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Statistical Inference of Concomitants Based on Morgenstern Family under Generalized Order Statistics

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Abstract: In this paper, the joint densities of the concomitants of generalized order statistics (GOS's) for exponential and power function subfamilies of Morgenstern family are derived. Statistical inference such as maximum likelihood (ML) and Bayesian estimation under different types of loss functions based on informative and non-informative priors for the distribution parameters, reliability and cumulative hazard functions are obtained. In addition, Bayesian prediction bounds, Bayes predictive estimator and approximate confidence intervals of the estimators are considered. Applications of these results are given for concomitants of order statistics are presented.

Keywords: Bayesian estimation, Bayesian prediction, Concomitants, Generalized order statistics, Maximum likelihood estimation, Morgenstern family

1 Introduction

The Morgenstern family has been discussed by Johnson and Kotz [13], provides a flexible family that can be used in such contexts, which is specified by the cumulative distribution function (cdf) and the probability density function (pdf), respectively, as follows:

$$F_{X,Y}(x,y) = F_X(x)F_Y(y)[1+\alpha(1-F_X(x))(1-F_Y(y))], \tag{1}$$

$$f_{X,Y}(x,y) = f_X(x)f_Y(y)[1+\alpha(2F_X(x)-1)(2F_Y(y)-1)], \tag{2}$$

where $-1 \le \alpha \le 1$, and $f_X(x)$, $f_Y(y)$, and $F_X(x)$, $F_Y(y)$ are the marginal pdf's and cdf's of X and Y respectively. The association parameter α is known as the dependence parameter of the random variables X and Y. If α is zero, then X and Y are independent.
The conditional pdf of Y given X is given by:

$$f_{Y|X}(y\,|\,x) = f_Y(y)[1+\alpha(2F_X(x)-1)(2F_Y(y)-1)], -1 \le \alpha \le 1. \tag{3}$$

The general theory of concomitants of order statistics has originally studied by David et al. [8]. Let (X_i,Y_i), $i=1,2,\cdots,n$, be n pairs of independent random variables from bivariate population with cdf $F(x,y)$. Let $X_{(r:n)}$ be the $r-th$ order statistics, then the Y value associated with $X_{(r:n)}$ is called the concomitant of the $r-th$ order statistics and is denoted by $Y_{[r:n]}$. Some times exact information are available only on the concomitants variable since the other variable is only ranked and not measured exactly, consider for example a group of patients ranked according to the value of their response to a treatment and subsequently the values of their blood test are observed only on those patients whose initial value exceeds a threshold, in this situation we have information only on the concomitants variable. For $1 \le r_1 < \cdots < r_k \le n$, the joint density for $Y_{[r_1:n]},\cdots,Y_{[r_k:n]}$ is given by:

$$\begin{aligned} g_{[r_1,\cdots,r_k:n]}(y_1,\cdots,y_k) &= g_{Y_{[r_1:n]},\cdots,Y_{[r_k:n]}}(y_1,\cdots,y_k) \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{x_k}\cdots\int_{-\infty}^{x_2}\prod_{i=1}^{k} f_{Y|X}(y_i|x_i) \\ &\quad \times f_{(r_1,\cdots,r_k:n)}(x_1,\cdots,x_k)\,\mathrm{d}x_1\cdots\mathrm{d}x_k, \end{aligned} \tag{4}$$

where $f_{(r_1,\cdots,r_k:n)}(x_1,\cdots,x_k) = f_{X_{(r_1:n)},\cdots,X_{(r_k:n)}}(x_1,\cdots,x_k)$. That is, in general, the joint concomitants of order statistics $Y_{[r_1:n]},\cdots,Y_{[r_k:n]}$ is dependent, where $f_{(r_1,\cdots,r_k:n)}(x_1,\cdots,x_k)$ is the joint density of $X_{(r_1:n)},\cdots,X_{(r_k:n)}$.
Kamps [14] has introduced the GOS's, the joint density function of the first r GOS's

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$X_{(1,n,m,k)}, X_{(2,n,m,k)}, \cdots, X_{(r,n,m,k)}$, $1 \le r \le n$ is given by:

$$\begin{aligned} f_{(1,2,\cdots,r;n,m,k)}(x_1,\cdots,x_r) &= f_{X_{(1,n,m,k)},\cdots,X_{(r,n,m,k)}}(x_1,\cdots,x_r) \\ &= c_{r-1}\Big(\prod_{i=1}^{r} f_X(x_i)\Big) \\ &\quad \times \Big(\prod_{i=1}^{r-1}(1-F_X(x_i))^m\Big)(1-F_X(x_r))^{\gamma_r-1}, \\ x_1 \le x_2 &\le \ldots \le x_r, \end{aligned} \tag{5}$$

with parameters $n \in \mathbb{N}$, $k > 0$, $m \in \mathbb{R}$, such that $\gamma_r = k + (n-r)(m+1) > 0$, $c_{r-1} = \prod_{j=1}^{r}\gamma_j$ for all $1 \le r \le n$.

In Bayesian approach, the performance depends on the prior information about the unknown parameters and the loss function. The prior information can be expressed by the experimenter, who has some beliefs about the unknown parameters and their statistical distributions. Traditionally, most authors have used squared error (SE) loss function which is symmetric, although the use of symmetric loss functions may be inappropriate. One disadvantage when using SE loss is that it penalizes overestimation or underestimation. Overestimation of a parameter can lead to more severe or less severe consequences than underestimation, or vice versa. Subsequently, the use of an asymmetrical loss function, which associates greater importance to overestimation or underestimation, can be considered for the estimation of the parameter, many authors have studied the SE loss in Bayesian inference, see for example, Calabria and Pulcini [6], Singh et al. [22] and Jaheen ([11] and [12]).

A very useful asymmetric loss function known as the linear exponential (LINEX) loss function, has been introduced by Varian [25] and has been widely used by several authors. This function rises approximately exponentially on one side of zero and approximately linearly on the other side.

The general entropy (GE) loss is also asymmetric loss function which is used in several papers, for example, Dey et al. [9], Dey and Liu [10] and Soliman ([23] and [24]).

In many practical problems, one wishes to use a previous sample to predict another future random sample drawn from the same population which known as two-sample prediction. One way to do this is to construct an interval which will contain the future sample with a specified probability, this interval is called a prediction interval. two-sample Bayesian prediction bounds have been discussed by several authors, including Ali Mousa [1], Balakrishnan et al. [3], Ali Mousa and Al-Sagheer [2] and Raqab et al. [20].

In this article, we consider the classical and Bayesian inference of the distribution parameters, reliability and cumulative hazard functions for concomitants of GOS's based on some well-known subfamilies of Morgenstern family. The Bayes estimators are obtained under symmetric SE, asymmetric LINEX and GE loss functions using informative and non-informative priors. In addition, we consider this Bayesian approach to predict a future sample based on the observed sample. Furthermore, the approximate confidence intervals of the estimators are presented. The organization of the article is as follows: In Section 2, joint densities for concomitants of GOS's based on exponential and power function subfamilies of Morgenstern family are obtained. In Section 3, we derive non-Bayesian and Bayesian estimation under different types of loss functions based on informative and non-informative priors, and study the Bayesian prediction of two-sample, also, the approximate confidence bounds for the model parameters are constructed. Application of this results based on order statistics as a special case of GOS's is applied in Section 4. Finally, some conclusions and comments are given in Section 5.

2 Joint Densities for Concomitants of Exponential and Power Function Subfamilies of Morgenstern Family

Based on Morgenstern family, the density function of the concomitant of the $r-th$ GOS's $Y_{[r,n,m,k]}$, $1 \le r \le n$, is given by:

$$g_{[r,n,m,k]}(y) = f_Y(y)\left[1+\alpha C^*(r,n,m,k)(2F_Y(y)-1)\right], \tag{6}$$

where $C^*(r,n,m,k) = 1-2C(r,n,m,k) = 1-2\frac{\prod_{i=1}^{r}\gamma_i}{\prod_{i=1}^{r}(\gamma_i+1)}$, see Beg and Ahsanullah [5]. Mohie El-Din et al. [15] have proposed the joint density of the concomitants of GOS's for Morgenstern family, from (3), (4) and (5), the joint density of the first r concomitants of GOS's $Y_{[1,n,m,k]}, Y_{[2,n,m,k]}, \cdots, Y_{[r,n,m,k]}$, $1 \le r \le n$, for Morgenstern family is given by:

$$\begin{aligned} g_{[1,2,\cdots,r,n,m,k]}(y_1,\cdots,y_r) &= \int_{-\infty}^{\infty}\int_{-\infty}^{x_k}\cdots\int_{-\infty}^{x_2}\prod_{i=1}^{r} f_{Y|X}(y_i|x_i) \\ &\quad \times f_{(1,2,\cdots,r,n,m,k)}(x_1,\cdots,x_r)\mathrm{d}x_1\cdots\mathrm{d}x_r \\ &= \left(\prod_{i=1}^{r} f_Y(y_i)\right)\left[1+\alpha\sum_{i=1}^{r}C_i(2F_Y(y_i)-1)\right. \\ &\quad +\alpha^2\sum_{i,j=1,i\neq j}^{r}C_{ij}(2F_Y(y_i)-1)(2F_Y(y_j)-1) \\ &\quad +\alpha^3\sum_{i,j,k=1,i\neq j\neq k}^{r}C_{ijk}(2F_Y(y_i)-1) \\ &\quad \times(2F_Y(y_j)-1)(2F_Y(y_k)-1)+\cdots+\alpha^{r-1} \\ &\quad \times\sum_{i_1,i_2,\cdots,i_r=1,i_1\neq i_2\neq\cdots\neq i_r}^{r}C_{i_1i_2\cdots i_r} \\ &\quad \times\prod_{i=i_1}^{i_r}(2F_Y(y_i)-1) \\ &\quad \left.+\alpha^r C^*\prod_{i=1}^{r}(2F_Y(y_i)-1)\right], \end{aligned} \tag{7}$$

where all the constants C's are constant functions of γ's. In this section we choose two distributions that belong to different families to apply it in our work. The pdf and cdf of exponential distribution are, respectively, given by:

$$f_1(y) = \lambda e^{-\lambda y},\ \lambda > 0,\ 0 \le y < \infty, \tag{8}$$

$$F_1(y) = 1-e^{-\lambda y},\ \lambda > 0,\ 0 \le y < \infty, \tag{9}$$

also, the reliability function and cumulative hazard function are, respectively, given by:

$$R_1(y) = R_1 = e^{-\lambda y},\ \lambda > 0,\ 0 \le y < \infty, \tag{10}$$
$$H_1(y) = H_1 = \lambda y,\ \lambda > 0,\ 0 \le y < \infty. \tag{11}$$

