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# Spectral Solutions for Multi-Term Fractional Initial Value Problems Using a New Fibonacci Operational Matrix of Fractional Integration

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**Abstract:** This paper is concerned with deriving an operational matrix of fractional-order integration of Fibonacci polynomials. As an application of this matrix, a spectral algorithm for solving some fractional-order initial value problems is exhibited and implemented. The key idea for obtaining the suggested spectral numerical solutions for these equations is actually based on utilizing the developed Fibonacci operational matrix along with the application of tau method in order to reduce the fractional-order differential equation with its initial conditions into a system of linear algebraic equations in the unknown expansion coefficients which can be efficiently solved. Some illustrative examples are included aiming to ascertain the efficiency and applicability of the presented algorithm. The numerical results reveal that the proposed algorithm is easy and applicable.

**Keywords:** Fibonacci polynomials, operational matrix of integration, tau method, fractional-order differential equations

## 1 Introduction

Fractional calculus is an important branch of mathematical analysis. Several types of differential equations of fractional-order appear in problems related to several fields of applied science such as fluid mechanics, biology, engineering, electromagnetism and physics [1, 2]. So, due to the frequent importance of such equations, there is a great focus on obtaining solutions of them through different techniques. Solving fractional-order differential equations analytically is not always possible, and therefore the employment of different numerical techniques for solving such equations is necessary.

There are extensive studies in the numerical methods for solving various kinds of fractional-order differential equations. Some of the proposed numerical methods for solving such equations are: the Adomian's decomposition method [3, 4]; finite difference method [5, 6]; Taylor collocation method [7], variational iteration method [8], higher order numerical methods [9]. The approach of utilizing operational matrices for solving fractional-order fractional differential equations is followed in a variety of papers. For example, Saadatmandi and Dehghan in [10] introduced and used a Legendre operational matrix of fractional derivatives for solving some fractional-order differential equations. Also, Rostamy et al. in [11] introduced a Bernstein operational matrix of fractional derivatives for handling these equations.

The employment of spectral methods (see, for instance, Boyd [12] and Canuto et al. [13]) is powerful in handling various types of differential equations. Spectral methods are a class of techniques in which the numerical solution is expressed in terms of certain "basis functions", which may be expressed in terms of various polynomials, which are often orthogonal. These methods take on a global approach unlike finite element methods which use a local approach, and this of course leads to excellent error properties, and high convergence. Spectral methods have been used by many authors for solving ordinary, partial and fractional differential equations. For some articles concerning spectral solutions of ordinary differential equations, see for example [14–17], and for some other articles concerning solutions of fractional-order

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differential equations, see for example [18, 19].

As an alternative approach to differentiating solution expansions is to integrate the differential equation $q$ times, where $q$ is the order of the equation. This approach has the advantage that the resulted algebraic systems are cheaper in solving than those obtained by the differentiated forms. This approach is followed in a number of papers. For example, Abd-Elhameed et al. in [20] and Doha et al. in [21] solved the integrated forms of third- and fifth-order elliptic differential equations using general parameters generalized Jacobi polynomials. Also, this approach is followed in solving fractional-order differential equation, see for example [22].

The operational matrices of derivatives of various orthogonal polynomials are widely used for solving various kinds of differential equations. To be more precise, these operational matrices are used for solving both linear and nonlinear ordinary and fractional differential equations. For example [23–29].

The Fibonacci polynomials and their related polynomials are of interest. In addition, the celebrated Fibonacci numbers and golden ratio appear in several applications in different fields of applied science. For properties and applications of Fibonacci polynomials and some other generalizations, one can be refereed to the important book of Koshy [30]. Many authors were interested in investigating Fibonacci polynomials and their generalizations from a theoretical point of view, see, for example [31–33], however the use of these polynomials numerically is rare. For example, the authors in [34] employed these polynomials for solving some boundary value problems through the collocation method. Also, a collocation tau method using Fibonacci technique is followed in [35] for handling some ordinary differential equations.

