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Existence of Positive Solution to a Class of Fractional Differential Equations with Three Point Boundary Conditions

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Abstract: This article is concerned with the existence and uniqueness of positive solution to a class of fractional order differential equation with three point boundary conditions of the type

$$
\begin{aligned}
&{}^{c}D^{\alpha}u(t) = f(t,u(t),{}^{c}D^{\alpha-1}u(t)), 1<\alpha\le 2, t\in J=[0,1]\\
&u(0)=\gamma u(\eta),\ \ {}^{c}D^{\beta}u(1)=\gamma^{c}D^{\beta}u(\eta), 0<\beta<1, \eta\in(0,1),
\end{aligned}
$$

where $f : J\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is nonlinear continuous function and D^{α}_{0+} represents Caputo's fractional derivative of order α. We use some results from fixed point theory to obtain the existence and uniqueness results. We provide an example to show the applicability of our results

Keywords: Fractional differential equations; Three point boundary conditions, Fixed point theorems, Existence and uniqueness results.

1 Introduction

Fractional differential equations have extensive applications in real life problems. These applications can be found in various scientific and engineering disciplines such as physics, chemistry, biology, viscoelasticity, control theory, signal processing etc, for detail we refer [1,2,3,4,5]. Moreover, most of the authors studied fractional differential equations as an object of mathematical investigations, we refer the readers to [5,6,7,8,9,10,11,12,13,14,15,16] and the references therein for the recent development in the theory of fractional differential equations. It is worthwhile to mention that Caputo's fractional derivatives play important role in applied problems as it provides known physical interpretation for initial and boundary conditions. On the other hand, the Riemann-Liouville derivatives of fractional order do not provide physical interpretations in most of the cases for initial and boundary conditions. Existence theory for real world problems which can be modeled by of fractional differential equations with multi-point boundary conditions have attracted the attention of many researchers and is a rapidly growing area of investigation, we refer the readers to [17,18,19,20,21,22,23,24,25,26,27]. The purpose of this paper is to study existence and uniqueness of solution for boundary value problem of the form

$$
\begin{aligned}
&{}^{c}D^{\alpha}u(t) = f(t,u(t),{}^{c}D^{\alpha-1}u(t)),\\
&u(0)=\gamma u(\eta), {}^{c}D^{\beta}u(1)=\gamma\,{}^{c}D^{\beta}u(\eta),
\end{aligned}
\tag{1}
$$

where $1<\alpha\le 2$, $0<\beta<1$, $\gamma\in,(0,1)$ and $t\in J=[0,1]$. We developed the necessary and sufficient conditions for the existence of positive solution to the above class of fractional differential equations with the help of classical fixed point theory. We also provide an example for the illustrations of our main results.

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2 Preliminaries

We recall some basic definitions and lemmas of fixed point theory and functional analysis, which are required for our main work [1,2,3,4,5,6,7].

Definition 1. *The Banach space of all continuous functions from* $J \to \mathbb{R}$ *with the usual norm* $\|u\|_\infty = sup\{|u(t)| : 0 \le t \le 1\}$, *is denoted by* $C(J,\mathbb{R})$. *The Banach space of functions* $u : J \to \mathbb{R}$ *that are Lebesgue integrable with the norm* $\|u\|_{L^1} = \int_0^1 |u(t)|dt$ *is denoted by* $L^1(J,\mathbb{R})$. *Further, we know that [28] the space* $\widetilde{C}(J,\mathbb{R}) = \{u \in C(J,\mathbb{R}) :^c D^{\alpha-1}u \in C(J,\mathbb{R})\}$ *is a Banach space under the norm* $\|u\|_{\widetilde{C}} = \max\{\|u\|_\infty, \|^cD^{\alpha-1}u\|_\infty\}$.

Definition 2. *The fractional integral of order* $\alpha \in \mathbb{R}_+$ *of a function* $h \in L^1([a,b],\mathbb{R})$ *is defined by*

$$I^\alpha_{a+}h(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}h(s)\,ds.$$

When $a = 0$, we write $I^q h(t) = [h * \varphi_\alpha](t)$, where $\varphi_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$ for $t > 0$, $\varphi_\alpha(t) = 0$ for $t \le 0$ and $\varphi_a \to \delta(t)$ as $\alpha \to 0$, where δ is the delta function.