The pdf and cdf of power function distribution are, respectively, given by:

$$f_2(y) = a y^{a-1},\ a > 0,\ 0 \le y \le 1, \tag{12}$$
$$F_2(y) = y^{a},\ a > 0,\ 0 \le y \le 1, \tag{13}$$

also, the reliability function and cumulative hazard function are, respectively, given by:

$$R_2(y) = R_2 = 1 - y^{a},\ a > 0,\ 0 \le y \le 1, \tag{14}$$
$$H_2(y) = H_2 = -\log(1 - y^{a}),\ a > 0,\ 0 \le y \le 1. \tag{15}$$

From (7), (8) and (9), the joint density of the concomitants of GOS's $Y_{[1,n,m,k]}, Y_{[2,n,m,k]}, \cdots, Y_{[r,n,m,k]}$, $1 \le r \le n$, for exponential subfamily of Morgenstern family is given by:

$$\begin{aligned}
g_{1[1,2,\cdots,r;n,m,k]}(y_1,\cdots,y_r) = \lambda^r \Bigg[& e^{-\lambda \sum_{i=1}^{r} y_i} + \alpha \sum_{i=1}^{r} C_i \Big(e^{-\lambda \sum_{t=1}^{r} y_t} \\
& -2e^{-\lambda(\sum_{t=1}^{r} y_t + y_i)} \Big) + \cdots + \alpha^{r-1} \\
& \times \sum_{t,i_1,i_2,\cdots,i_r=1, t\neq i_1\neq i_2\neq\cdots\neq i_r}^{r} C_{i_1 i_2 \cdots i_r} \Big[e^{-\lambda(\sum_{i=i_1}^{i_r} y_i + y_t)} \\
& + (-2)^1 \sum_{i,j=i_1, i\neq j}^{i_r} e^{-\lambda(\sum_{j=i_1, i\neq j}^{i_r} y_j + 2y_i + y_t)} + \cdots \\
& + (-2)^{r-1} \sum_{i,j=i_1, i\neq j}^{i_r} e^{-\lambda(y_i + 2\sum_{j=i_1, i\neq j}^{i_r} y_j + y_t)} \\
& + (-2)^r e^{-\lambda(2\sum_{i=i_1}^{i_r} y_i + y_t)} \Big] + \alpha^r C^* \Big[e^{-\lambda \sum_{i=1}^{r} y_i} \\
& + (-2)^1 \sum_{i,j=1, i\neq j}^{r} e^{-\lambda(\sum_{j=1, i\neq j}^{r} y_j + 2y_i)} + \cdots \\
& + (-2)^{r-1} \sum_{i,j=1, i\neq j}^{r} e^{-\lambda(y_i + 2\sum_{j=1, i\neq j}^{r} y_j)} \\
& + (-2)^r e^{-2\lambda \sum_{i=1}^{r} y_i} \Big] \Bigg].
\end{aligned} \tag{16}$$

From (7), (12) and (13), the joint density of the concomitants of GOS's $Y_{[1,n,m,k]}, Y_{[2,n,m,k]}, \cdots, Y_{[r,n,m,k]}$, $1 \le r \le n$, for power subfamily of Morgenstern family is given by:

$$\begin{aligned}
g_{2[1,2,\cdots,r;n,m,k]}(y_1,\cdots,y_r) = a^r \Bigg[& T_0\, e^{-a\sum_{i=1}^{r} A_i} + \alpha \sum_{i=1}^{r} C_i\, T_0 \\
& \times \Big(2e^{-a(A_i + \sum_{t=1}^{r} A_t)} - e^{-a\sum_{t=1}^{r} A_t} \Big) + \cdots + \alpha^{r-1} \\
& \times \sum_{t,i_1,i_2,\cdots,i_r=1, t\neq i_1\neq i_2\neq\cdots\neq i_r}^{r} C_{i_1 i_2 \cdots i_r}\, T_r \\
& \times \Big[(-1)^r e^{-a(\sum_{i=i_1}^{i_r} A_i + A_t)} + (-1)^{r-1}(2)^1 \\
& \times \sum_{i,j=i_1, i\neq j}^{i_r} e^{-a(\sum_{j=i_1, i\neq j}^{i_r} A_j + 2A_i + A_t)} + \cdots \\
& + (-1)^1 (2)^{r-1} \sum_{i,j=i_1, i\neq j}^{i_r} e^{-a(A_i + 2\sum_{j=i_1, i\neq j}^{i_r} A_j + A_t)} \\
& + (2)^r e^{-a(2\sum_{i=i_1}^{i_r} A_i + A_t)} \Big] + \alpha^r C^*\, T_0\, \Big[(-1)^r \\
& \times e^{-a\sum_{i=1}^{r} A_i} + (-1)^{r-1}(2)^1 \\
& \times \sum_{i,j=1, i\neq j}^{r} e^{-a(\sum_{j=1, i\neq j}^{r} A_j + 2A_i)} + \cdots \\
& + (-1)^1 (2)^{r-1} \sum_{i,j=1, i\neq j}^{r} e^{-a(A_i + 2\sum_{j=1, i\neq j}^{r} A_j)} \\
& + (2)^r e^{-2a\sum_{i=1}^{r} A_i} \Big] \Bigg],
\end{aligned} \tag{17}$$

where $A_i = \log y_i^{-1}$, $1 \le i \le r$, $T_0 = e^{\sum_{i=1}^{r} A_i}$, $T_r = e^{A_t + \sum_{i=i_1}^{i_r} A_i}$.

3 Classical and Bayesian Estimations for Concomitants of *GOS*'s

In this section, we study classical estimation such as ML estimation with its approximate confidence intervals and obtain Bayesian estimation using informative and non-informative priors under SE, LINEX and GE loss functions for exponential and power subfamily of Morgenstern family. Also, we focus on two-sample Bayesian prediction bounds and estimation for exponential subfamily of Morgenstern family.

3.1 Maximum likelihood estimation

Suppose that $\mathbf{y} = (y_1, y_2, \cdots, y_r)$ is a concomitants of GOS's sample. From (16), the log-likelihood function for exponential subfamily of Morgenstern family is given by:

$$\begin{aligned}
\ell(\lambda|\mathbf{y}) &= \log L(\lambda|\mathbf{y}) \\
&= \log \lambda^r + \log \Bigg[e^{-\lambda \sum_{i=1}^{r} y_i} + \alpha \sum_{i=1}^{r} C_i \Big(e^{-\lambda \sum_{t=1}^{r} y_t} - 2e^{-\lambda(\sum_{t=1}^{r} y_t + y_i)} \Big) \\
&\quad + \cdots + \alpha^r C^* \Big[e^{-\lambda \sum_{i=1}^{r} y_i} + \cdots + (-2)^r e^{-2\lambda \sum_{i=1}^{r} y_i} \Big] \Bigg].
\end{aligned} \tag{18}$$

To derive the ML estimation of the unknown parameter λ, say $\hat{\lambda}_{ML}$, we differentiate (18) with respect to λ and then solve the following non-linear equation numerically by using Newton-Raphson method:

$$(\sum_{i=1}^{r} y_i)e^{-\lambda \sum_{i=1}^{r} y_i} + \cdots + \alpha^r C^* \left[(\sum_{i=1}^{r} y_i)e^{-\lambda \sum_{i=1}^{r} y_i} + \cdots + (-2)^r (2\sum_{i=1}^{r} y_i)e^{-2\lambda (\sum_{i=1}^{r} y_i)} \right] = r\lambda^{-r-1} L(\lambda|\mathbf{y}). \tag{19}$$

The ML estimation of the reliability and cumulative hazard functions are given by replacing $\hat{\lambda}_{ML}$ in (10) and (11), respectively. The observed asymptotic variance of ML estimation for the parameter λ is given by dropping the expectation operator from the element of the inverse of the Fisher information matrix as follows:

$$Var(\hat{\lambda}_{ML}) = \left(-\frac{\partial^2 \ell(\lambda|\mathbf{y})}{\partial \lambda^2}\right)^{-1}.$$

The asymptotic normality of the ML estimator can be used to compute the approximate confidence interval for the parameter λ. Thus, $(1-\alpha_1)100\%$ confidence interval for λ becomes:

$$\hat{\lambda}_{ML} \pm z_{\frac{\alpha_1}{2}} \sqrt{Var(\hat{\lambda}_{ML})},$$

where $z_{\frac{\alpha_1}{2}}$ is a standard normal percentile.