The main aim of this article is to obtain new numerical solutions of some types of fractional-order initial value problems (IVPs). An operational matrix of fractional integration of Fibonacci polynomials is constructed, and then it is used together with the application of the spectral tau method for obtaining the proposed numerical solutions for some fractional-order differential equations. The paper is organized as follows. Section 2 is devoted for presenting some fundamental definitions of the fractional calculus and also some relevant properties of Fibonacci polynomials. In Section 3, the operational matrix of the fractional integration of Fibonacci polynomials in Caputo sense is constructed. In Section 4, we present and implement a numerical algorithm for solving fractional-order linear differential equations based on utilizing the introduced operational matrix along with the application of the spectral tau method. In Section 5, the proposed algorithm is tested through some numerical examples. A final conclusion is given in Section 6.

## 2 Preliminaries and Used Formulae

### *2.1 Some definitions and properties of fractional calculus*

In this section, we recall some well-known definitions and preliminary facts of the fractional calculus.

**Definition 1.** *The Riemann-Liouville fractional integral operator $I^{\alpha}$ of order $\alpha$ on the usual Lebesgue space $L_1[0,1]$ is defined as*

$$I^{\alpha} f(t) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau, & \alpha > 0; \\ f(t), & \alpha = 0. \end{cases} \tag{1}$$

The operator $I^{\alpha}$ satisfies the following identities:

$$(i)\, I^{\alpha} I^{\beta} = I^{\alpha+\beta}, \qquad (ii)\, I^{\alpha} I^{\beta} = I^{\beta} I^{\alpha}, \tag{2}$$

$$(iii) I^{\alpha} (t-a)^{\nu} = \frac{\Gamma(\nu+1)}{\Gamma(\nu+\alpha+1)} (t-a)^{\nu+\alpha}, \tag{3}$$

where $f \in L_1[0,1]$, $\alpha, \beta \geqslant 0$, and $\nu > -1$.

**Definition 2.** *The Riemann-Liouville fractional derivative of order $\alpha > 0$ is defined by*

$$(D^{\alpha} f)(t) = \left(\frac{d}{dt}\right)^{n} (I^{n-\alpha} f)(t), \, n-1 \leqslant \alpha < n, \quad n \in \mathbb{N}. \tag{4}$$

For more details on the mathematical properties of fractional derivatives and integrals, see for example, [36].

### 2.2 Some relevant properties of Fibonacci polynomials

The Fibonacci polynomials can be defined by many ways. For example, they can be generated with the aid of the recurrence relation

$$F_{n+2}(x) = x F_{n+1}(x) + F_n(x),\, n \geq 0,$$

with the initial values:

$$F_0(x) = 0, \quad F_1(x) = 1,$$

and they also can be generated by the following explicit form

$$F_n(x) = \frac{\left(x+\sqrt{x^2+4}\right)^n - \left(x-\sqrt{x^2+4}\right)^n}{2^n\sqrt{x^2+4}},$$

or by the following power form representation:

$$F_n(x) = \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-j-1}{j} x^{n-2j-1}. \tag{5}$$

The derivatives of the Fibonacci polynomials can be given in terms of their corresponding original polymelias. For example, Falcon and Plaza [37] found the following relation for the first derivative of Fibonacci polynomials

$$F'_{i+1}(x) = \sum_{m=0}^{\lfloor \frac{i-1}{2} \rfloor} (-1)^m\,(i-2m)\,F_{i-2m}(x), \quad i \geq 1, \tag{6}$$

where the notation $\lfloor z \rfloor$ denotes the largest integer less than or equal to $z$.
Also, the following inversion formula to (5) in which the polynomials $x^i$ is expressed in terms of the Fibonacci polynomials is of interest (see, Falcon and Plaza [37])

$$x^m = \sum_{i=0}^{\lfloor \frac{m}{2} \rfloor} (-1)^i \left[\binom{m}{i} - \binom{m}{i-1}\right] F_{m-2i+1}(x), \quad m \geq 0. \tag{7}$$