Definition 3. *The Caputo fractional order derivative of a function h on the interval* $[a,b]$ *is defined by*

$${}^cD^\alpha_{a+}h(t) = \frac{1}{\Gamma(n-\alpha)}\int_a^t (t-s)^{n-\alpha-1}h^{(n)}(s)\,ds,\ n = [\alpha]+1,$$

provided that the integral on the right converges.

For more details on the subject of fractional derivatives and integrals, we refer the readers to [2,3,4,5].

Lemma 1. *[14] The fractional order differential equation of order* $\alpha > 0$ *of the form*

$${}^cD^\alpha h(t) = 0,\ n-1 < \alpha \le n,$$

has a unique solution of the form

$$h(t) = C_0 + C_1t + C_2t^2 + \ldots + C_{n-1}t^{n-1},$$

where $C_i \in \mathbb{R}$, *for* $i = 0,1,\ldots n-1$.

Lemma 2. *[14]. The following result holds for a fractional derivative and integral of order* α

$$I^{\alpha c}D^\alpha h(t) = h(t) + C_0 + C_1t + C_2t^2 + \ldots + C_{n-1}t^{n-1},$$

for arbitrary $C_i \in \mathbb{R}$, *for* $i = 0,1,2,\ldots,n-1$.

3 Main Results

In this section, we study existence and uniqueness of the solution of the fractional differential equation (1).

Lemma 3. *For* $y(t) \in C[0,1]$, *the unique solution of*

$$\begin{aligned} &{}^cD^\alpha u(t) = y(t),\ 1 < \alpha \le 2, t \in J = [0,1],\\ &u(0) = \gamma u(\eta),\ {}^cD^\beta u(1) = \gamma\, {}^cD^\beta u(\eta),\\ &0 < \beta < 1,\ \gamma \in (0,1) \end{aligned} \tag{2}$$

is given by $u(t) = \int_0^1 G(t,s)y(s)ds$, $t \in [0,1]$, *where* $G(t,s)$ *is the Green's function and is given by*

$$G(t,s) = \begin{cases} \frac{1}{\Gamma\alpha}(t-s)^{\alpha-1} + \frac{1}{\Gamma\alpha}(\frac{\gamma}{1-\gamma})(\eta-s)^{\alpha-1} \\ + \frac{\Gamma(2-\beta)(\gamma\eta+(1-\gamma)t)}{1-\gamma\eta^{1-\beta}\Gamma(\alpha-\beta)} \\ \left[\gamma(\eta-s)^{\alpha-\beta-1} - (1-s)^{\alpha-\beta-1}\right], & 0 \le s \le t \le \eta, \\ \frac{1}{\Gamma\alpha}(\frac{\gamma}{1-\gamma})(\eta-s)^{\alpha-1} \\ + \frac{\Gamma(2-\beta)(\gamma\eta+(1-\gamma)t)}{(1-\gamma\eta^{1-\beta})(\Gamma(\alpha-\beta))} \\ \left[\gamma(\eta-s)^{\alpha-\beta-1} - (1-s)^{\alpha-\beta-1}\right], & 0 \le t \le s \le \eta. \end{cases} \tag{3}$$

Proof. In view of lemmama (2), we have

$$u(t) = I^\alpha y(t) + c_0 + c_1t,\ c_0, c_1 \in R \tag{4}$$

and

$${}^cD^\beta u(t) = I^{\alpha-\beta}y(t) + c_1\frac{t^{1-\beta}}{\Gamma(2-\beta)}.$$

The boundary conditions $u(0) = \gamma u(\eta)$, and ${}^cD^\beta u(1) = \gamma^c D^\beta u(\eta)$ yield that

$$c_1 = \frac{\Gamma(2-\beta)}{1-\gamma\eta^{1-\beta}}\{\gamma I^{\alpha-\beta}y(\eta) - I^{\alpha-\beta}y(1)\}$$

$$c_0 = \frac{\gamma}{1-\gamma}\left\{I^\alpha y(\eta) + \frac{\Gamma(2-\beta)}{1-\gamma\eta^{1-\beta}}(\gamma I^{\alpha-\beta}y(\eta) - I^{\alpha-\beta}y(1))\eta\right\}.$$

