^{\alpha-n}-\left(t\left(1-\frac{s}{t}\right)\right)^{\alpha-n}\\&=t^{\alpha-n}\left((1-s)^{\alpha-\beta-1}-\left(1-\frac{s}{t}\right)^{\alpha-n}\right).\end{aligned}$$

Note that, if $t=0$, then $s=0$, and thus, $\frac{\partial^{n-1}}{\partial t^{n-1}}g_1(t,s)=0$. If $0<s\le t<1$, then $\frac{1}{t}>1$, and since, s is positive, $\frac{s}{t}>s$. This implies that $1-\frac{s}{t}<1-s$, and, since $-1<\alpha-n\le 0$ and $n-1<\beta+1\le n$, $\left(1-\frac{s}{t}\right)^{\alpha-n}>(1-s)^{\alpha-n}\ge(1-s)^{\alpha-\beta-1}$. Therefore $(1-s)^{\alpha-\beta-1}-\left(1-\frac{s}{t}\right)^{\alpha-n}<0$, and, consequently, $\frac{\partial^{n-1}}{\partial t^{n-1}}g_1(t,s)<0$. Also note that

$$\frac{\partial^{n-1}}{\partial t^{n-1}}g_2(t,s)=\frac{(1-s)^{\alpha-1-\beta}t^{\alpha-n}}{\Gamma(\alpha-n+1)}>0,$$

since $(1-s)^{\alpha-1-\beta}t^{\alpha-n}>0$.

Since $\frac{\partial^{n-1}}{\partial t^{n-1}}g_1(t,s)<0$, $\frac{\partial^{n-2}}{\partial t^{n-2}}g_1(t,s)$ is a decreasing function of t. Hence, for $0\le\gamma\le s<1$,

$$\begin{aligned}\min_{\gamma\le t\le 1}\frac{\partial^{n-2}}{\partial t^{n-2}}g_1(t,s)&=\frac{\partial^{n-2}}{\partial t^{n-2}}g_1(1,s)\\&=\frac{(1-s)^{\alpha-1-\beta}-(1-s)^{\alpha-n+1}}{\Gamma(\alpha-n+2)}\\&=\frac{(1-s)^{\alpha-1-\beta}[1-(1-s)^{\beta-n+2}]}{\Gamma(\alpha-n+2)}\\&\ge\frac{(1-s)^{\alpha-1-\beta}[1-(1-\gamma)^{\beta-n+2}]}{\Gamma(\alpha-n+2)}\\&\ge\frac{(1-s)^{\alpha-1-\beta}[1-(1-\gamma)^{\beta-n+2}]\gamma^{\alpha-n+1}s^{\alpha-n+2}}{\Gamma(\alpha-n+2)}\\&=[1-(1-\gamma)^{\alpha-n+2}]\gamma^{\alpha-n+1}s\frac{(1-s)^{\alpha-1-\beta}s^{\alpha-n+1}}{\Gamma(\alpha-n+2)}\\&=[1-(1-\gamma)^{\beta-n+2}]\gamma^{\alpha-n+1}s\frac{\partial^{n-2}}{\partial t^{n-2}}G(s,s).\end{aligned}$$

Note that $\frac{\partial^{n-2}}{\partial t^{n-2}}g_2(t,s)$ is an increasing function of t.
Hence, for $0\le\gamma\le s<1$,

$$\begin{aligned}\min_{\gamma\le t\le 1}\frac{\partial^{n-2}}{\partial t^{n-2}}g_2(t,s)&=\frac{\partial^{n-2}}{\partial t^{n-2}}g_2(\gamma,s)\\&=\frac{(1-s)^{\alpha-1-\beta}\gamma^{\alpha-n+1}}{\Gamma(\alpha-n+2)}\\&\ge\frac{(1-s)^{\alpha-1-\beta}\gamma^{\alpha-n+1}[1-(1-\gamma)^{\beta-n+2}]s^{\alpha-n+2}}{\Gamma(\alpha-n+2)}\\&=[1-(1-\gamma)^{\beta-n+2}]\gamma^{\alpha-n+1}s\frac{\partial^{n-2}}{\partial t^{n-2}}G(s,s).\end{aligned}$$