In what follows, it is more convenient to write the power form representation of the Fibonacci polynomials (5) and its inversion formula (7) in the following two forms:

$$F_i(x) = \sum_{\substack{k=0 \\ (k+i)\ odd}}^{i} \frac{\left(\frac{i+k-1}{2}\right)!}{k!\,\left(\frac{i-k-1}{2}\right)!} x^k,\ i \geq 0, \tag{8}$$

and

$$x^m = m! \sum_{\substack{k=1 \\ (k+m)\ odd}}^{m+1} \frac{(-1)^{\frac{m-k+1}{2}}\,k}{\left(\frac{m-k+1}{2}\right)!\,\left(\frac{m+k+1}{2}\right)!} F_k(x),\ m \geq 0. \tag{9}$$

## 3 Fibonacci Operational Matrix of Integration

Let $y(x)$ be a square Lebesgue integrable function on $(0,1)$, and assume that it can be expressed in terms of the linearly independent Fibonacci polynomials as:

$$y(x) = \sum_{k=1}^{\infty} a_k\,F_k(x),$$

and let $y_n(x)$ be an approximation to $y(x)$, that is

$$y(x) \approx y_n(x) = \sum_{k=1}^{n+1} a_k\,F_k(x) = \mathbf{A}^T\,\Phi(x), \tag{10}$$

where

$$\mathbf{A}^T = [a_1, a_2, \ldots, a_{n+1}], \tag{11}$$

and

$$\Phi(x) = [F_1(x), F_2(x), \ldots, F_{n+1}(x)]^T. \tag{12}$$

Now, we state the following lemma, from which the operational matrix of integration of Fibonacci polynomials can be introduced.

**Lemma 1.***The following integral formula holds (see [37])*

$$\int_0^x F_{i+1}(t)\,dt = \frac{1}{i+1}\begin{cases} F_{i+2}(x)+F_i(x), & \text{if } i \text{ even},\\ F_{i+2}(x)+F_i(x)-2, & \text{if } i \text{ odd}. \end{cases} \tag{13}$$

The integral vector of the vector $\Phi(x)$ can be written in the form

$$\int_0^x \Phi(t)\,dt \approx \mathbf{G}^{(1)}\,\Phi(x), \tag{14}$$

where $\mathbf{G}^{(1)}$ is the $(n+1)\times(n+1)$ operational matrix of integration whose nonzero elements can be written explicitly as follows:

$$g_{ij} = \begin{cases} \frac{1}{i+1}, & j = i+1,\\ \frac{1}{i+1}, & j = i-1,\ i \geq 2,\\ \frac{-2}{i+1}, & j = 0,\ i \text{ odd},\ i \geq 3,\\ \frac{-1}{2}, & j = 0, i = 1,\\ 0, & otherwise. \end{cases}$$

For example if $n = 4$, then

$$\mathbf{G}^{(1)} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0\\ -\frac{1}{2} & 0 & \frac{1}{2} & 0 & 0\\ 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0\\ -\frac{1}{2} & 0 & \frac{1}{4} & 0 & \frac{1}{4}\\ 0 & 0 & 0 & \frac{1}{5} & 0 \end{pmatrix}_{5\times 5}.$$

### 3.1 A Fibonacci Operational Matrix of Fractional Integration

The main purpose of this section is to find the operational matrix of fractional integration of Fibonacci polynomials. From relation (14), it is easy to see that, for every positive integer $r$, we have

$$J^{(r)}\,\Phi(x) \approx \mathbf{G}^{(r)}\,\Phi(x), \tag{15}$$

where $J^{(r)}$ is the repeated $r$ times integration of $\Phi(x)$.
Now, we are going to state and prove an important theorem concerning the operational matrix of fractional integration of Fibonacci polynomials.