From (17), the log-likelihood function for power subfamily of Morgenstern family is given by:

$$\begin{aligned}\ell(a|\mathbf{y}) = \log L(a|\mathbf{y}) &= \log a^r + \log \Big[T_0\, e^{-a\sum_{i=1}^{r} A_i} \\ &+ \alpha \sum_{i=1}^{r} C_i\, T_0\, \left(2e^{-a(A_i+\sum_{t=1}^{r} A_t)} - e^{-a\sum_{t=1}^{r} A_t}\right) + \cdots \\ &+ \alpha^r C^*\, T_0\, \left[(-1)^r e^{-a\sum_{i=1}^{r} A_i} + \cdots + (2)^r e^{-2a\sum_{i=1}^{r} A_i}\right]\Big].\end{aligned} \tag{20}$$

To derive the ML estimation of the unknown parameter a, say $\hat{a}_{ML}$, we differentiate (20) with respect to a and then solve the following non-linear equation numerically by using Newton-Raphson method:

$$\begin{aligned}&T_0(\sum_{i=1}^{r} A_i)\, e^{-a\sum_{i=1}^{r} A_i} + \cdots + \alpha^r C^*\, T_0\, \Big[(-1)^r (\sum_{i=1}^{r} A_i)e^{-a\sum_{i=1}^{r} A_i} + \cdots \\ &+ (2)^r (2\sum_{i=1}^{r} A_i)e^{-2a\sum_{i=1}^{r} A_i}\Big] = r\, a^{-r-1} L(a|\mathbf{y}).\end{aligned} \tag{21}$$

The ML estimation of the reliability and cumulative hazard functions are given by replacing $\hat{a}_{ML}$ in (14) and (15), respectively. The observed asymptotic variance of ML estimation for the parameter a is given by dropping the expectation operator from the element of the inverse of the Fisher information matrix as follows:

$$Var(\hat{a}_{ML}) = \left(-\frac{\partial^2 \ell(a|\mathbf{y})}{\partial a^2}\right)^{-1}.$$

The asymptotic normality of the ML estimator can be used to compute the approximate confidence interval for the parameter a. Thus, $(1-\alpha_2)100\%$ confidence interval for a becomes:

$$\hat{a}_{ML} \pm z_{\frac{\alpha_2}{2}} \sqrt{Var(\hat{a}_{ML})},$$

where $z_{\frac{\alpha_2}{2}}$ is a standard normal percentile.

3.2 Bayesian estimation

If we have enough information about the parameter we should use informative prior, otherwise it is better to consider non-informative prior. In this paper we study both informative and non-informative priors under different types of loss functions as follows:

3.2.1 Informative prior

For exponential and power subfamilies of Morgenstern family, suppose that λ and a have a natural conjugate gamma prior, with hyperparameters ν_1, β_1 and ν_2, β_2, respectively:

$$\pi_1(\lambda) = \frac{\beta_1^{\nu_1}}{\Gamma(\nu_1)} \lambda^{\nu_1 - 1} e^{-\beta_1 \lambda}, \nu_1, \beta_1 > 0, \lambda > 0, \tag{22}$$

$$\pi_2(a) = \frac{\beta_2^{\nu_2}}{\Gamma(\nu_2)} a^{\nu_2 - 1} e^{-\beta_2 a}, \nu_2, \beta_2 > 0, a > 0. \tag{23}$$

Combining the likelihood function, (16), and prior function, (22), then the posterior density of λ given $\mathbf{y}$ is given by:

$$\begin{aligned}\pi_1^*(\lambda|\mathbf{y}) &= \frac{L(\lambda|\mathbf{y})\pi_1(\lambda)}{\int_0^\infty L(\lambda|\mathbf{y})\pi_1(\lambda) d\lambda} \\ &= \frac{\lambda^{r+\nu_1-1}}{D_1} \Big[e^{-\lambda(\sum_{i=1}^{r} y_i + \beta_1)} + \alpha \sum_{i=1}^{r} C_i \left(e^{-\lambda(\sum_{t=1}^{r} y_t + \beta_1)} \right. \\ &\left. -2e^{-\lambda(\sum_{t=1}^{r} y_t + y_i + \beta_1)} \right) + \cdots \\ &+ \alpha^r C^* \left[e^{-\lambda(\sum_{i=1}^{r} y_i + \beta_1)} + \cdots + (-2)^r e^{-\lambda(2\sum_{i=1}^{r} y_i + \beta_1)} \right] \Big],\end{aligned} \tag{24}$$

where

$$\begin{aligned}D_1 = \Gamma(r+\nu_1) \Big[&(\sum_{i=1}^{r} y_i + \beta_1)^{-(r+\nu_1)} + \cdots + \alpha^r C^* \Big[(\sum_{i=1}^{r} y_i + \beta_1)^{-(r+\nu_1)} \\ &+ \cdots + (-2)^r (2\sum_{i=1}^{r} y_i + \beta_1)^{-(r+\nu_1)} \Big]\Big].\end{aligned} \tag{25}$$

Combining the likelihood function, (17), and prior function, (23), then the posterior density of a given $\mathbf{y}$ is given by:

$$\begin{aligned}\pi_2^*(a|\mathbf{y}) &= \frac{L(a|\mathbf{y})\pi_2(a)}{\int_0^\infty L(a|\mathbf{y})\pi_2(a) da} \\ &= \frac{a^{r+\nu_2-1}}{D_2} \Big[T_0\, e^{-a(\sum_{i=1}^{r} A_i + \beta_2)} + \alpha \sum_{i=1}^{r} C_i\, T_0\, \left(2e^{-a(A_i + \sum_{t=1}^{r} A_t + \beta_2)} \right. \\ &\left. -e^{-a(\sum_{t=1}^{r} A_t + \beta_2)} \right) + \cdots \\ &+ \alpha^r C^*\, T_0\, \left[(-1)^r e^{-a(\sum_{i=1}^{r} A_i + \beta_2)} + \cdots + (2)^r e^{-a(2\sum_{i=1}^{r} A_i + \beta_2)} \right] \Big],\end{aligned} \tag{26}$$

where

$$\begin{aligned}D_2 = \Gamma(r+\nu_2) \Big[&T_0\, (\sum_{i=1}^{r} A_i + \beta_2)^{-(r+\nu_2)} + \cdots \\ &+ \alpha^r C^* T_0 \Big[(-1)^r (\sum_{i=1}^{r} A_i + \beta_2)^{-(r+\nu_2)} + \cdots \\ &+ (-2)^r (2\sum_{i=1}^{r} A_i + \beta_2)^{-(r+\nu_2)} \Big]\Big].\end{aligned} \tag{27}$$

Bayes estimator of λ and a based on squared error loss function

From (24), the Bayes estimator $\hat{\lambda}_{BS}$ under SE loss function is given by:

$$\begin{aligned}\hat{\lambda}_{BS} &= E_{\lambda}(\lambda|\mathbf{y}) = \int_0^{\infty} \lambda\ \pi_1^*(\lambda|\mathbf{y})d\lambda \\ &= \frac{1}{D_1}\Gamma(r+\nu_1+1)\Bigg[(\sum_{i=1}^{r} y_i+\beta_1)^{-(r+\nu_1+1)}+\cdots \\ &\quad +\alpha^r C^*\Bigg[(\sum_{i=1}^{r} y_i+\beta_1)^{-(r+\nu_1+1)}+\cdots \\ &\quad +(-2)^r(2\sum_{i=1}^{r} y_i+\beta_1)^{-(r+\nu_1+1)}\Bigg]\Bigg].\end{aligned} \tag{28}$$

From (26), the Bayes estimator $\hat{a}_{BS}$ under SE loss function is given by:

$$\begin{aligned}\hat{a}_{BS} &= E_a(a|\mathbf{y}) = \int_0^{\infty} a\ \pi_2^*(a|\mathbf{y})da \\ &= \frac{1}{D_2}\Gamma(r+\nu_2+1)\Bigg[T_0\ (\sum_{i=1}^{r} A_i+\beta_2)^{-(r+\nu_2+1)}+\cdots \\ &\quad +\alpha^r C^* T_0\Bigg[(-1)^r(\sum_{i=1}^{r} A_i+\beta_2)^{-(r+\nu_2+1)}+\cdots \\ &\quad +(-2)^r(2\sum_{i=1}^{r} A_i+\beta_2)^{-(r+\nu_2+1)}\Bigg]\Bigg].\end{aligned} \tag{29}$$

Bayes estimator of λ and a based on linear exponential loss function

From (24), the Bayes estimator $\hat{\lambda}_{BL}$ under LINEX loss function is given by:

$$\begin{aligned}\hat{\lambda}_{BL} &= \frac{-1}{p_1}\log\left[\int_0^{\infty} e^{-p_1\lambda}\pi_1^*(\lambda|\mathbf{y})d\lambda\right] \\ &= \frac{-1}{p_1}\log\Bigg[\frac{1}{D_1}\Gamma(r+\nu_1)\Bigg[(\sum_{i=1}^{r} y_i+\beta_1+p_1)^{-(r+\nu_1)}+\cdots \\ &\quad +\alpha^r C^*\Bigg[(\sum_{i=1}^{r} y_i+\beta_1+p_1)^{-(r+\nu_1)}+\cdots \\ &\quad +(-2)^r(2\sum_{i=1}^{r} y_i+\beta_1+p_1)^{-(r+\nu_1)}\Bigg]\Bigg]\Bigg].\end{aligned} \tag{30}$$