Thus, $\min_{\gamma\le t\le 1}\frac{\partial^{n-2}}{\partial t^{n-2}}G(\beta;t,s)\ge[1-(1-\gamma)^{\beta-n+2}]\gamma^{\alpha-n+1}s\frac{\partial^{n-2}}{\partial t^{n-2}}G(\beta;s,s)$ for all $\gamma\le s<1$, which proves (12).

We need an additional assumption on a.

(H3)There exists a $\gamma\in(0,1)$ and an $m>0$ such that $a(t)>m$ a. e. on $[\gamma,1]$.

We first use the contractive portion of Krasnosel'skii's fixed point theorem.

Theorem 5. *Suppose that (H1) and (H2) are satisfied and that there exists a $\gamma \in (0,1)$ such that (H3) is satisfied. Let $A, B \in \mathbb{R}$ with $0 \le A \le \dfrac{1}{\overline{G}_{n-2}M}$ and*

$$B \ge \left[m[1-(1-\gamma)^{\beta-n+2}]\gamma^{\alpha-n+1} \int_\gamma^1 s \frac{\partial^{n-2}}{\partial t^{n-2}} G(\beta; s,s)\, \mathrm{d}s \right]^{-1}.$$

If there exist positive constants r and R with $r < R$ and $Br < AR$, and if f satisfies

(A1) $f(x_0, x_1, \dots, x_{n-2}) \le AR$ *for all* $(x_0, x_1, \dots, x_{n-2}) \in [0,R]^{n-1}$, *and*
(A2) $f(x_0, x_1, \dots, x_{n-2}) \ge Br$ *for all* $(x_0, x_1, \dots, x_{n-2}) \in [0,r]^{n-1}$,

then (9), (10) has at least one positive solution u with $r < \|u\| < R$.

Proof. Define the cone

$$\mathscr{K} = \{u \in \mathscr{B} : u^{(n-2)}(t) \ge 0 \text{ for all } t \in [0,1]\}. \tag{13}$$

Define the open set $\Omega_2 = \{u \in \mathscr{B} : \|u\| < R\}$. Let $u \in K \cap \partial\Omega_2$. Then assumption (A1) and (7) give

$$\begin{aligned}
|Tu^{(n-2)}|(t) &= \left| \int_0^1 \frac{\partial^{n-2}}{\partial t^{n-2}} G(t,s) a(s) f(u(s), u'(s), \dots, u^{(n-2)}(s))\, \mathrm{d}s \right| \\
&\le \int_0^1 \left| \frac{\partial^{n-2}}{\partial t^{n-2}} G(\beta; t,s) \right| |a(s)| |f(s, u(s), \dots, u^{(n-2)}(s))| \,\mathrm{d}s \\
&\le MAR \int_0^1 \left| \frac{\partial^{n-2}}{\partial t^{n-2}} G(\beta; t,s) \right| \mathrm{d}s \\
&\le MAR\overline{G}_{n-2} \\
&\le R \\
&= \|u\|.
\end{aligned}$$

So, $\|Tu\| \le \|u\|$ for all $u \in \mathscr{K} \cap \partial\Omega_2$.