**Theorem 1.***Let $\Phi(x)$ be the Fibonacci polynomial vector defined in Eq.* (12)*. For every $\alpha > 0$, one has*

$$I^{\alpha}\,\Phi(x) = x^{\alpha}\,\mathbf{G}^{(\alpha)}\,\Phi(x), \tag{16}$$

*where $\mathbf{G}^{(\alpha)}$ is the $(n+1)\times(n+1)$ Fibonacci operational matrix of fractional integration of order $\alpha$, and it is defined by:*

$$\mathbf{G}^{(\alpha)} = \left(g_{i,j}^{(\alpha)}\right), \tag{17}$$

*where*

$$g_{i,j}^{(\alpha)} = \begin{cases} \xi_{\alpha}(i,j), & i \geq j;\\ 0, & otherwise. \end{cases}$$

*and*

$$\xi_\alpha(i,j) = j! \sum_{\substack{k=1 \\ (i+k)\, odd, (j+k)\, odd}}^{i} \frac{(-1)^{\frac{k-j+1}{2}}\, k!\, \left(\frac{i+k-1}{2}\right)!}{\left(\frac{i-k-1}{2}\right)!\, \left(\frac{k-j+1}{2}\right)!\, \left(\frac{k+j+1}{2}\right)!\, \Gamma(k+1+\alpha)}. \tag{18}$$

*Explicitly, $\boldsymbol{G}^{(\alpha)}$ has the following form*

$$\boldsymbol{G}^{(\alpha)} = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \xi_\alpha(\lceil\alpha\rceil,1) & \xi_\alpha(\lceil\alpha\rceil,\lceil\alpha\rceil) & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \xi_\alpha(i,1) & \dots & \xi_\alpha(i,i) & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \xi_\alpha(n+1,1) & \xi_\alpha(n+1,2) & \xi_\alpha(n+1,3) & \dots & \xi_\alpha(n+1,n+1) \end{pmatrix}. \tag{19}$$

*Proof.*If we apply the integral operator $I^\alpha$ to Eq. (8), and make use of the identity (3), then we get

$$I^\alpha F_i(x) = \sum_{\substack{k=1 \\ (k+i)\ even}}^{i} \frac{\left(\frac{i+k}{2}\right)!}{\left(\frac{i-k}{2}\right)!\ \Gamma(k+1+\alpha)} x^{k+\alpha}, \tag{20}$$

and accordingly Eq. (9) enables one to turn Eq. (20) into the form

$$I^\alpha F_i(x) = x^\alpha \sum_{j=1}^{i} \xi_\alpha(i,j) F_j(x), \tag{21}$$

where $\xi_\alpha(i,j)$ is given in (18). We rewrite Eq. (21) in the following vector form:

$$I^\alpha F_i(x) = x^\alpha\, [\xi_\alpha(i,1), \xi_\alpha(i,2), \dots, \xi_\alpha(i,i), 0, 0, \dots, 0]\ \Phi(x), \quad 1 \le i \le n+1. \tag{22}$$

Finally, Eq. (22) leads to the desired result.

## 4 Spectral Solutions for Fractional-Order Differential Equation Through the Fibonacci Operational Matrix

In this section, we describe in detail how the operational matrix of fractional integration of Fibonacci polynomials can be employed for solving linear multi-order fractional initial value problems.
Consider the following linear fractional differential equation:

$$u^{(\alpha)}(x) = \sum_{i=1}^{k-1} \varepsilon_i\, u^{(\alpha_i)}(x) + \varepsilon_k\, u(x) + f(x), \qquad x \in (0,1), \tag{23}$$

subject to the following initial conditions:

$$u^{(r)}(0) = a_r, \qquad r = 0,1,\dots,m-1, \tag{24}$$

where $\varepsilon_i$ are known constants and $\alpha \in (m-1,m]$, $\quad 0 < \alpha_1 < \alpha_2 < \dots < \alpha_{k-1} < \alpha$, and $u^{(\alpha)}(x) = D^\alpha u(x)$ is the Riemann-Liouville fractional derivative of order $\alpha$ of $u(x)$ and $f(x)$ is the source function. For the existence and uniqueness of the solution of Eq. (23), see [38]. Now apply the operator $I^\alpha$ to Eq. (23) to get

$$u(x) - \sum_{j=0}^{m-1} u^{(j)}(0^+)\, \frac{x^j}{j!} = \sum_{i=1}^{k-1} \varepsilon_i\, I^{\alpha-\alpha_i} \left[ u(x) - \sum_{j=0}^{\lceil\alpha_i\rceil-1} u^{(j)}(0^+)\, \frac{x^j}{j!} \right] + \varepsilon_k\, I^\alpha\, u(x) + I^\alpha\, f(x). \tag{25}$$

If we make use of the initial conditions (24) together with relation (3), then we get

$$u(x) = \sum_{i=1}^{k-1} \varepsilon_i\, I^{\alpha-\alpha_i}\, u(x) + \varepsilon_k\, I^\alpha\, u(x) + g(x), \tag{26}$$

where

$$g(x)=I^{\alpha}\,f(x)+\sum_{j=0}^{m-1}a_j\,\frac{x^j}{j!}+\sum_{i=1}^{k-1}\sum_{j=0}^{\lceil\alpha_i\rceil-1}\varepsilon_i\,a_j\,\frac{x^{j+\alpha-\alpha_i}}{\Gamma(j+1+\alpha-\alpha_i)}.$$

Now, if $u(x)$ is approximated as

$$u(x)\approx u_n(x)=\mathbf{A}^T\,\Phi(x), \tag{27}$$

then based on Theorem 1, the following approximations can be obtained

$$I^{\alpha}\,u(x)\approx x^{\alpha}\,\mathbf{A}^T\,\mathbf{G}^{(\alpha)}\,\Phi(x), \tag{28}$$

and

$$I^{\alpha-\alpha_i}\,u(x)\approx x^{\alpha-\alpha_i}\,\mathbf{A}^T\,\mathbf{G}^{(\alpha-\alpha_i)}\,\Phi(x), \tag{29}$$

and therefore the residual of (26) is given by

$$R(x)=\mathbf{A}^T\,\Phi(x)-\varepsilon_k\,x^{\alpha}\,\mathbf{A}^T\,\mathbf{G}^{(\alpha)}\,\Phi(x)-\sum_{i=1}^{k-1}\varepsilon_i\,x^{\alpha-\alpha_i}\,\mathbf{A}^T\,\mathbf{G}^{(\alpha-\alpha_i)}\,\Phi(x)-g(x). \tag{30}$$

Now, the application of the spectral tau method to (26) (see, for example [16]) leads to

$$\int_0^1 R(x)\,F_i(x)\,dx=0,\quad i=1,2,\dots n+1. \tag{31}$$

Now Eqs. (31) generate a linear algebraic system in the unknown expansion coefficients $a_i$ of dimension $(n+1)$. This system can be solved through any suitable solver such as the Gaussian elimination procedure, and consequently the approximate solution can be obtained.

## 5 Numerical Examples

In this section, we illustrate some numerical examples accompanied with comparisons to ascertain the efficiency of the proposed algorithm for handling multi-term fractional-order IVPs. In all examples, we apply the Fibonacci operational matrix method (FOMM) which is implemented in the previous section.