From (26), the Bayes estimator $\hat{a}_{BL}$ under LINEX loss function is given by:

$$\begin{aligned}\hat{a}_{BL} &= \frac{-1}{p_2}\log\left[\int_0^{\infty} e^{-p_2 a}\pi_2^*(a|\mathbf{y})da\right] \\ &= \frac{-1}{p_2}\log\Bigg[\frac{1}{D_2}\Gamma(r+\nu_2)\Bigg[T_0\ (\sum_{i=1}^{r} A_i+\beta_2+p_2)^{-(r+\nu_2)}+\cdots \\ &\quad +\alpha^r C^* T_0\Bigg[(-1)^r(\sum_{i=1}^{r} A_i+\beta_2+p_2)^{-(r+\nu_2)}+\cdots \\ &\quad +\ (-2)^r(2\sum_{i=1}^{r} A_i+\beta_2+p_2)^{-(r+\nu_2)}\Bigg]\Bigg]\Bigg].\end{aligned} \tag{31}$$

Bayes estimator of λ and a based on general entropy loss function

From (24), the Bayes estimator $\hat{\lambda}_{BG}$ under GE loss function is given by:

$$\begin{aligned}\hat{\lambda}_{BG} &= \left[E_{\lambda}(\lambda^{-l_1}|\mathbf{y})\right]^{\frac{-1}{l_1}} = \left[\int_0^{\infty} \lambda^{-l_1}\pi_1^*(\lambda|\mathbf{y})d\lambda\right]^{\frac{-1}{l_1}} \\ &= \Bigg[\frac{1}{D_1}\Gamma(r+\nu_1-l_1)\Bigg[(\sum_{i=1}^{r} y_i+\beta_1)^{-(r+\nu_1-l_1)}+\cdots \\ &\quad +\alpha^r C^*\Bigg[(\sum_{i=1}^{r} y_i+\beta_1)^{-(r+\nu_1-l_1)}+\cdots \\ &\quad +\ (-2)^r(2\sum_{i=1}^{r} y_i+\beta_1)^{-(r+\nu_1-l_1)}\Bigg]\Bigg]\Bigg]^{\frac{-1}{l_1}}.\end{aligned} \tag{32}$$

From (26), the Bayes estimator $\hat{a}_{BG}$ under GE loss function is given by:

$$\begin{aligned}\hat{a}_{BG} &= \left[E_a(a^{-l_2}|\mathbf{y})\right]^{\frac{-1}{l_2}} = \left[\int_0^{\infty} a^{-l_2}\pi_2^*(a|\mathbf{y})da\right]^{\frac{-1}{l_2}} \\ &= \Bigg[\frac{1}{D_2}\Gamma(r+\nu_2-l_2)\Bigg[T_0\ (\sum_{i=1}^{r} A_i+\beta_2)^{-(r+\nu_2-l_2)}+\cdots \\ &\quad +\alpha^r C^* T_0\Bigg[(-1)^r(\sum_{i=1}^{r} A_i+\beta_2)^{-(r+\nu_2-l_2)}+\cdots \\ &\quad +\ (-2)^r(2\sum_{i=1}^{r} A_i+\beta_2)^{-(r+\nu_2-l_2)}\Bigg]\Bigg]\Bigg]^{\frac{-1}{l_2}}.\end{aligned} \tag{33}$$

Remark 3.1. It can be shown that, when $l_1 = 1$ and $l_2 = 1$, the Bayesian estimate in (32) and (33) coincides with the Bayesian estimate under the weighted SE loss function, respectively. Similarly, when $l_1 = -1$ and $l_2 = -1$, the Bayesian estimate in (32) and (33) coincides with the Bayesian estimate under the SE loss function, respectively.

3.2.2 Non-informative prior

For exponential and power subfamilies of Morgenstern family, suppose that λ and a have the Jeffreys vague prior, respectively:

$$\pi_3(\lambda) \propto \frac{1}{\lambda}, \tag{34}$$

$$\pi_4(a) \propto \frac{1}{a}. \tag{35}$$

Combining the likelihood function, (16), and prior function, (34), then the posterior density of λ given $\mathbf{y}$ is given by:

$$\pi_3^*(\lambda|\mathbf{y}) = \frac{\lambda^{r-1}}{Z_1}\left[e^{-\lambda\sum_{i=1}^r y_i} + \alpha\sum_{i=1}^r C_i\left(e^{-\lambda\sum_{t=1}^r y_t} - 2e^{-\lambda(\sum_{t=1}^r y_t + y_i)}\right)\right. \\ \left. + \cdots + \alpha^r C^*\left[e^{-\lambda\sum_{i=1}^r y_i} + \cdots + (-2)^r e^{-2\lambda\sum_{i=1}^r y_i}\right]\right], \quad (36)$$

where

$$Z_1 = \Gamma(r)\left[(\sum_{i=1}^r y_i)^{-r} + \cdots + \alpha^r C^*\left[(\sum_{i=1}^r y_i)^{-r} + \cdots + (-2)^r(2\sum_{i=1}^r y_i)^{-r}\right]\right]. \quad (37)$$

Combining the likelihood function, (17), and prior function, (35), then the posterior density of a given $\mathbf{y}$ is given by:

$$\pi_4^*(a|\mathbf{y}) = \frac{a^{r-1}}{Z_2}\left[T_0\, e^{-a\sum_{i=1}^r A_i} + \alpha\sum_{i=1}^r C_i\, T_0\left(2e^{-a(A_i+\sum_{t=1}^r A_t)} - e^{-a\sum_{t=1}^r A_t}\right)\right. \\ \left. + \cdots + \alpha^r C^*\, T_0\left[(-1)^r e^{-a\sum_{i=1}^r A_i} + \cdots + (2)^r e^{-2a\sum_{i=1}^r A_i}\right]\right], \quad (38)$$

where

$$Z_2 = \Gamma(r)\left[T_0\,(\sum_{i=1}^r A_i)^{-r} + \cdots + \alpha^r C^* T_0\left[(-1)^r(\sum_{i=1}^r A_i)^{-r} + \cdots\right.\right. \\ \left.\left. + (-2)^r(2\sum_{i=1}^r A_i)^{-r}\right]\right]. \quad (39)$$

Bayes estimator of λ and a based on squared error loss function

From (36), the Bayes estimator $\hat{\lambda}_{BS}$ under SE loss function is given by:

$$\hat{\lambda}_{BS} = \frac{1}{Z_1}\Gamma(r+1)\left[(\sum_{i=1}^r y_i)^{-(r+1)} + \cdots + \alpha^r C^*\left[(\sum_{i=1}^r y_i)^{-(r+1)} + \cdots\right.\right. \\ \left.\left. + (-2)^r(2\sum_{i=1}^r y_i)^{-(r+1)}\right]\right]. \quad (40)$$

From (38), the Bayes estimator $\hat{a}_{BS}$ under SE loss function is given by:

$$\hat{a}_{BS} = \frac{1}{Z_2}\Gamma(r+1)\left[T_0\,(\sum_{i=1}^r A_i)^{-(r+1)} + \cdots + \alpha^r C^* T_0\left[(-1)^r(\sum_{i=1}^r A_i)^{-(r+1)}\right.\right. \\ \left.\left. + \cdots + (-2)^r(2\sum_{i=1}^r A_i)^{-(r+1)}\right]\right]. \quad (41)$$

Bayes estimator of λ and a based on linear exponential loss function

From (36), the Bayes estimator $\hat{\lambda}_{BL}$ under LINEX loss function is given by:

$$\hat{\lambda}_{BL} = \frac{-1}{p_1}\log\left[\frac{1}{Z_1}\Gamma(r)\left[(\sum_{i=1}^r y_i + p_1)^{-r} + \cdots + \alpha^r C^*\left[(\sum_{i=1}^r y_i + p_1)^{-r} +\right.\right.\right. \\ \left.\left.\left. + \cdots\ (-2)^r(2\sum_{i=1}^r y_i + p_1)^{-r}\right]\right]\right]. \quad (42)$$

From (38), the Bayes estimator $\hat{a}_{BL}$ under LINEX loss function is given by:

$$\hat{a}_{BL} = \frac{-1}{p_2}\log\left[\frac{1}{Z_2}\Gamma(r)\left[T_0\,(\sum_{i=1}^r A_i + p_2)^{-r} + \cdots + \alpha^r C^* T_0\right.\right. \\ \left.\left. \times\left[(-1)^r(\sum_{i=1}^r A_i + p_2)^{-r} + \cdots + (-2)^r(2\sum_{i=1}^r A_i + p_2)^{-r}\right]\right]\right]. \quad (43)$$

Bayes estimator of λ and a based on general entropy loss function

From (36), the Bayes estimator $\hat{\lambda}_{BG}$ under GE loss function is given by:

$$\hat{\lambda}_{BG} = \left[\frac{1}{Z_1}\Gamma(r-l_1)\left[(\sum_{i=1}^r y_i)^{-(r-l_1)} + \cdots + \alpha^r C^*\left[(\sum_{i=1}^r y_i)^{-(r-l_1)} + \cdots\right.\right.\right. \\ \left.\left.\left. + (-2)^r(2\sum_{i=1}^r y_i)^{-(r-l_1)}\right]\right]\right]^{\frac{-1}{l_1}}. \quad (44)$$

From (38), the Bayes estimator $\hat{a}_{BG}$ under GE loss function is given by:

$$\hat{a}_{BG} = \left[\frac{1}{Z_2}\Gamma(r-l_2)\left[T_0\,(\sum_{i=1}^r A_i)^{-(r-l_2)} + \cdots + \alpha^r C^* T_0\left[(-1)^r(\sum_{i=1}^r A_i)^{-(r-l_2)}\right.\right.\right. \\ \left.\left.\left. + \cdots + (-2)^r(2\sum_{i=1}^r A_i)^{-(r-l_2)}\right]\right]\right]^{\frac{-1}{l_2}}. \quad (45)$$

Remark 3.2. It can be shown that, when $l_1 = 1$ and $l_2 = 1$, the Bayesian estimate in (44) and (45) coincides with the Bayesian estimate under the weighted SE loss function, respectively. Similarly, when $l_1 = -1$ and $l_2 = -1$, the Bayesian estimate in (44) and (45) coincides with the Bayesian estimate under the SE loss function, respectively.