Thus (4) implies that

$$\begin{aligned} u(t) &= I^\alpha y(t) + \frac{\gamma}{1-\gamma}I^\alpha y(\eta) + \frac{\Gamma(2-\beta)}{1-\gamma\eta^{1-\beta}}(\gamma\eta \\ &+ (1-\gamma)t)\left\{\gamma I^{\alpha-\beta}y(\eta) - I^{\alpha-\beta}y(1)\right\} \\ &= \frac{1}{\Gamma\alpha}\int_0^t (t-s)^{\alpha-1}y(s)ds \\ &+ \frac{1}{\Gamma\alpha}(\frac{\gamma}{1-\gamma})\int_0^\eta (\eta-s)^{\alpha-1}y(s)ds \\ &+ \frac{\Gamma(2-\beta)}{(1-\gamma\eta^{1-\beta})(\Gamma(\alpha-\beta))}(\gamma\eta+(1-\gamma)t) \\ &\left\{\gamma\int_0^\eta (\eta-s)^{\alpha-\beta-1}y(s)ds - \int_0^1 (1-s)^{\alpha-\beta-1}y(s)ds\right\} \\ &= \int_0^1 G(t,s)y(s)ds, \end{aligned}$$

where the Green's function $G(t,s)$ is given in (3).

If $\alpha-\beta<1$ then the green function $G(t,s)$ become unbounded but the function $t:\to\int_0^1 G(t,s)y(s)ds$ is continuous on $J=[0,1]$ so attain its spermium value say

$$G^*=\sup_{t\in[0,1]}\int_0^\eta \mid G(t,s)\mid ds.$$

In view of Lemma (3), we write the BVP (1) as an integral equation of the form

$$\int_0^1 G(t,s)f(s,u(s),{}^cD^{\alpha-1}u(s))ds, t\in J.$$

Define an operator $T:\tilde{C}(J,R)\longrightarrow\tilde{C}(J,R)$ by

$$Tu(t)=\int_0^1 G(t,s)f(s,u(s),{}^cD^{\alpha-1}u(s))ds, t\in J, \quad (5)$$

then solutions of the BVP (1) are fixed points of T. Note that

$$\begin{aligned}({}^cD^{\alpha-1}Tu)t&=I^{\alpha-(\alpha-1)}y(t)+\frac{\Gamma(2-\beta)t^{2-\alpha}}{(1-\gamma\eta^{1-\beta})\Gamma(3-\alpha)}\\&\left\{\gamma I^{\alpha-\beta}y(\eta)-I^{\alpha-\beta}y(1)\right\}\\&=\int_0^t f(s,u(s),{}^cD^{\alpha-1}u(s))ds\\&+\frac{\Gamma(2-\beta)t^{2-\alpha}}{(1-\gamma\eta^{1-\beta})\Gamma(3-\alpha)(\Gamma(\alpha-\beta))}\\&\left\{\gamma\int_0^\eta(\eta-s)^{\alpha-\beta-1}f(s,u(s),{}^cD^{\alpha-1}u(s))ds\right.\\&\left.-\int_0^1(1-s)^{\alpha-\beta-1}f(s,u(s),{}^cD^{\alpha-1}u(s))ds\right\}.\end{aligned} \quad (6)$$

Theorem 1. *Further assume*

(A_1) *$f:J\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is continuous;*

(A_2) *There exist $p\in C(J,\mathbb{R}^+)$ and $\psi:[0,\infty]\to(0,\infty)$ is continuous and non-decreasing such that*

$$\mid f(t,u,z)\mid\le p(t)\psi(\mid z\mid) \text{ for all } t\in J,u,z\in\mathbb{R};$$

(A_3) *There exist constant $r>0$ such that*

$$r\ge\max\left\{G^*p^*\psi(r),p^*\psi(r)\left(\frac{\Gamma(3-\alpha)\Gamma(\alpha-\beta+1)(1-\gamma\eta^{1-\beta})}{\Gamma(3-\alpha)\Gamma(\alpha-\beta+1)(1-\gamma\eta^{1-\beta})}\right)+p^*\psi(r)\left(\frac{\Gamma(2-\beta)(\gamma\eta^{\alpha-\beta}-1)}{\Gamma(3-\alpha)\Gamma(\alpha-\beta+1)(1-\gamma\eta^{1-\beta})}\right)\right\},$$

where $p^*=\sup\{p(s),s\in J\}$, then the BVP (1) has at least one solution on J with $|u(t)|<r$ for each $t\in J$.