*Example 1.* [39] Consider the linear fractional-order IVP:

$$u^{(\frac{3}{2})}(x)+3\,u(x)=3x^3+\frac{8}{\sqrt{\pi}}x^{\frac{3}{2}},\qquad x\in(0,1), \tag{32}$$

subject to the initial conditions

$$u(0)=u'(0)=0, \tag{33}$$

with the exact smooth solution $u(x)=x^3$. If we apply the integral operator $I^{\frac{3}{2}}$ to both sides of Eq. (32) and make use of the initial conditions, then we get

$$u(x)+3\,I^{\frac{3}{2}}\,u(x)=x^3+\frac{64}{105\,\sqrt{\pi}}x^{\frac{9}{2}}. \tag{34}$$

The residual of Eq. (34) is given by

$$R(x)=\mathbf{A}^T\,\Phi(x)+3\,x^{\frac{3}{2}}\,\mathbf{A}^T\,\mathbf{G}^{(\frac{3}{2})}\,\Phi(x)-x^3-\frac{64}{105\,\sqrt{\pi}}x^{\frac{9}{2}},$$

where the matrix $\mathbf{G}^{(\frac{3}{2})}$ is given by

$$\mathbf{G}^{(\frac{3}{2})}=\frac{4}{3\,\sqrt{\pi}}\begin{pmatrix}1&0&0&0\\0&\frac{2}{5}&0&0\\\frac{27}{35}&0&\frac{8}{35}&0\\0&\frac{52}{105}&0&\frac{16}{105}\end{pmatrix}.$$

The application of tau method to Eq. (34) leads to the following linear system of four equations:

$$\int_0^1 R(x)\,F_i(x)\,dx = 0, \qquad i = 1,2,3,4. \tag{35}$$

The solution of this system yields

$$a_0 = 0, \quad a_1 = -2, \quad a_3 = 0, \quad a_4 = 1,$$

and consequently

$$u_3(x) = (0,-2,0,1)\,.\,\left(1,x,1+x^2,2x+x^3\right)^T = x^3,$$

which is the exact solution.

*Example 2.* [40] Consider the linear fractional-order IVP:

$$u^{(2)}(x) + 3\,u^{(1)}(x) + 2\,u^{(0.1379)}(x) + u^{(0.0159)}(x) + 5\,u(x) = f(x), \qquad x \in (0,1), \tag{36}$$

subject to the initial conditions

$$u(0) = 1, \quad u'(0) = 0, \tag{37}$$

where $f(x)$ is chosen such that the exact solution of (36) is $u(x) = 1 + \dfrac{x^2}{2}$. If we apply the operator $I^2$ to both sides of Eq. (36), and make use of the initial conditions, then the following equation is obtained

$$u(x) + 3\,I^{(1)}\,u(x) + 2\,I^{(1.862)}\,u(x) + I^{(1.9841)}\,u(x) + 5\,I^{(2)}\,u(x) = g(x), \tag{38}$$

where

$$g(x) = 6 + 3x + 1.13137\,x^{1.8621} + 0.507365\,x^{1.9841} + 3x^2 + 0.5\,x^3 + 0.102352\,x^{3.8621} + 0.0426753\,x^{3.9841}.$$

The residual of Eq. (38) is given by

$$\begin{aligned} R(x) = {} & \mathbf{A}^T\,\Phi(x) + 3\,\mathbf{A}^T\,\mathbf{G}^{(1)}\,\Phi(x) + 2\,x^{1.8621}\,\mathbf{A}^T\,\mathbf{G}^{(1.8621)}\,\Phi(x) \\ & + x^{1.9841}\,\mathbf{A}^T\,\mathbf{G}^{(1.9841)}\,\Phi(x) + 5\,\mathbf{A}^T\,\mathbf{G}^{(2)}\,\Phi(x) - g(x), \end{aligned}$$

where the involved matrices can be written explicitly as

$$\mathbf{G}^{(1)} = \begin{pmatrix} 0 & 1 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{3} & 0 \end{pmatrix}, \qquad \mathbf{G}^{(1.1862)} = \begin{pmatrix} 0.565684 & 0 & 0 \\ 0 & 0.197647 & 0 \\ 0.463332 & 0 & 0.102352 \end{pmatrix},$$

$$\mathbf{G}^{(2)} = \begin{pmatrix} -\frac{1}{2} & 0 & \frac{1}{2} \\ 0 & -\frac{1}{3} & 0 \\ -\frac{1}{6} & 0 & \frac{1}{6} \end{pmatrix}, \qquad \mathbf{G}^{(1.9841)} = \begin{pmatrix} 0.507365 & 0 & 0 \\ 0 & 0.170023 & 0 \\ 0.422014 & 0 & 0.0853506 \end{pmatrix}.$$