Remark 3.3. It can be shown that, the estimation of reliability and cumulative hazard functions can be obtained by replacing the estimators of the distribution parameters in (10), (14) and (11), (15), respectively.

3.3 Bayesian prediction

For two sample prediction, the data from a past (informative) sample are used to make prediction on future observations (or functions of such observations) of a future sample from the same population. Consider the concomitants of *GOS*'s $Y_{[1,n,m,k]}, Y_{[2,n,m,k]}, \cdots, Y_{[n,n,m,k]}$ from a random sample of size n , and $W_{[1,n_1,m,k]}, W_{[2,n_1,m,k]}, \cdots, W_{[n_1,n_1,m,k]}$ is another independent sample of size n_1 from the same population. In this case, if we want to find the prediction bounds for $W_{[s,n_1,m,k]}$ from the second sample, $1 \le s \le n_1$, from (6), the density function of $W_{[s,n,m,k]}$ is given by:

$$g(w_s|\theta) = f_W(w)\,[1 + \alpha^* C^*(s, n_1, m, k)(2F_W(w) - 1)]. \quad (46)$$

To find the Bayesian prediction bounds and Bayes predictive estimator for $W_{[s,n_1,m,k]}$ in a future random sample based on exponential subfamily of Morgenstern family, from (8), (9), (24), (36) and (46), the Bayes predictive density function of $W_{[s,n_1,m,k]}$, $1 \le s \le n_1$, becomes:

$$g_i(w_s|\mathbf{y}) = \int_0^\infty g_i(w_s|\lambda)\pi_i^*(\lambda|\mathbf{y})\mathrm{d}\lambda,\ i = 1, 3. \quad (47)$$

Hence, the predictive survival function for the $s-th$ future concomitants of GOS's is given by:

$$P_i[W_{[s,n,m,k]} \geq \xi|\mathbf{y}] = \int_{\xi}^{\infty} g_i(w_s|\mathbf{y})\mathrm{d}w_s,\ i = 1,3. \quad (48)$$

So the lower and upper $100\tau\%$ prediction bounds $[L(x),U(x)]$ for Y_s is obtained by equating (48) to $(1+\tau)/2$ and $(1-\tau)/2$, respectively. Now, the predictive estimator of $W_{[s,n_1,m,k]}$, $1 \leq s \leq n_1$, under SE loss function can be obtained as:

$$\tilde{w}_s = E(W_s|\mathbf{y}) = \int_0^{\infty} w_s g_i(w_s|\mathbf{y})\mathrm{d}w_s,\ i = 1,3. \quad (49)$$

4 Numerical results

The main idea of applying a numerical example is to determine the value of the association parameter α. In this section, for type-II censored sample (with $m = 0$ and $k = 1$), in order to illustrate all the inferential results established in the preceding sections, we use real life data for exponential and power subfamilies of Morgenstern family, as follows:

1. We consider the data given in Nelson [19] for exponential subfamily of Morgenstern family. The original data consists of 60 times to breakdown in minutes of an insulating fluid subjected to high-voltage stress. The data is partitioned by Nelson [19] into six groups, each with ten insulating fluids. These data have been analyzed by Balakrishnan et al. [4] by assuming two-parameter exponential distribution. We introduce here the data from groups 4 (group X) and 6 (group Y), as shown in Table 4.1. Based on this data, we computed the ML and Bayesian estimates of λ under the SE, LINEX and GE loss functions using informative (with $\nu_1 = 0.1$, $\beta_1 = 0.2$) and non-informative prior. Also, we computed the ML, Bayesian estimates of the reliability (with $y = 1.5$) and cumulative hazard (with $y = 1.5$) functions, the 95% approximate confidence intervals and the two-sample Bayesian prediction estimation and lower and upper limits of 95% prediction intervals (with $n_1 = 10$, $\alpha^* = \alpha$). We have fitted the exponential distribution to this data, the results are as follows: $\lambda_1 = 0.306996$, $\lambda_2 = \lambda = 0.211706$ for group X and group Y, respectively. Since the correlation coefficient of group X and group Y is $\rho = 0.018289$, for exponential subfamily of Morgenstern family, $\rho = \frac{\alpha}{4}$ then $\alpha = 0.0731559$.
2. We consider the data given in Crowder and Hand [7] for power function subfamily of Morgenstern family. The original data consists of twelve hospital patients were given a special diet. Measurements of plasma ascorbic acid were taken twice before treatment, three times during, and twice after, at week numbers $1,2,6,10,14,15,16$. A question of interest is whether there is any treatment effect. We introduce here the data from weeks 1 (X) and 16 (Y), as shown in Table 4.2, with ignoring some values considered as outliers. Based on this data, we computed the ML and Bayesian estimates of a under the SE, LINEX and GE loss functions using informative (with $\nu_2 = 0.1$, $\beta_2 = 0.2$) and non-informative prior. Also, we computed the ML, Bayesian estimates of the reliability (with $y = 0.5$) and cumulative hazard (with $y = 0.5$) functions and the 95% approximate confidence intervals. We have fitted the power distribution to this data, the results are as follows: $a_1 = 0.22$, $a_2 = a = 0.59$ for data sets X and Y, respectively. Since the correlation coefficient of data sets X and Y is $\rho = 0.218537$, for power subfamily of Morgenstern family, $\rho = 0.275198\alpha$ then $\alpha = 0.794682$.

Table 4.1

$r =$	1	2	3	4	5	6	7	8	9	10
X_i	1.17	3.87	2.80	0.70	3.82	0.02	0.50	3.72	0.06	3.57
Y_i	2.12	3.97	1.56	1	1.83	8.71	2.10	7.21	3.83	5.13
$X_{(i:n)}$	0.02	0.06	0.50	0.70	1.17	2.80	3.57	3.72	3.82	3.87
$Y_{[i:n]}$	8.71	3.83	2.10	1	2.12	1.56	5.13	7.21	1.83	3.97

Observations of order statistics and its concomitants from exponential distribution, $n = 10$.

Table 4.2

$r =$	1	2	3	4	5	6	7	8	9	10
X_i	0.22	0.18	0.3	0.54	0.16	0.3	0.7	0.31	0.6	0.73
Y_i	0.59	0.7	0.36	0.56	0.4	0.88	0.41	0.4	0.67	0.87
$X_{(i:n)}$	0.16	0.18	0.22	0.3	0.3	0.31	0.54	0.6	0.7	0.73
$Y_{[i:n]}$	0.4	0.7	0.59	0.36	0.88	0.4	0.56	0.67	0.41	0.87

Observations of order statistics and its concomitants from power function distribution, $n = 10$.

5 Conclusion and comments

Based on exponential and power function distributions, which are belongs to different families, the joint densities of the concomitants of GOS's for these subfamilies of Morgenstern family have been discussed. The statistical inference procedure for the unknown parameters of the given distributions such as ML estimation with its approximate confidence intervals, Bayes estimators and two-sample Bayesian prediction bounds and estimation are presented. Our applications of these results in those models are given for concomitants of order statistics, and the estimations are conducted on the basis of complete and type-II censored samples. From the results in Tables 4.3 to 4.9, we observe the following.