Next, define the open set $\Omega_1 = \{u \in \mathscr{B} : \|u\| < r\}$. Let $u \in K \cap \partial\Omega_1$. Then, using (H1)-(H3), assumption (A2) and (12), we have that

$$\begin{aligned}
Tu^{(n-2)}(t) &\ge \int_0^1 \frac{\partial^{n-2}}{\partial t^{n-2}} G(t,s) a(s) f(u(s), u'(s), \dots, u^{(n-2)}(s))\, \mathrm{d}s \\
&\ge \int_\gamma^1 \frac{\partial^{n-2}}{\partial t^{n-2}} G(\beta; t,s) a(s) f(u(s), u'(s), \dots, u^{(n-2)}(s))\, \mathrm{d}s \\
&\ge mBr \int_\gamma^1 \frac{\partial^{n-2}}{\partial t^{n-2}} G(\beta; t,s)\, \mathrm{d}s \\
&\ge mBr \int_\gamma^1 [1-(1-\gamma)^{\beta-n+2}]\gamma^{\alpha-n+1} s \frac{\partial^{n-2}}{\partial t^{n-2}} G(\beta; s,s)\, \mathrm{d}s \\
&= mBr[1-(1-\gamma)^{\beta-n+2}]\gamma^{\alpha-n+1} \int_\gamma^1 s \frac{\partial^{n-2}}{\partial t^{n-2}} G(\beta; s,s)\, \mathrm{d}s \\
&\ge r \\
&= \|u\|.
\end{aligned}$$

Therefore, $\|Tu\| \ge \|u\|$ for all $u \in K \cap \partial\Omega_1$. Since $0 \in \Omega_1 \subset \Omega_2$, the contractive part of Kraznosel'skii's Theorem gives the existence of at least one fixed point of T in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$. So, there exists at least one solution of u of (9), (10) with $r < \|u\| \le R$.

The expansive part of Krasnosel'skii's fixed point theorem can also be applied. The proof is similar to Theorem 5 and is therefore omitted.

Theorem 6. *Suppose that (H1) and (H2) are satisfied and that there exists $\gamma \in (0,1)$ such that (H3) is satisfied. Let $A, B \in \mathbb{R}$ with $0 \le A \le \dfrac{1}{\overline{G}_{n-2}M}$ and*

$$B \ge \left[m[1-(1-\gamma)^{\beta-n+2}]\gamma^{\alpha-n+1} \int_\gamma^1 s \frac{\partial^{n-2}}{\partial t^{n-2}} G(\beta; s,s)\, \mathrm{d}s \right]^{-1}.$$

If there exist positive constants r and R such that $r < R$ and $Ar < BR$, and if f satisfies

(A3) $f(x_0, x_1, \ldots, x_{n-2}) \ge BR$ for all $(x_0, x_1, \ldots, x_{n-2}) \in [0,R]^{n-1}$ and
(A4) $f(x_0, x_1, \ldots, x_{n-2}) \le Ar$ for all $(x_0, x_1, \ldots, x_{n-2}) \in [0,r]^{n-1}$.

then (9), (10) has at least one positive solution u with $r < \|u\| < R$.

As is shown in the following theorems, Krasnosel'skii's fixed point theorem can be used to show the existence of finitely many solutions to (9), (10).

Theorem 7. *Suppose that (H1) and (H2) are satisfied and that there exists $\gamma \in (0,1)$ such that (H3) is satisfied. Let $A, B \in \mathbb{R}$ with $0 \le A \le \dfrac{1}{\overline{G}_{n-2}M}$ and*

$$B \ge \left[m[1-(1-\gamma)^{\beta-n+2}]\gamma^{\alpha-n+1} \int_\gamma^1 s \frac{\partial^{n-2}}{\partial t^{n-2}} G(s,s)\, \mathrm{d}s \right]^{-1}.$$

If there exists a $k \in \mathbb{N}$ such that there are positive constants r_i and R_i for $i = 1, 2, \ldots, k$ such that $r_1 < R_1 < r_2 < R_2 < \cdots < r_k < R_k$ and $Br_i < AR_i$, and if f satisfies

(A5) $f(x_0, x_1, \ldots, x_{n-2}) \le AR_i$ for all $(x_0, x_1, \ldots, x_{n-2}) \in [0,R_i]^{n-1}$, and
(A6) $f(x_0, x_1, \ldots, x_{n-2}) \ge Br_i$ for all $(x_0, x_1, \ldots, x_{n-2}) \in [0,r_i]^{n-1}$,

then (9), (10) has at least k positive solutions u_i, where u_i satisfies $r_i < \|u\| < R_i$.