Proof. We use Schauder fixed point theorem. We prove that T is continuous. Choose r as defined in (A_3) and define $D=\{u\in\tilde{C}(J,\mathbb{R}),||u||_{\tilde{C}}\le r\}$ a closed subset of $\tilde{C}(J,\mathbb{R})$. If $\{u_n\}$ converges to u in $\tilde{C}(J,\mathbb{R})$, then there exist $\delta>0$ such that

$$||u_n||_{\tilde{C}}\le\delta,\ \ ||u||_{\tilde{C}}\le\delta.$$

For all $t\in J$, we have

$$\begin{aligned}&|Tu_n(t)-Tu(t)|\\&\le\int_0^\eta|G(t,s)[f(s,u_n(s),D^{\alpha-1}u_n(s))-f(s,u(s),D^{\alpha-1}u(s))]|ds,\end{aligned}$$

which in view of the continuity of f and Lebesague dominated convergence Theorem implies

$$||Tu_n(t)-Tu(t)||_\infty\to 0 \text{ as } n\to\infty. \quad (7)$$

Also

$$\begin{aligned}&||{}^cD^{\alpha-1}Tu_n(t)-{}^cD^{\alpha-1}Tu(t)||\\&\le\int_0^t\mid f(s,u_n(s),D^{\alpha-1}u_n(s))-f(s,\bar{u}(s)D^{\alpha-1}\bar{u}(s))\mid ds+\\&\frac{\Gamma(2-\beta)t^{2-\alpha}}{(1-\gamma\eta^{1-\beta})\Gamma(3-\alpha)(\Gamma(\alpha-\beta))}\\&\left\{\gamma\int_0^\eta(\eta-s)^{\alpha-\beta-1}|f(s,u_n(s),{}^cD^{\alpha-1}u_n(s))-f(s,\bar{u}(s),{}^cD^{\alpha-1}\bar{u}(s))|ds\right\}\\&-\frac{\Gamma(2-\beta)t^{2-\alpha}}{(1-\gamma\eta^{1-\beta})\Gamma(3-\alpha)(\Gamma(\alpha-\beta))}\\&\left\{\int_0^1(1-s)^{\alpha-\beta-1}|f(s,u_n(s),{}^cD^{\alpha-1}u_n(s))-f(s,\bar{u}(s),{}^cD^{\alpha-1}\bar{u}(s))|ds\right\}.\end{aligned}$$

By Lebesgue dominated convergence Theorem, we obtain

$$||{}^cD^{\alpha-1}Tu_n(t)-{}^cD^{\alpha-1}Tu(t)||_\infty\to 0\ \ as\ \ n\to\infty. \quad (8)$$

From (7) and (8), it follows that

$$||Tu_n(t)-Tu(t)||_{\tilde{C}}\to 0 \text{ as } n\to\infty,$$

implies that T is continuous. Now, we show that $T(D)$ is bounded.
Let $u(t)\in D$ then for each $t\in J$,we have

$$\begin{aligned}|Tu(t)|&\le\int_o^t(|G(t,s)||f(s,u(s),{}^cD^{\alpha-1}u(s))|)ds\\&\le G^\star p^\star\Psi(||{}^cD^{\alpha-1}u(t)||)\\&\le G^\star p^\star\Psi\max\{||u(t)||_\infty,||{}^cD^{\alpha-1}u(t)||_\infty)\}\\&\le G^\star p^\star\Psi(||u||_{\tilde{C}})\\&\le G^\star p^\star\Psi(r).\end{aligned}$$

Further

$$\begin{aligned}|{}^cD^{\alpha-1}Tu(t)|&\le\int_0^t|f(s,u(s),{}^cD^{\alpha-1}u(s))|)ds\\&+\frac{\Gamma(2-\beta)t^{2-\alpha}}{(1-\gamma\eta^{1-\beta})\Gamma(3-\alpha)(\Gamma(\alpha-\beta))}\\&\left\{\int_0^\eta(\eta-s)^{\alpha-\beta-1}|f(s,u(s),{}^cD^{\alpha-1})|ds\right.\\&\left.-\int_0^1(1-s)^{\alpha-\beta-1}|f(s,u(s),{}^cD^{\alpha-1})|ds\right\}\\&\le p^\star\Psi(||u||_{\tilde{C}})\\&\left\{1+\frac{\Gamma(2-\beta)(\eta^{\alpha-1})}{\Gamma(3-\alpha)\Gamma(\alpha-\beta+1)(1-\gamma\eta^{\alpha-\beta})}\right\}.\end{aligned}$$