The application of tau method to (38) yields the system

$$\int_0^1 R(x)\,F_i(x)\,dx = 0, \qquad i = 1,2,3, \tag{39}$$

which can be solved to give

$$a_0 = \tfrac{1}{2}, \quad a_1 = 0, \quad a_3 = \tfrac{1}{2},$$

and consequently

$$u_2(x) = \left(\tfrac{1}{2},0,\tfrac{1}{2}\right).\left(1,x,1+x^2\right)^T = 1 + \frac{x^2}{2},$$

which is the exact solution.

**Table 1:** Comparison between different solutions of Example 3 for $\alpha = 0.5$

| $x$ | [42] BPOM | [43] Exact | [41] ADM | FOMM |
|---|---|---|---|---|
| 0.1 | 0.039754 | 0.039750 | 0.039874 | 0.039750 |
| 0.2 | 0.157043 | 0.157036 | 0.158512 | 0.157035 |
| 0.3 | 0.347373 | 0.347370 | 0.353625 | 0.347369 |
| 0.4 | 0.604699 | 0.604695 | 0.622083 | 0.604695 |
| 0.5 | 0.921768 | 0.921768 | 0.960047 | 0.921767 |
| 0.6 | 1.290458 | 1.290457 | 1.363093 | 1.290456 |
| 0.7 | 1.702007 | 1.702008 | 1.826257 | 1.702008 |
| 0.8 | 2.147286 | 2.147287 | 2.344224 | 2.147286 |
| 0.9 | 2.616998 | 2.617001 | 2.911278 | 2.617000 |
| 1.0 | 3.101902 | 3.101906 | 3.521462 | 3.101905 |

**Table 2:** Comparison between different solutions of Example 3 for $\alpha = 1.5$

| $x$ | [42] BPOM | [43] Exact | [41] ADM | FOMM |
|---|---|---|---|---|
| 0.1 | 0.033510 | 0.033507 | 0.036478 | 0.033507 |
| 0.2 | 0.125226 | 0.125221 | 0.140640 | 0.125220 |
| 0.3 | 0.267611 | 0.267609 | 0.307485 | 0.267608 |
| 0.4 | 0.455439 | 0.455435 | 0.533284 | 0.455435 |
| 0.5 | 0.684336 | 0.684335 | 0.814757 | 0.684334 |
| 0.6 | 0.950395 | 0.950393 | 1.148840 | 0.950392 |
| 0.7 | 1.249959 | 1.249959 | 1.532571 | 1.249960 |
| 0.8 | 1.579558 | 1.579557 | 1.963033 | 1.579558 |
| 0.9 | 1.935832 | 1.935832 | 2.437331 | 1.935831 |
| 1.0 | 2.315526 | 2.315526 | 2.952567 | 2.315526 |

*Example 3.* [41–43] Consider the fractional-order IVP:

$$u^{(2)}(x) + u^{(\alpha)}(x) + u(x) = 8, \quad x \in (0,1), \quad \alpha \in (0,2), \tag{40}$$

subject to the initial conditions

$$u(0) = 1, \quad u'(0) = 0.$$

A series solution of this problem is given in [43]. We apply FOMM with $n = 7$. In Tables 1 and 2, we display a comparison between the results obtained by the application of FOMM with those obtained by the using the following methods:

- –Exact series solution [43].
- –Adomian decomposition method ADM [41].
- –Block pulse operational method BPOM [42],

for the cases $\alpha = 0.5$ and $\alpha = 1.5$.