Proof. Define open sets $\Omega_{2_i} = \{u \in \mathscr{B} : \|u\| < R_i\}$ for $i = 1, \ldots, k$ and $\Omega_{1_i} = \{u \in \mathscr{B} : \|u\| < r_i\}$ for $i = 1, \ldots, k$. Then a proof similar to the proof of Theorem 5 shows the existence of at least one fixed point of T in $K \cap (\overline{\Omega}_{2_i} \backslash \Omega_{1_i})$ for each i. So, there exists at least one solution of u_i of (9), (10) with $r_i < \|u\| \le R_i$ for each $i = 1, \ldots, k$.

The expansive part of Krasnosel'skii's fixed point theorem can also be applied to show the solutions of finitely many solutions. The proof is omitted since it is similar to the proof above.

Theorem 8. *Suppose that (H1) and (H2) are satisfied and that there exists $\gamma \in (0,1)$ such that (H3) is satisfied. Let $A, B \in \mathbb{R}$ with $0 \le A \le \dfrac{1}{\overline{G}_{n-2}M}$ and*

$$B \ge \left[m[1-(1-\gamma)^{\beta-n+2}]\gamma^{\alpha-n+1} \int_\gamma^1 s \frac{\partial^{n-2}}{\partial t^{n-2}} G(s,s)\, \mathrm{d}s \right]^{-1}.$$

If there exists a $k \in \mathbb{N}$ such that there are positive constants r_i and R_i for $i = 1, 2, \ldots, k$ such that $r_1 < R_1 < r_2 < R_2 < \cdots < r_k < R_k$ and $Br_i < AR_i$, and if f satisfies

(A7) $f(x_0, x_1, \ldots, x_{n-2}) \ge BR_i$ for all $(x_0, x_1, \ldots, x_{n-2}) \in [0,R_i]^{(n-1)}$, and
(A8) $f(x_0, x_1, \ldots, x_{n-2}) \le Ar_i$ for all $(x_0, x_1, \ldots, x_{n-2}) \in [0,r_i]^{(n-1)}$,

then (9), (10) has at least k positive solutions u_i, where u_i satisfies $r_i < \|u\| < R_i$.

Krasnosel'skii's fixed point theorem can be also used to show the existence of infinitely many solutions to (9), (10). The proofs are similar to the proofs of the theorems above and are therefore omitted.

Theorem 9. *Suppose that (H1) and (H2) are satisfied and that there exists $\gamma \in (0,1)$ such that (H3) is satisfied. Let $A, B \in \mathbb{R}$ with $0 \le A \le \dfrac{1}{\overline{G}_{n-2}M}$ and*

$$B \ge \left[m[1-(1-\gamma)^{\beta-n+2}]\gamma^{\alpha-n+1} \int_\gamma^1 s \frac{\partial^{n-2}}{\partial t^{n-2}} G(s,s)\, \mathrm{d}s \right]^{-1}.$$

If there are positive constants r_i and R_i for $i = 1, 2, \ldots$ such that $r_1 < R_1 < r_2 < R_2 < \cdots$ and $Br_i < AR_i$, and if f satisfies

(A9) $f(x_0,x_1,\ldots,x_{n-2}) \leq AR_i$ *for all* $(x_0,x_1,\ldots,x_{n-2}) \in [0,R_i]^{n-1}$, *and*
(A10) $f(x_0,x_1,\ldots,x_{n-2}) \geq Br_i$ *for all* $(x_0,x_1,\ldots,x_{n-2}) \in [0,r_i]^{n-1}$,

then (9), (10) *has infinitely many positive solutions* u_i, *where* u_i *satisfies* $r_i < \|u\| < R_i$.