Hence, it follows that $||Tu(t)||_{\tilde{C}} \leq r$, where
$r \geq \max\Bigg\{ G^* p^* \psi(r),$
$p^*\psi(r)\left(\frac{\Gamma(3-\alpha)\Gamma(\alpha-\beta+1)(1-\gamma\eta^{1-\beta})}{\Gamma(3-\alpha)\Gamma(\alpha-\beta+1)(1-\gamma\eta^{1-\beta})}\right)$
$+p^*\psi(r)\left(\frac{\Gamma(2-\beta)(\gamma\eta^{\alpha-\beta}-1)}{\Gamma(3-\alpha)\Gamma(\alpha-\beta+1)(1-\gamma\eta^{1-\beta})}\right)\Bigg\},$
which yields $T(D) \subseteq D$. Finally, we prove that T map D into equi-continuous set of $\tilde{C}(J,\mathbb{R})$. For $t_1,t_2 \in J, t_1 < t_2$ and $u(t) \in D$, we have

$$\begin{aligned}
&|Tu(t_2) - Tu(t_1)| \\
&\leq \int_o^1 |G(t_2,s) - G(t_1,s)||f(s,u(s),{}^cD^{\alpha-1}u(s))|ds \\
&\leq p^\star\Psi(||u||_{\tilde{C}}) \int_o^1 |G(t_2,s) - G(t_1,s)|ds
\end{aligned}$$

which implies that $||Tu(t_2) - Tu(t_1)|| \to 0$ as $t_2 \to t_1$. Moreover,

$$\begin{aligned}
&||T^cD^{\alpha-1}u(t_2) - T^cD^{\alpha-1}u(t_1)|| \\
&\leq |\int_0^{t_2} f(s,u(s),{}^cD^{\alpha-1}u(s)ds) - \int_0^{t_1} f(s,u(s),{}^cD^{\alpha-1}u(s)ds| \\
&+ \frac{(t_2^{2-\alpha} - t_1^{2-\alpha})\Gamma(2-\beta)}{\Gamma(3-\alpha)\Gamma(\alpha-\beta)(1-\gamma\eta^{1-\beta})} \\
&\Bigg\{\int_0^\eta (\eta-s)^{\alpha-\beta-1} f(s,u(s),{}^cD^{\alpha-1}u(s))ds \\
&- \int_0^1 (1-s)^{\alpha-\beta-1} f(s,u(s),{}^cD^{\alpha-1}u(s)ds\Bigg\} \\
&\leq p^*\Psi(||{}^cD^{\alpha-1}u(s))||_\infty \{(t_2-t_1) \\
&+ \frac{(t_2^{2-\alpha} - t_1^{2-\alpha})\Gamma(2-\beta)}{\Gamma(3-\alpha)\Gamma(\alpha-\beta)(1-\gamma\eta^{1-\beta})}\{\gamma\eta^{\alpha-\beta} - 1\} \\
&\leq p^*\Psi(||u_||_{\tilde{c}}) \\
&\left\{t_2 - t_1 + \frac{(t_2^{2-\alpha} - t_1^{2-\alpha})\Gamma(2-\beta)(\gamma\eta^{\alpha-\beta}-1)}{\Gamma(3-\alpha)\Gamma(\alpha-\beta+1)(1-\gamma\eta^{1-\beta})}\right\}
\end{aligned}$$

which implies that $|T({}^cD^{\alpha-1}u)t_2 - T({}^cD^{\alpha-1}u)t_1| \longrightarrow 0$ as $t_2 \longrightarrow t_1$. Hence $||Tu(t_2) - Tu(t_1)||_{\tilde{C}} \longrightarrow 0$ as $t_2 \longrightarrow t_1$. By Arzela Ascoli Theorem, it follows that T is completely continuous. Hence by Schauder's fixed point theorem T has a fixed point u in D.