*Example 4.* [41]Consider the following fractional IVP:

$$u^{(2.2)}(x) + 1.3\,u^{(1.5)}(x) + 2.6\,u(x) = \sin(2x), \qquad x \in (0,1), \tag{41}$$

subject to

$$u(0) = u'(0) = u''(0) = 0.$$

Eq. (41) have the following series solution (see, [41])

$$\begin{aligned} u_s(x) = {} & \frac{28561}{3600000}x^6 + \frac{2}{\Gamma(4.2)}x^{3.2} - \frac{13}{5\,\Gamma(4.9)}x^{3.9} + \frac{169}{50\,\Gamma(5.6)}x^{4.6} - \frac{8}{\Gamma(6.2)}x^{5.2} \\ & - \frac{2197}{500\,\Gamma(6.3)}x^{5.3} - \frac{26}{5\,\Gamma(6.4)}x^{5.4} + \frac{52}{5\,\Gamma(6.9)}x^{5.9}. \end{aligned} \tag{42}$$

**Table 3:** Maximum absolute error of Example 4

| $n$ | 3 | 5 | 7 | 9 | 11 | 13 | 15 |
|---|---|---|---|---|---|---|---|
| $E$ | $5.2\,.10^{-4}$ | $2.0\,.10^{-6}$ | $3.4\,.10^{-8}$ | $5.9\,.10^{-10}$ | $4.7\,.10^{-12}$ | $1.6\,.10^{-14}$ | $1.2\,.10^{-16}$ |

We apply FOMM for various values of $n$. In Table 3 we list the maximum absolute error $E$ of Example 4 for different values of $n$, where

$$E = \max_{x\in[0,1]} |u_n(x) - u_s(x)|,$$

Moreover, Figure 1 illustrates a comparison between our solution for $n = 3,5,7$ with the series solution obtained in [41].

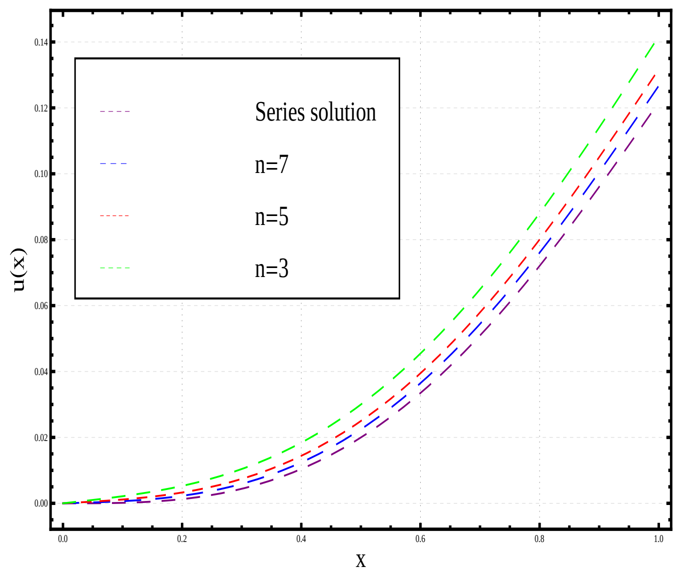


**Fig.1.** Different solutions of Example 4.

## 6 Conclusion

In this paper, we have introduced a novel operational matrix of fractional integration of Fibonacci polynomials. This matrix is utilized to solve fractional-order differential equations. The spectral tau method is applied to convert the fractional differential equation into a system of linear algebraic equations which can be efficiently solved. To the best of our knowledge, this is the first time in which such operational matrix of fractional integration is employed for handling fractional-order differential equations. Moreover, we do believe that some other generalizations for this work can be done to handle various ordinary differential equations as well as fractional differential equations. The illustrated numerical examples in this paper indicate the high efficiency and applicability of the proposed algorithm.

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