Theorem 10. *Suppose that (H1) and (H2) are satisfied and that there exists* $\gamma \in (0,1)$ *such that (H3) is satisfied. Let* $A,B \in \mathbb{R}$ *with* $0 \leq A \leq \dfrac{1}{\overline{G}_{n-2}M}$ *and* $B \geq \left[m[1-(1-\gamma)^{\beta-n+2}]\gamma^{\alpha-n+1}\int_{\gamma}^{1} s\frac{\partial^{n-2}}{\partial t^{n-2}}G(s,s)\,\mathrm{d}s\right]^{-1}$.
If there are positive constants r_i *and* R_i *for* $i = 1,\ 2,\ \ldots$ *such that* $r_1 < R_1 < r_2 < R_2 < \cdots$ *and* $Br_i < AR_i$, *and if* f *satisfies*

(A11) $f(x_0,x_1,\ldots,x_{n-2}) \geq BR_i$ *for all* $(x_0,x_1,\ldots,x_{n-2}) \in [0,R_i]^{(n-1)}$, *and*
(A12) $f(x_0,x_1,\ldots,x_{n-2}) \leq Ar_i$ *for all* $(x_0,x_1,\ldots,x_{n-2}) \in [0,r_i]^{(n-1)}$,

then (9), (10) *has infinitely many positive solutions* u_i, *where* u_i *satisfies* $r_i < \|u\| < R_i$.

7 The Leggett-Williams Fixed Point Theorem

In this section, we will consider (9) and (10) along with the Banach space $\mathscr{B}$, the cone $\mathscr{K}$, and the operator T defined in the previous section. To again show the existence of multiple solutions, we will use the Leggett-Williams fixed point theorem, as in [6]. In order to do this, for α a positive concave functional, we define the following subsets of $\mathscr{K}$:

$$\mathscr{K}_c = \{u \in \mathscr{K} : \|u\| < c\},$$

$$\mathscr{K}_a = \{u \in \mathscr{K} : \|u\| < a\},$$

$$\mathscr{K}(\alpha,b,d) = \{u \in \mathscr{K} : b \leq \alpha(u),\ \|u\| \leq d\},\ \text{and}$$

$$\mathscr{K}(\alpha,b,c) = \{u \in \mathscr{K} : b \leq \alpha(u),\ \|u\| \leq c\}.$$

Theorem 11(Leggett-Williams [15]). *Suppose that* $T : \overline{\mathscr{K}}_c \to \overline{\mathscr{K}}_c$ *is completely continuous, and suppose there exists a concave positive functional* α *on* $\mathscr{K}$ *such that* $\alpha(u) \leq \|u\|$ *for* $u \in \overline{\mathscr{K}}_c$. *Suppose there exist constants* $0 < a < b < d \leq c$ *such that*

1. $\{u \in \mathscr{K}(\alpha,\ b,\ d) : \alpha(u) > b\} \neq \emptyset$ *and* $\alpha(Tu) > b$ *if* $u \in \mathscr{K}(\alpha,\ b,\ d)$;
2. $\|Tu\| < u$ *if* $u \in \mathscr{K}_a$; *and*
3. $\alpha(Tu) > b$ *for* $u \in \mathscr{K}(\alpha,\ b,\ c)$ *with* $\|Tu\| > d$.

Then T *has at least three fixed points* $u_1,\ u_2,$ *and* u_3 *such that* $\|u_1\| < a$, $b < \alpha(u_2)$, *and* $\|u_3\| > a$ *with* $\alpha(u_3) < b$.