Theorem 2. *Assuming that the following hold*

(A_4) $f : J\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ *is continuous;*

(A_5) *There exists constant* $k > 0$ *such that for each* $t \in J$ *and all* $x,y,\bar{x},\bar{y} \in \mathbb{R}$,

$$|f(t,x,y) - f(t,\bar{x},\bar{y})| \leq k(|x-\bar{x}| + |y-\bar{y}|),$$

holds. Further, if

$$\max\left\{2G^*k, 2k\left(\frac{\Gamma(3-\alpha)\Gamma(\alpha-\beta+1)(1-\gamma\eta^{1-\beta})}{\Gamma(3-\alpha)\Gamma(\alpha-\beta+1)(1-\gamma\eta^{1-\beta})} + \frac{\Gamma(2-\beta)(\gamma\eta^{\alpha-\beta}-1)}{\Gamma(3-\alpha)\Gamma(\alpha-\beta+1)(1-\gamma\eta^{1-\beta})}\right)\right\} < 1,$$

then the BVP (1) *has a unique solution on* J.

Proof. For the uniqueness of solutions, we use Banach contraction principle. We show that the operator $T : \tilde{C}(J,\mathbb{R}) \longrightarrow \tilde{C}(J,\mathbb{R})$ is contraction mapping with fixed point $u(t)$. In view of the continuity of f and G and the proof of Theorem (1), it follows from (5) and (6) that Tu and ${}^cD^{\alpha-1}Tu$ are both continuous on J. Let $u,\bar{u} \in \tilde{C}(J,\mathbb{R})$ then for each $t \in J$, we have

$$\begin{aligned}
&|\ Tu(t) - T\bar{u}(t)\ | \leq \sup_{t\in j} G(t,s) \\
&\int_0^\eta |\ f(s,u(s),{}^cD^{\alpha-1}u(s)) - f(s,\bar{u}(s),{}^cD^{\alpha-1}\bar{u}(s))\ |\ ds \\
&\leq G^*k\{|\ u-\bar{u}\ | + |\ D^{\alpha-1}u - D^{\alpha-1}\bar{u}\ |\} \leq 2G^*k \parallel u-\bar{u} \parallel_{\tilde{C}},
\end{aligned}$$

and

$$\begin{aligned}
&|\ D^{\alpha-1}Tu(t) - D^{\alpha-1}T\bar{u}(t)\ | \\
&\leq \int_0^t |\ f(s,u(s),D^{\alpha-1}u(s)) - f(s,\bar{u}(s)D^{\alpha-1}\bar{u}(s))\ |\ ds \\
&+ \frac{\Gamma(2-\beta)t^{2-\alpha}}{(1-\gamma\eta^{1-\beta})\Gamma(3-\alpha)(\Gamma(\alpha-\beta))}\gamma\int_0^\eta (\eta-s)^{\alpha-\beta-1} \\
&|f(s,u(s),{}^cD^{\alpha-1}u(s)) - f(s,\bar{u}(s),{}^cD^{\alpha-1}\bar{u}(s))|ds \\
&- \frac{\Gamma(2-\beta)t^{2-\alpha}}{(1-\gamma\eta^{1-\beta})\Gamma(3-\alpha)(\Gamma(\alpha-\beta))} \\
&\int_0^1 (1-s)^{\alpha-\beta-1}|f(s,u(s),{}^cD^{\alpha-1}u(s)) - f(s,\bar{u}(s),{}^cD^{\alpha-1}\bar{u}(s))|ds.
\end{aligned}$$