Table 4.3

r	$\hat{\lambda}_{ML}$		$\hat{\lambda}_{BS}$	$\hat{\lambda}_{BL}$						$\hat{\lambda}_{BG}$					
				$p_1=-3$	$p_1=-2$	$p_1=-1$	$p_1=1$	$p_1=2$	$p_1=3$	$l_1=-3$	$l_1=-2$	$l_1=-1$	$l_1=1$	$l_1=2$	$l_1=3$
1	0.109689	IP	0.119995	0.146373	0.135917	0.127285	0.113728	0.108263	0.103443	0.212706	0.16694	0.119995	0.0106336	-	-
		NIP	0.111375	0.136574	0.12654	0.118302	0.10544	0.10028	0.0957388	0.204917	0.15868	0.111375	-	-	-
2	0.15363	IP	0.160083	0.182567	0.174152	0.166717	0.154115	0.148708	0.143777	0.22973	0.195419	0.160083	0.0826677	0.0246623	-
		NIP	0.154707	0.176838	0.168538	0.161222	0.148856	0.143562	0.13874	0.225099	0.190421	0.154707	0.0761975	-	-
3	0.198809	IP	0.203536	0.227429	0.218644	0.210725	0.196969	0.19094	0.185378	0.265313	0.234841	0.203536	0.136611	0.0982304	0.0393309
		NIP	0.199504	0.223289	0.214532	0.20665	0.192986	0.187008	0.181498	0.261957	0.231156	0.199504	0.131723	0.0925076	-
4	0.249754	IP	0.253316	0.280929	0.270846	0.261687	0.245627	0.23853	0.231953	0.312346	0.283189	0.253316	0.190302	0.155934	0.117145
		NIP	0.250165	0.277834	0.267717	0.258541	0.24248	0.235393	0.228831	0.309854	0.280376	0.250165	0.18637	0.151465	0.111709
5	0.276114	IP	0.278862	0.305215	0.295702	0.286951	0.271357	0.264366	0.257834	0.33158	0.305506	0.278862	0.223164	0.193463	0.161574
		NIP	0.276371	0.302819	0.293263	0.28448	0.268852	0.261855	0.255321	0.329625	0.303289	0.276371	0.220057	0.157582	0.164008
6	0.305908	IP	0.308037	0.334568	0.325056	0.31624	0.300379	0.293209	0.286475	0.356997	0.332755	0.308037	0.256686	0.229652	0.201217
		NIP	0.306045	0.33271	0.323142	0.314282	0.298361	0.29117	0.284421	0.355475	0.331003	0.306045	0.254167	0.226826	0.198025
7	0.283092	IP	0.284936	0.303872	0.29719	0.29089	0.279296	0.273944	0.268856	0.323896	0.304586	0.284936	0.244312	0.223109	0.201057
		NIP	0.283174	0.302161	0.295458	0.289141	0.277525	0.272166	0.267073	0.322431	0.302976	0.283174	0.24222	0.220829	0.198561
8	0.250759	IP	0.252381	0.265066	0.26065	0.256427	0.248502	0.244778	0.241199	0.282606	0.267613	0.252381	0.221016	0.204746	0.187946
		NIP	0.250818	0.263509	0.25909	0.254864	0.246939	0.243216	0.239639	0.28122	0.26614	0.250818	0.219256	0.202875	0.18595
9	0.267762	IP	0.269154	0.281921	0.277487	0.273235	0.265231	0.261456	0.257821	0.297893	0.283628	0.269154	0.23943	0.224081	0.208308
		NIP	0.267773	0.280556	0.276115	0.271859	0.263847	0.260071	0.256435	0.296674	0.28233	0.267773	0.237872	0.222425	0.206545
10	0.267236	IP	0.268429	0.279737	0.275828	0.272061	0.264924	0.261537	0.258263	0.294208	0.281406	0.268429	0.241841	0.228156	0.214141
		NIP	0.267191	0.27851	0.274596	0.270826	0.263683	0.260295	0.25702	0.293101	0.280235	0.267191	0.24046	0.226697	0.212598
Bayesian prediction															
r		(L_{W_1},U_{W_1})	Width	$\tilde{w}_1$	(L_{W_2},U_{W_2})	Width	$\tilde{w}_2$	(L_{W_3},U_{W_3})	Width	$\tilde{w}_3$	(L_{W_4},U_{W_4})	Width	$\tilde{w}_4$	-	-
5	IP	(0.0859445,19.1543)	19.0684	4.3469	(0.0870242,19.2448)	19.1578	4.3767	(0.0881307,19.3345)	19.2464	4.4065	(0.0892648,19.4234)	19.3341	4.4363	-	-
5	NIP	(0.086724,19.4875)	19.4008	4.40829	(0.0878135,19.5799)	19.4921	4.43851	(0.0889299,19.6714)	19.5825	4.46873	(0.0900743,19.7622)	19.6721	4.49895	-	-
7	IP	(0.0840483,16.7046)	16.6206	3.97063	(0.0851048,16.7792)	16.6941	3.99785	(0.0861875,16.8531)	16.7669	4.02507	(0.0872973,16.9263)	16.839	4.05229	-	-
7	NIP	(0.0845735,16.8771)	16.7925	4.00492	(0.0856366,16.9527)	16.8671	4.03238	(0.086726,17.0276)	16.9409	4.05983	(0.0878428,17.1017)	17.0139	4.08729	-	-
8	IP	(0.0948665,18.1903)	18.0954	4.38916	(0.0960592,18.2701)	18.174	4.41925	(0.0972815,18.349)	18.2517	4.44934	(0.0985344,18.4272)	18.3287	4.47943	-	-
8	NIP	(0.0954597,18.3598)	18.2643	4.42439	(0.0966599,18.4405)	18.3438	4.45473	(0.0978898,18.5203)	18.4224	4.48506	(0.0991505,18.5993)	18.5001	4.51539	-	-

Different types of estimation of λ based on informative (IP) and non-informative (NIP) priors. Also, a 95% two-sample Baysian prediction intervals and estimation of w_s, $s=1,2,3,4$, at $r=5$, $r=7$ and $r=8$.

Table 4.4

r	$\hat{R}_{1(ML)}$		$\hat{R}_{1(BS)}$	$\hat{R}_{1(BL)}$						$\hat{R}_{1(BG)}$					
				$p_1=-3$	$p_1=-2$	$p_1=-1$	$p_1=1$	$p_1=2$	$p_1=3$	$l_1=-3$	$l_1=-2$	$l_1=-1$	$l_1=1$	$l_1=2$	$l_1=3$
1	0.848289	IP	0.835276	0.802872	0.815564	0.826192	0.843166	0.850106	0.856274	0.726833	0.778482	0.835276	0.984176	-	-
		NIP	0.846147	0.814761	0.827116	0.8374	0.853713	0.860347	0.866227	0.735374	0.788187	0.846147	-	-	-
2	0.79418	IP	0.78653	0.760446	0.770105	0.778742	0.793603	0.800065	0.806005	0.708507	0.745926	0.78653	0.883378	0.963682	-
		NIP	0.792898	0.767009	0.776618	0.785187	0.799888	0.806265	0.812118	0.713446	0.75154	0.792898	0.891994	-	-
3	0.742143	IP	0.736899	0.710957	0.720388	0.728996	0.744194	0.750955	0.757246	0.671683	0.703096	0.736899	0.814715	0.862996	0.94271
		NIP	0.74137	0.715386	0.724845	0.733465	0.748654	0.755397	0.761666	0.675072	0.706993	0.74137	0.820711	0.870436	-
4	0.687543	IP	0.683879	0.656132	0.666131	0.675346	0.691812	0.699216	0.706149	0.625929	0.653911	0.683879	0.751674	0.79144	0.838855
		NIP	0.687119	0.659185	0.669265	0.67854	0.695086	0.702514	0.709463	0.628273	0.656676	0.687119	0.756120	0.796763	0.845723
5	0.660888	IP	0.658169	0.63266	0.641752	0.650232	0.665621	0.672637	0.67926	0.608128	0.632384	0.658169	0.71552	0.748118	0.784773
		NIP	0.660633	0.634938	0.644104	0.652646	0.668126	0.675176	0.681826	0.609914	0.63449	0.660633	0.718862	0.789486	0.781913
6	0.632002	IP	0.629987	0.605408	0.614108	0.622283	0.637266	0.644157	0.650696	0.585379	0.607057	0.629987	0.680431	0.70859	0.739467
		NIP	0.631873	0.607098	0.615874	0.624114	0.639198	0.64613	0.652704	0.586717	0.608654	0.631873	0.683007	0.7116	0.743016
7	0.654006	IP	0.6522	0.633936	0.640321	0.646401	0.657741	0.663043	0.668122	0.615178	0.633257	0.6522	0.693178	0.715579	0.739645
		NIP	0.653926	0.635565	0.641987	0.648099	0.659491	0.664813	0.669912	0.616531	0.634788	0.653926	0.695357	0.71803	0.742419
8	0.686507	IP	0.684839	0.671931	0.676397	0.680695	0.688835	0.692694	0.696423	0.654483	0.669369	0.684839	0.717829	0.735563	0.754335
		NIP	0.686446	0.673503	0.677982	0.682293	0.690452	0.694319	0.698054	0.655846	0.67085	0.686446	0.719727	0.73763	0.756597
9	0.66922	IP	0.667824	0.655156	0.659528	0.663748	0.671765	0.67558	0.679273	0.639647	0.653481	0.667824	0.698273	0.714536	0.731643
		NIP	0.669209	0.656499	0.660887	0.66512	0.673161	0.676985	0.680687	0.640817	0.654754	0.669209	0.69990	0.716313	0.733581
10	0.669748	IP	0.66855	0.657306	0.661172	0.664918	0.672075	0.675498	0.678823	0.643192	0.655663	0.66855	0.695752	0.710182	0.72527
		NIP	0.669793	0.6585170	0.6623940	0.6661510	0.673327	0.676757	0.68009	0.644261	0.656815	0.669793	0.697195	0.711738	0.72695

Different types of estimation of R_1 based on informative (IP) and non-informative (NIP) priors.