Theorem 12. *Suppose that (H1) and (H2) are satisfied and that there exists* $\gamma \in (0,1)$ *such that (H3) is satisfied. Define the continuous positive concave functional* $\alpha : \mathscr{B} \to \mathscr{B}$ *by* $\alpha(u) = \min\limits_{\gamma \leq t \leq 1} |u^{(n-2)}(t)|$, *and let* $0 < A \leq \dfrac{1}{M\overline{G}_{n-2}}$ *and*

$$B \geq \left[m[1-(1-\gamma)^{\beta-1}]\gamma^{\alpha-n+1}\int_{0}^{\gamma} s\frac{\partial^{n-2}}{\partial t^{n-2}}G(\beta;s,s)\,\mathrm{d}s\right]^{-1}.$$

Let $a,\ b,$ *and* c *be such that* $0 < a < b < c$. *Assume that the following hold:*

(L1) $f(u(t),u'(t),\ldots,u^{(n-2)}(t)) < Aa$ *for all* $(t,u^{(n-2)}(t)) \in [0,1] \times [0,a]$,
(L2) $f(u(t),u'(t),\ldots,u^{(n-2)}(t)) > Bb$ *for all* $(t,u^{(n-2)}(t)) \in [\gamma,1] \times [b,c]$,
(L3) $f(u(t),u'(t),\ldots,u^{(n-2)}(t)) \leq Ac$ *for all* $(t,u^{(n-2)}(t)) \in [0,1] \times [0,c]$.

Then (9), (10) *has at least three positive solutions* $u_1,\ u_2,\ u_3 \in K$ *satisfying*

$$\|u_1\| < a,$$

$$b < \alpha(u_2),\ \text{and}$$

$$a < \|u_3\| \text{ with } \alpha(u_3) < b.$$

Proof. Let $u \in \mathscr{K}_c$. Then $\|u\| < c$ and by $(L3)$ and (7),

$$\begin{aligned}
|Tu^{(n-2)}|(t) &= \left| \int_0^1 \frac{\partial^{n-2}}{\partial t^{n-2}} G(\beta;t,s) a(s) f(u(s),u'(s),\ldots,u^{(n-2)}(s))\,\mathrm{d}s \right| \\
&\le \int_0^1 \frac{\partial^{n-2}}{\partial t^{n-2}} G(\beta;t,s)|a(s)||f(u(s),u'(s),\ldots,u^{(n-2)}(s))|\,\mathrm{d}s \\
&\le M \int_0^1 \frac{\partial^{n-2}}{\partial t^{n-2}} G(\beta;t,s)|f(u(s),u'(s),\ldots,u^{(n-2)}(s))|\,\mathrm{d}s \\
&< AcM \int_0^1 \frac{\partial^{n-2}}{\partial t^{n-2}} G(\beta;t,s)\,\mathrm{d}s \\
&\le AcM\overline{G}_{n-2} \\
&= c.
\end{aligned}$$

Hence, $\|Tu\| < c$ and $T : \mathscr{K}_c \to \mathscr{K}_c$.

Similarly, let $u \in \mathscr{K}_a$. Then $\|u\| < a$, and by $(L1)$ and (7),

$$\begin{aligned}
|Tu^{(n-2)}|(t) &= \left| \int_0^1 \frac{\partial^{n-2}}{\partial t^{n-2}} G(t,s) a(s) f(u(s),u'(s),\ldots,u^{(n-2)}(s))\,\mathrm{d}s \right| \\
&\le \int_0^1 \left| \frac{\partial^{n-2}}{\partial t^{n-2}} G(\beta;t,s) \right| |a(s)||f(u(s),u'(s),\ldots,u^{(n-2)}(s))|\,\mathrm{d}s \\
&\le M \int_0^1 \frac{\partial^{n-2}}{\partial t^{n-2}} G(\beta;t,s)|f(u(s),u'(s),\ldots,u^{(n-2)}(s))|\,\mathrm{d}s \\
&< MAa \int_0^1 \frac{\partial^{n-2}}{\partial t^{n-2}} G(\beta;t,s)\,\mathrm{d}s \\
&= M\overline{G}_{n-2}Aa \\
&= a.
\end{aligned}$$

So, $T : \mathscr{K}_a \to \mathscr{K}_a$.