Which implies that

$$\begin{aligned}
&|\ D^{\alpha-1}Tu(t) - D^{\alpha-1}T\bar{u}(t)\ | \\
&\leq k\{|\ u-\bar{u}\ | + |\ D^{\alpha-1}u - D^{\alpha-1}\bar{u}\ |\}t \\
&+ \frac{\Gamma(2-\beta)t^{2-\alpha}}{(1-\gamma\eta^{1-\beta})(\Gamma(3-\alpha))(\Gamma(\alpha-\beta))} \\
&\left(\frac{(\eta-s)^{\alpha-\beta}}{\alpha-\beta}\ |_0^\eta - \frac{(1-s)^{\alpha-\beta}}{\alpha-\beta}\ |_0^1\right) \\
&(|\ u-\bar{u}\ | + |\ D^{\alpha-1}u - D^{\alpha-1}\bar{u}\ |)kt \\
&\leq 2k \parallel u-\bar{u} \parallel_{\tilde{C}} \\
&+ 2\frac{\Gamma(2-\beta)k\cdot}{\Gamma(3-\alpha)\Gamma(\alpha-\beta+1)(1-\gamma\eta^{1-\beta})} \\
&(\gamma\eta^{\alpha-\beta} - 1) \parallel u-\bar{u} \parallel_{\tilde{C}} \\
&\leq 2k\frac{\Gamma(3-\alpha)\Gamma(\alpha-\beta+1)(1-\gamma\eta^{1-\beta})}{\Gamma(3-\alpha)\Gamma(\alpha-\beta+1)(1-\gamma\eta^{1-\beta})} \\
&+ 2k\frac{\Gamma(2-\beta)(\gamma\eta^{\alpha-\beta}-1)}{\Gamma(3-\alpha)\Gamma(\alpha-\beta+1)(1-\gamma\eta^{1-\beta})} \\
&\parallel u-\bar{u} \parallel_{\tilde{C}}.
\end{aligned}$$

It follows that

$$\parallel Tu(t) - T\bar{u}(t) \parallel_{\tilde{C}}\ \leq\ d \parallel u-\bar{u} \parallel_{\tilde{C}},$$

where

$$d = \max\left\{2G^*k, 2k\frac{\Gamma(3-\alpha)\Gamma(\alpha-\beta+1)(1-\gamma\eta^{1-\beta})}{\Gamma(3-\alpha)\Gamma(\alpha-\beta+1)(1-\gamma\eta^{1-\beta})} + 2k\frac{\Gamma(2-\beta)(\gamma\eta^{\alpha-\beta}-1)}{\Gamma(3-\alpha)\Gamma(\alpha-\beta+1)(1-\gamma\eta^{1-\beta})}\right\} < 1$$

$\Rightarrow T$ is contraction mapping.
By Banach fixed point theorem the BVP (1) has a unique solution.

Example 1. Consider the following boundary value problem

$$\begin{aligned}
&{}^cD_t^{3/2}u(t) \\
&= \frac{1}{20\cos t+7}\left(\frac{1}{1+3|u(t)|+4|{}^cD^{1/2}u(t)|}\right), t \in [0,1] \\
&u(0) = \frac{1}{2}u(\frac{1}{2}), {}^cD^{1/2}u(1) = \frac{1}{2}{}^cD^{\frac{1}{2}}u(\frac{1}{2}).
\end{aligned}$$

Here $\alpha = 3/2, \beta = \gamma = \eta = \frac{1}{2}$ and $f(t,u,v) = \frac{1}{20\cos t+7}\left(\frac{1}{1+3|u(t)|+4|v|}\right)$, with $v = {}^c D^{1/2}u(t)$. We find that

$$|f(t,u,v) - f(t,\bar{u},\bar{v})| \leq \frac{4}{27}(|u-\bar{u}| - |v-\bar{v}|),$$

which is condition (A_1) of Theorem (2) with $(k = \frac{4}{27})$. Also $G^* = \sup_{t\in j}\int_0^\eta | G(t,s) | ds = 1.404$ and

$$2k\frac{\Gamma(3-\alpha)\Gamma(\alpha-\beta+1)(1-\gamma\eta^{1-\beta})}{\Gamma(3-\alpha)\Gamma(\alpha-\beta+1)(1-\gamma\eta^{1-\beta})} + 2k\frac{\Gamma(2-\beta)(\gamma\eta^{\alpha-\beta}-1)}{\Gamma(3-\alpha)\Gamma(\alpha-\beta+1)(1-\gamma\eta^{1-\beta})} = 0.46 < 1,$$

$$\max\left\{2G^*k, 2k\left(\frac{\Gamma(3-\alpha)\Gamma(\alpha-\beta+1)(1-\gamma\eta^{1-\beta})}{\Gamma(3-\alpha)\Gamma(\alpha-\beta+1)(1-\gamma\eta^{1-\beta})} + \frac{\Gamma(2-\beta)(\gamma\eta^{\alpha-\beta}-1)}{\Gamma(3-\alpha)\Gamma(\alpha-\beta+1)(1-\gamma\eta^{1-\beta})}\right)\right\} < 1.$$

Hence by Theorem (2), boundary value problem has a unique solution on $J \in [0,1]$.

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