1. Increasing the value of the estimated parameters does not depend on the increasing of r.
2. For fixed r, $\hat{\lambda}_{BL}$, $\hat{H}_{1(BL)}$ decreases as p_1 increases, $\hat{\lambda}_{BG}$, $\hat{H}_{1(BG)}$ decreases as l_1 increases, and $\hat{R}_{1(BL)}$, $\hat{R}_{1(BG)}$ increases as p_1, l_1 increases, respectively.
3. For fixed r, $\hat{a}_{BL}$, $\hat{R}_{2(BL)}$ decreases as p_2 increases, $\hat{a}_{BG}$, $\hat{R}_{2(BL)}$ decreases as l_2 increases, and $\hat{H}_{2(BL)}$, $\hat{H}_{2(BG)}$ increases as p_2, l_2 increases, respectively.
4. At $l_1=-1$, we find that $\hat{\lambda}_{BG}=\hat{\lambda}_{BS}$, $\hat{R}_{1(BG)}=\hat{R}_{1(BS)}$, and $\hat{H}_{1(BG)}=\hat{H}_{1(BS)}$.
5. At $l_2=-1$, we find that $\hat{a}_{BG}=\hat{a}_{BS}$, $\hat{R}_{2(BG)}=\hat{R}_{2(BS)}$, and $\hat{H}_{2(BG)}=\hat{H}_{2(BS)}$.
6. The Bayes estimators of the considered parameters under LINEX and GE loss functions give more accurate results through the fitted values than ML estimator and those under SE loss function by increasing and decreasing the values of p_1, p_2 and l_1, l_2 , respectively.
7. We can compute two-sample Bayesian prediction bounds and estimation for only exponential distribution, not for power function distribution.

Table 4.5

r	$\hat{H}_{1(ML)}$		$\hat{H}_{1(BS)}$	$\hat{H}_{1(BL)}$						$\hat{H}_{1(BG)}$					
				$p_1=-3$	$p_1=-2$	$p_1=-1$	$p_1=1$	$p_1=2$	$p_1=3$	$l_1=-3$	$l_1=-2$	$l_1=-1$	$l_1=1$	$l_1=2$	$l_1=3$
1	0.164534	IP	0.179993	0.21956	0.203876	0.190928	0.170592	0.162395	0.155164	0.319059	0.25041	0.179993	0.0159504	-	-
		NIP	0.167063	0.204861	0.18981	0.177453	0.15816	0.15042	0.143608	0.307375	0.23802	0.167063	-	-	-
2	0.230445	IP	0.240125	0.273851	0.261228	0.250076	0.231173	0.223062	0.215665	0.344595	0.293129	0.240125	0.124002	0.0369935	-
		NIP	0.232061	0.265257	0.252807	0.241833	0.223284	0.215343	0.20811	0.337649	0.285632	0.232061	0.114296	-	-
3	0.298214	IP	0.305304	0.341144	0.327966	0.316087	0.295454	0.28641	0.278067	0.39797	0.352262	0.305304	0.204917	0.147346	0.0589964
		NIP	0.299256	0.334934	0.321798	0.309975	0.289479	0.280512	0.272247	0.392936	0.346734	0.299256	0.197585	0.138761	-
4	0.374631	IP	0.379974	0.421394	0.406269	0.392531	0.368441	0.357795	0.34793	0.468519	0.424784	0.379974	0.285453	0.233901	0.175718
		NIP	0.375248	0.416751	0.4015760	0.387812	0.36372	0.35309	0.343247	0.464781	0.420564	0.375248	0.279555	0.227198	0.167564
5	0.414171	IP	0.418293	0.457823	0.443553	0.430427	0.407036	0.396549	0.386751	0.49737	0.458259	0.418293	0.334746	0.290195	0.242361
		NIP	0.414557	0.454229	0.439894	0.42672	0.403278	0.392783	0.382982	0.494438	0.454934	0.414557	0.330086	0.236373	0.246012
6	0.458862	IP	0.462056	0.501852	0.487584	0.47436	0.450569	0.439813	0.429712	0.535496	0.499133	0.462056	0.385029	0.344478	0.301826
		NIP	0.459068	0.499065	0.484713	0.471423	0.447542	0.436755	0.426631	0.533213	0.496505	0.459068	0.38125	0.340239	0.297038
7	0.424638	IP	0.427404	0.455808	0.445785	0.436335	0.418944	0.410916	0.403284	0.485844	0.456879	0.427404	0.366468	0.334664	0.301586
		NIP	0.424761	0.453242	0.443187	0.433711	0.416288	0.408249	0.40061	0.483647	0.454464	0.424761	0.36333	0.331244	0.297842
8	0.376139	IP	0.378572	0.397599	0.390975	0.384641	0.372753	0.367167	0.361799	0.423909	0.40142	0.378572	0.331524	0.307119	0.281919
		NIP	0.376227	0.395264	0.388635	0.38229	0.370409	0.364824	0.359459	0.42183	0.39921	0.376227	0.328884	0.304313	0.278925
9	0.401643	IP	0.403731	0.422881	0.41623	0.409853	0.397847	0.392184	0.386732	0.44684	0.425442	0.403731	0.359145	0.336122	0.312462
		NIP	0.401659	0.420834	0.414173	0.407789	0.395771	0.390107	0.384653	0.445011	0.423495	0.401659	0.356808	0.333638	0.309818
10	0.400854	IP	0.402643	0.419606	0.413742	0.408092	0.397386	0.392306	0.387395	0.441312	0.422109	0.402643	0.362762	0.342234	0.321212
		NIP	0.400787	0.417765	0.411894	0.406239	0.395525	0.390443	0.38553	0.439652	0.420353	0.400787	0.36069	0.340046	0.318897

Different types of estimation of H_1 based on informative (IP) and non-informative (NIP) priors.

Table 4.6

r	$\hat{a}_{ML}$		$\hat{a}_{BS}$	$\hat{a}_{BL}$						$\hat{a}_{BG}$					
				$p_2=-3$	$p_2=-2$	$p_2=-1$	$p_2=1$	$p_2=2$	$p_2=3$	$l_2=-3$	$l_2=-2$	$l_2=-1$	$l_2=1$	$l_2=2$	$l_2=3$
1	1.47018	IP	1.29106	-	-	2.93741	0.936102	0.752904	0.636922	2.00859	1.65668	1.29106	0.205836	-	-
		NIP	1.44615	-	-	-	0.998582	0.786094	0.656488	2.30436	1.88248	1.44615	-	-	-
2	2.05156	IP	1.84219	-	-	2.9345	1.42241	1.18014	1.01789	2.42163	2.13701	1.84219	1.1679	0.480606	-
		NIP	2.03682	-	-	3.70402	1.52706	1.24839	1.06676	2.6986	2.37319	2.03682	1.26552	-	-
3	2.13499	IP	1.96701	-	13.0941	2.65199	1.60941	1.37885	1.21456	2.40521	2.18958	1.96701	1.48068	1.1834	0.594202
		NIP	2.11848	-	-	2.98061	1.70446	1.44619	1.26557	2.60021	2.36301	2.11848	1.5842	1.25398	-
4	1.78813	IP	1.69121	7.13138	2.6577	2.02987	1.46744	1.30491	1.18003	1.98986	1.84278	1.69121	1.36532	1.18035	0.957866
		NIP	1.77124	-	2.92677	2.15584	1.52462	1.34846	1.21458	2.08875	1.93235	1.77124	1.4247	1.22722	0.986245
5	2.03781	IP	1.93355	5.59315	2.97207	2.30396	1.68613	1.5052	1.36547	2.22388	2.0805	1.93355	1.62321	1.45431	1.26789
		NIP	2.02635	-	3.25722	2.44612	1.75377	1.55762	1.40773	2.3346	2.18234	2.02635	1.69695	1.51746	1.3186
6	1.82173	IP	1.74586	3.06874	2.37526	1.99734	1.56206	1.42003	1.30608	1.97769	1.86317	1.74586	1.49931	1.36714	1.22539
		NIP	1.80908	3.35466	2.51129	2.08477	1.61079	1.45915	1.3384	2.05191	1.93196	1.80908	1.55076	1.4121	1.26306
7	1.78216	IP	1.71749	2.72197	2.23552	1.93206	1.555	1.42639	1.32138	1.92377	1.82175	1.71749	1.49998	1.38497	1.26389
		NIP	1.77196	2.8996	2.33972	2.00437	1.5981	1.46165	1.35092	1.98689	1.8806	1.77196	1.54522	1.42521	1.2987
8	1.79365	IP	1.7364	2.61394	2.20412	1.93356	1.58415	1.46192	1.36102	1.92681	1.83249	1.7364	1.53739	1.43337	1.3252
		NIP	1.78714	2.7577	2.29544	1.99936	1.62496	1.49574	1.38966	1.985	1.88699	1.78714	1.58026	1.47205	1.35944
9	1.62094	IP	1.58175	2.21664	1.93613	1.73507	1.4598	1.35979	1.27583	1.74759	1.6653	1.58175	1.41007	1.32138	1.23023
		NIP	1.62016	2.30749	2.00006	1.78347	1.4913	1.38625	1.29849	1.79177	1.70663	1.62016	1.44242	1.35055	1.25609
10	1.72041	IP	1.67999	2.31657	2.03851	1.83605	1.55488	1.45165	1.36459	1.84013	1.7606	1.67999	1.51486	1.42992	1.343
		NIP	1.72027	2.40722	2.10375	1.88622	1.58824	1.47991	1.38895	1.88582	1.80362	1.72027	1.54952	1.46165	1.3717

Different types of estimation of a based on informative prior. based on informative (IP) and non-informative (NIP) priors.

One-sample Bayesian prediction bounds and estimation can not be computed.

