Journal of Analysis & Number Theory
An International Journal

http://dx.doi.org/10.18576/jant/040102

A New Class of Hermite Poly-Genocchi Polynomials

*Waseem A. Khan**

Department of Mathematics, Integral University, Lucknow-226026, India

Received: 9 Sep. 2015, Revised: 5 Oct. 2015, Accepted: 13 Oct. 2015
Published online: 1 Jan. 2016

Abstract: In this paper, we introduce a new class of Hermite poly-Genocchi polynomials and we give some identities of those polynomials related to the Stirling numbers of the second kind. The concepts of poly-Bernoulli numbers $B_n^{(k)}(a,b)$, poly-Bernoulli polynomials $B_n^{(k)}(x,a,b)$ of Jolany et al, Hermite-Bernoulli polynomials ${}_HB_n(x,y)$ of Dattoli et al and ${}_HB_n^{(\alpha)}(x,y)$ of Pathan et al are generalized to the one ${}_HG_n^{(k)}(x,y)$. Some implicit summation formulae and general symmetry identities are derived by using different analytical means and applying generating functions. These results extend some known summations and identities of Hermite poly-Genocchi numbers and polynomials.

Keywords: Hermite polynomials, poly-Genocchi polynomials, Hermite poly-Genocchi polynomials, summation formulae, symmetric identities.
2010 Mathematics Subject Classification.: 33C45, 33C99, 05A10, 05A15

1 Introduction

The 2-variable Kampe de Feriet generalization of the Hermite polynomials [13] and [15] reads

$$H_n(x,y) = n!\sum_{r=0}^{[\frac{n}{2}]}\frac{y^r x^{n-2r}}{r!(n-2r)!} \tag{1.1}$$

These polynomials are usually defined by the generating function

$$e^{xt+yt^2} = \sum_{n=0}^{\infty} H_n(x,y)\frac{t^n}{n!} \tag{1.2}$$

and reduce to the ordinary Hermite polynomials $H_n(x)$ (see [1]) when $y=-1$ and x is replaced by $2x$.
The classical Bernoulli polynomials $B_n(x)$, the classical Euler polynomials $E_n(x)$ and the classical Genocchi polynomials $G_n(x)$, together with their familiar generalizations $B_n^{(\alpha)}(x)$, $E_n^{(\alpha)}(x)$ and $G_n^{(\alpha)}(x)$ of (real or complex) order α are usually defined by means of the following generating functions (see for details [2],[36], pp.532-533 and [38], p.61; see also [41] and the references cited therein):

$$\left(\frac{t}{e^t-1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} B_n^{(\alpha)}\frac{t^n}{n!} \qquad (|\,t\,|<2\pi;1^{\alpha}=1) \tag{1.3}$$

$$\left(\frac{2}{e^t+1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} E_n^{(\alpha)}\frac{t^n}{n!} \qquad (|\,t\,|<\pi;1^{\alpha}=1) \tag{1.4}$$

and

$$\left(\frac{2t}{e^t+1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} G_n^{(\alpha)}\frac{t^n}{n!} \qquad (|\,t\,|<\pi;1^{\alpha}=1) \tag{1.5}$$

So that obviously the classical Bernoulli polynomials $B_n(x)$, the classical Euler polynomials $E_n(x)$ and the classical Genocchi polynomials $G_n(x)$ are given respectively by

$$B_n(x) = B_n^{(1)}(x), E_n(x) = E_n^{(1)}(x)$$

and

$$G_n(x) = G_n^{(1)}(x) \qquad (n\varepsilon N) \tag{1.6}$$

For the classical Bernoulli numbers B_n, the classical Euler numbers E_n and the classical Genocchi numbers G_n

$$B_n^1(0) = B_n(0) = B_n, E_n^1(0) = E_n(0) = E_n$$

and

$$G_n^1(0) = G_n(0) = G_n, \tag{1.7}$$

respectively.

* Corresponding author e-mail: waseem08_khan@rediffmail.com

The history of Genocchi numbers can be traced back to Italian mathematician Angelo Genocchi (1817-1889). From Genocchi to the present time, Genocchi numbers have been extensively studied in many different context in such branches of Mathematics as, for instance, elementary number theory, complex analytic number theory, Homotopy theory (stable Homotopy groups of spheres), differential topology (differential structures on spheres), theory of modular forms (Eisenstein series), p-adic analytic number theory (p-adic L-functions), quantum physics (quantum Groups). The works of Genocchi numbers and their combinatorial relations have received much attention [3,4,5,6,7,8,9,10,11,12,14,16,17,29,39,40].

In [25], Kaneko introduced and studied poly-Bernoulli numbers which generalize the classical Bernoulli numbers. poly-Bernoulli numbers $B_n^{(k)}$ with $k\varepsilon z$ and $n\varepsilon N$, appear in the following power series

$$\frac{Li_k(1-e^{-t})}{1-e^{-t}}=\sum_{n=0}^{\infty}B_n^{(k)}\frac{t^n}{n!} \tag{1.8}$$

where $k\varepsilon z$ and

$$Li_k=\sum_{m=1}^{\infty}\frac{z^m}{m^k},|z|<1$$

so for $k\leq 1$,

$$Li_k=-\ln(1-z),\ \ Li_0(z)=\frac{z}{1-z},\ \ Li_{-1}=\frac{z}{(1-z)^2},\ldots$$

Moreover when $k\geq 1$, the left hand side of (1.8) can be written in the form

$$e^t\frac{1}{e^t-1}\int_0^t\frac{1}{e^t-1}\cdots\int_0^t\frac{1}{e^t-1}\int_0^t\frac{t}{e^t-1}dtdt\cdots dt=\sum_{n=0}^{\infty}B_n^{(k)}\frac{t^n}{n!}$$

In the special case, one can see

$$B_n^{(1)}=B_n.$$

Recently, Jolany et al [21,22] generalized the concept of poly-Bernoulli polynomials is defined as follows.