Let d be a constant such that $b < d \le c$. Then, for $u(t) = \dfrac{d}{(n-2)!}t^{n-2}$, $\alpha(u) = d > b$ and $u \in \mathscr{K}(\alpha,\ b,\ d)$. Thus $\mathscr{K}(\alpha,\ b,\ d) \neq \emptyset$. Hence, $\|Tu\| < u$ if $u \in \mathscr{K}_a$, and condition $(B2)$ of Theorem 11 holds.

Let $u \in \mathscr{K}(\alpha,\ b,\ d)$. Then $\|u\| \le d \le c$ and $\alpha(u) = \min\limits_{\gamma \le t \le 1} |u^{(n-2)}(t)| = \min\limits_{\gamma \le t \le 1} u^{(n-2)}(t) \ge b$. Now, by $(L2)$ and (12),

$$\begin{aligned}
\alpha(Tu) &= \min_{\gamma \le t \le 1} \int_0^1 \frac{\partial^{n-2}}{\partial t^{n-2}} G(t,s) a(s) f(u(s),u'(s),\ldots,u^{(n-2)}(s))\,\mathrm{d}s \\
&> \min_{\gamma \le t \le 1} \int_\gamma^1 \frac{\partial^{n-2}}{\partial t^{n-2}} G(\beta;t,s) a(s) f(u(s),u'(s),\ldots,u^{(n-2)}(s))\,\mathrm{d}s \\
&> Bbm[1-(1-\gamma)^{\beta-1}]\gamma^{\alpha-n+1} \int_0^1 s \frac{\partial^{n-2}}{\partial t^{n-2}} G(\beta;s,s)\,\mathrm{d}s \\
&= b.
\end{aligned}$$

Thus, for all $u \in \mathscr{K}(\alpha,\ b,\ d)$, we have that $\alpha(Tu) > b$. So, condition $(B1)$ of Theorem 11 holds.

Finally, let $u \in \mathscr{K}(\alpha,\ b,\ c)$ with $\|Tu\| > d$. Then $\|u\| \leq c$ and $\alpha(u) = \min_{0\leq t\leq 1} |u^{(n-2)}(t)| = \min_{\gamma\leq t\leq 1} u^{(n-2)}(t) \geq b$. From assumption $(L2)$ and (12),

$$\begin{aligned}
\alpha(Tu) &= \min_{\gamma\leq t\leq 1}\int_0^1 \frac{\partial^{n-2}}{\partial t^{n-2}} G(\beta;t,s)a(s)f(u(s),u'(s),\ldots,u^{(n-2)}(s))\,\mathrm{d}s \\
&> \min_{\gamma\leq t\leq 1}\int_\gamma^1 \frac{\partial^{n-2}}{\partial t^{n-2}} G(\beta;t,s)a(s)f(u(s),u'(s),\ldots,u^{(n-2)}(s))\,\mathrm{d}s \\
&> Bbm[1-(1-\gamma)^{\beta-1}]\gamma^{\alpha-n+1}\int_0^1 s\frac{\partial^{n-2}}{\partial t^{n-2}} G(\beta;s,s)\,\mathrm{d}s \\
&= b.
\end{aligned}$$

This shows that condition $(B3)$ of Theorem 11 holds.

Thus, from Theorem 11, T has at least three fixed points $u_1,\ u_2,\ u_3$ such that $\|u_1\| < a,\ b < \alpha(u_2)$, and $a < \|u_3\|$ with $\alpha(u_3) < b$. These fixed points are solutions of (9), (10).

8 Conclusions

Here it was shown how four classical fixed point theorems, the Contraction Mapping Principle, Schauder's fixed point theorem, Krasnosel'skii's fixed point theorem and the Leggett-Williams fixed point theorem, can be used to show the existence of a unique, at least one, at least three, finitely many and infinitely many positive solutions of a fractional boundary value problem. Important properties of the Green's function associated with the boundary value problem were developed in order to apply these fixed point theorems. These properties may also be useful when applying other fixed point theorems to this boundary value problem.

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