8. The Bayesian prediction provides lower and upper limits of 95% prediction intervals, it is observed that the prediction intervals tend be wider when s increase, also for Bayesian prediction estimation. Even though, the concomitants of ordered random variables may be in order or not, with our data it is not in order, our results that have been obtained are in order.
9. For considering various values for the hyperparameters in the informative priors of Bayes estimators, the results did not change the obtained conclusions.
10. For the relation between ρ and α, we can use it as a standard to see wether the data fit the distribution or not.

The proposed procedures for the estimation and prediction problems may be considered for other models, and for some other distributions.

Conflict of interest

The authors declare that they have no conflict of interest.

Table 4.7

r	$\hat{R}_{2(ML)}$		$\hat{R}_{2(BS)}$	$\hat{R}_{2(BL)}$						$\hat{R}_{2(BG)}$					
				$p_2=-3$	$p_2=-2$	$p_2=-1$	$p_2=1$	$p_2=2$	$p_2=3$	$l_2=-3$	$l_2=-2$	$l_2=-1$	$l_2=1$	$l_2=2$	$l_2=3$
1	0.639063	IP	0.591349	-	-	0.869458	0.477357	0.406592	0.356916	0.751484	0.682832	0.591349	0.132964	-	-
		NIP	0.633001	-	-	-	0.499508	0.420088	0.365579	0.79755	0.728783	0.633001	-	-	-
2	0.758777	IP	0.721102	-	-	0.869194	0.626911	0.558691	0.506162	0.813355	0.77265	0.721102	0.554931	0.283323	-
		NIP	0.7563	-	-	0.923268	0.653016	0.579082	0.52261	0.845958	0.806982	0.7563	0.584051	-	-
3	0.772331	IP	0.744217	-	0.999886	0.8409	0.672268	0.615475	0.569096	0.811218	0.780785	0.744217	0.64168	0.559687	0.337589
		NIP	0.769711	-	-	0.873309	0.693164	0.633011	0.584065	0.835086	0.805615	0.769711	0.66649	0.58071	-
4	0.710453	IP	0.690333	0.992868	0.841528	0.755123	0.638377	0.595254	0.558658	0.748237	0.721216	0.690333	0.611852	0.558756	0.485182
		NIP	0.707043	-	0.868491	0.775598	0.652429	0.607289	0.569101	0.764916	0.737998	0.707043	0.627503	0.5728	0.49521
5	0.756467	IP	0.738216	0.979285	0.872556	0.797494	0.689241	0.647719	0.611892	0.785935	0.763568	0.738216	0.675388	0.63507	0.584733
		NIP	0.754525	-	0.895413	0.816496	0.703474	0.660289	0.623096	0.801749	0.779682	0.754525	0.691563	0.6507	0.599076
6	0.717118	IP	0.701844	0.880816	0.807258	0.749539	0.661333	0.626295	0.595582	0.746104	0.725128	0.701844	0.646277	0.612341	0.572318
		NIP	0.714627	0.902243	0.824601	0.764266	0.672581	0.636293	0.604541	0.758835	0.737927	0.714627	0.65867	0.624236	0.583341
7	0.709252	IP	0.695923	0.848433	0.787655	0.737945	0.659671	0.627939	0.59984	0.736435	0.717122	0.695923	0.646442	0.617103	0.58358
		NIP	0.707189	0.865991	0.802451	0.750756	0.669688	0.636922	0.607958	0.747718	0.728429	0.707189	0.657357	0.627635	0.593508
8	0.711559	IP	0.699882	0.836648	0.782983	0.738218	0.666479	0.63699	0.610693	0.73699	0.71922	0.699882	0.655492	0.629735	0.600906
		NIP	0.710254	0.85214	0.796294	0.749889	0.675781	0.645401	0.618345	0.747387	0.729629	0.710254	0.665578	0.63953	0.610266
9	0.674876	IP	0.665924	0.784858	0.7386840	0.6996	0.636456	0.610361	0.587013	0.702201	0.684721	0.665924	0.623707	0.599848	0.573751
		NIP	0.674701	0.797988	0.75001	0.709516	0.644308	0.617442	0.593449	0.711183	0.693625	0.674701	0.63205	0.607857	0.581323
10	0.696538	IP	0.687915	0.799256	0.756585	0.719912	0.659643	0.634397	0.611655	0.720703	0.704875	0.687915	0.65007	0.628849	0.6058
		NIP	0.696508	0.811481	0.767347	0.729485	0.667423	0.641489	0.618157	0.72941	0.713545	0.696508	0.658376	0.636922	0.613564

Different types of estimation of R_2 based on informative (IP) and non-informative (NIP) priors.

Table 4.8

r	$\hat{H}_{2(ML)}$		$\hat{H}_{2(BS)}$	$\hat{H}_{2(BL)}$						$\hat{H}_{2(BG)}$					
				$p_2=-3$	$p_2=-2$	$p_2=-1$	$p_2=1$	$p_2=2$	$p_2=3$	$l_2=-3$	$l_2=-2$	$l_2=-1$	$l_2=1$	$l_2=2$	$l_2=3$
1	0.447753	IP	0.525348	-	-	0.139886	0.739491	0.899945	1.03025	0.285705	0.381507	0.525348	2.01768	-	-
		NIP	0.457284	-	-	-	0.69413	0.86729	1.00627	0.22621	0.31637	0.45728	-	-	-
2	0.276047	IP	0.326975	-	-	0.140189	0.46695	0.582158	0.680899	0.206588	0.25793	0.326975	0.588911	1.26117	-
		NIP	0.279318	-	-	0.0798362	0.426153	0.546311	0.64892	0.167286	0.214454	0.279318	0.537768	-	-
3	0.258342	IP	0.295422	-	0.000114369	0.173283	0.397099	0.485361	0.563707	0.209218	0.247456	0.295422	0.443665	0.580377	1.08593
		NIP	0.261741	-	-	0.135466	0.366489	0.457268	0.537743	0.180221	0.21615	0.261741	0.405729	0.543504	-
4	0.341853	IP	0.370581	0.00715804	0.172536	0.280875	0.448827	0.518768	0.582218	0.290036	0.326817	0.370581	0.491265	0.582043	0.723231
		NIP	0.346664	-	0.140998	0.254121	0.427053	0.498751	0.563696	0.26799	0.303814	0.346664	0.466007	0.557113	0.702773
5	0.279097	IP	0.303519	0.020933	0.136328	0.226282	0.372165	0.434299	0.491199	0.240881	0.269754	0.303519	0.392469	0.454019	0.536599
		NIP	0.281667	-	0.110471	0.202733	0.351724	0.415078	0.473055	0.22096	0.248869	0.281667	0.368802	0.429707	0.512366
6	0.332514	IP	0.354044	0.126906	0.214111	0.288297	0.413498	0.467933	0.518217	0.29289	0.321407	0.354044	0.436526	0.490466	0.55806
		NIP	0.335994	0.102871	0.192855	0.268839	0.396633	0.452097	0.503286	0.27597	0.30391	0.335994	0.417533	0.471227	0.538984
7	0.343544	IP	0.362517	0.164364	0.238695	0.303886	0.416013	0.465312	0.511079	0.305934	0.332509	0.362517	0.436272	0.48272	0.538573
		NIP	0.346457	0.143881	0.220084	0.286674	0.400943	0.451108	0.497649	0.29073	0.316865	0.346457	0.419529	0.465797	0.521705
8	0.340297	IP	0.356843	0.178352	0.244644	0.303517	0.405747	0.451001	0.493161	0.305181	0.329587	0.356843	0.422369	0.462456	0.509317
		NIP	0.342132	0.160004	0.227787	0.28783	0.391886	0.437883	0.480708	0.291172	0.315218	0.342132	0.407099	0.447021	0.49386
9	0.393226	IP	0.40658	0.242252	0.302886	0.357239	0.451839	0.493705	0.532709	0.353535	0.378744	0.40658	0.472075	0.511079	0.555561
		NIP	0.393486	0.225661	0.287668	0.343172	0.439578	0.48217	0.521805	0.340826	0.365824	0.393486	0.458786	0.497815	0.542449
10	0.361634	IP	0.37409	0.224074	0.27894	0.328626	0.416056	0.45508	0.491587	0.327528	0.349735	0.37409	0.430676	0.463865	0.501206
		NIP	0.361676	0.208894	0.264816	0.315416	0.404331	0.443964	0.481012	0.315519	0.33751	0.361676	0.417979	0.451108	0.48847

Different types of estimation of H_2 based on informative (IP) and non-informative (NIP) priors.

Table 4.9

	$ACI(\lambda)$			$ACI(a)$		
r	Lower	Upper	Width	Lower	Upper	Width
1	-0.109807	0.329185	0.438992	-0.816423	3.75678	4.573203
2	-0.0632868	0.370547	0.433834	-0.29809	4.40121	4.6993
3	-0.0298703	0.427488	0.457358	0.103234	4.16675	4.06352
4	0.00151447	0.497994	0.49648	0.282846	3.29341	3.01056
5	0.031165	0.521063	0.489898	0.457839	3.61778	3.15994
6	0.0586269	0.553189	0.494562	0.494468	3.14899	2.65452
7	0.0718598	0.494324	0.422464	0.547041	3.01728	2.47024
8	0.0763153	0.425203	0.348887	0.606246	2.98105	2.37481
9	0.0923379	0.443186	0.350848	0.568122	2.67376	2.10564
10	0.10158	0.432892	0.331312	0.657024	2.7838	2.12677

A 95% approximate confidence intervals (ACI) and their lengths of the parameters λ and a.

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