Let $a,b,c>0$ and $a\neq b$. The generalized poly-Bernoulli numbers $B_n^{(k)}(a,b)$, the generalized poly-Bernoulli polynomials $B_n^{(k)}(x,a,b)$ and the polynomials $B_n^{(k)}(x,a,b,c)$ are appeared in the following series respectively

$$\frac{Li_k(1-(ab)^{-t})}{b^t-a^{-t}}=\sum_{n=0}^{\infty}B_n^{(k)}(a,b)\frac{t^n}{n!},|t|<\frac{2\pi}{|\ln a+\ln b|} \tag{1.9}$$

$$\frac{Li_k(1-(ab)^{-t})}{b^t-a^{-t}}e^{xt}=\sum_{n=0}^{\infty}B_n^{(k)}(x,a,b)\frac{t^n}{n!},|t|<\frac{2\pi}{|\ln a+\ln b|} \tag{1.10}$$

$$\frac{Li_k(1-(ab)^{-t})}{b^t-a^{-t}}c^{xt}=\sum_{n=0}^{\infty}B_n^{(k)}(x,a,b,c)\frac{t^n}{n!},|t|<\frac{2\pi}{|\ln a+\ln b|} \tag{1.11}$$

One can easily see that

$$B_n^{(k)}(0,1,e)=B_n^{(k)},B_n^{(k)}(x)=1+x$$

and

$$B_n^{(k)}(x)=B_n^{(k)}(e^{x+1},e^x) \tag{1.12}$$

where $B_n^{(k)}$ are generalized poly-Bernoulli numbers. For more information about poly-Bernoulli numbers and poly-Bernoulli polynomials, we refer to [18] to [23].

Very recently, Pathan et al [30] to [35] introduced the generalized Hermite-Bernoulli polynomials of two variables ${}_HB_n^{(\alpha)}(x,y)$ is defined by

$$\left(\frac{t}{e^t-1}\right)^{\alpha}e^{xt+yt^2}=\sum_{n=0}^{\infty}{}_HB_n^{(\alpha)}(x,y)\frac{t^n}{n!} \tag{1.13}$$

which is essentially a generalization of Bernoulli numbers, Bernoulli polynomials, Hermite polynomials and Hermite-Bernoulli polynomials ${}_HB_n(x,y)$ introduced by Dattoli et al [15, p.386(1.6)] in the form

$$\left(\frac{t}{e^t-1}\right)e^{xt+yt^2}=\sum_{n=0}^{\infty}{}_HB_n(x,y)\frac{t^n}{n!} \tag{1.14}$$

The Stirling number of the first kind is given by

$$(x)_n=x(x-1)\cdots(x-n+1)=\sum_{l=0}^{n}S_1(n,l)x^l,(n\geq 0) \tag{1.15}$$

and the Stirling number of the second kind is defined by generating function to be

$$(e^t-1)^n=n!\sum_{l=n}^{\infty}S_2(l,n)\frac{t^l}{l!} \tag{1.16}$$

In this paper, we first give definitions of the Hermite poly-Genocchi polynomials ${}_HG_n^{(k)}(x,y)$ and we give some formulae of those polynomials related to the Stirling numbers of the second kind. Some implicit summation formulae and general symmetry identities are derived by using different analytical means and applying generating functions. These results extend some known summations and identities of generalized Hermite-Bernoulli polynomials studied by Dattoli et al, Zhang et al, Yang, Khan, Pathan and Khan.

2 A new class of Hermite poly-Genocchi polynomials

Now, we define the Hermite poly-Genocchi polynomials as follows

$$\frac{2Li_k(1-e^{-t})}{e^t+1}e^{xt+yt^2}=\sum_{n=0}^{\infty}{}_HG_n^{(k)}(x,y)\frac{t^n}{n!},(k\varepsilon z) \tag{2.1}$$

so that

$$ {}_HG_n^{(k)}(x,y) = \sum_{m=0}^{n} \binom{n}{m} G_{n-m}^{(k)} H_m(x,y) \qquad (2.2) $$

when $x = y = 0$, $G_n^{(k)} = G(0,0)$ are called the poly-Genocchi numbers. By (2.1), we easily get $G_0^{(k)} = 0$. For $k = 1$, from (2.1), we have

$$ \frac{2Li_1(1-e^{-t})}{e^t+1} e^{xt+yt^2} = \sum_{n=0}^{\infty} {}_HG_n(x,y)\frac{t^n}{n!} \qquad (2.3) $$

Thus by (2.1) and (2.3), we get

$$ {}_HG_n^{(k)}(x,y) = {}_HG_n(x,y), (n \geq 0). $$

For $y = 0$ in (2.1), the result reduces to the poly-Genocchi polynomials Kim et al [28.,p.Eq.(4)4776] is defined as

$$ \frac{2Li_k(1-e^{-t})}{e^t+1} e^{xt} = \sum_{n=0}^{\infty} G_n^{(k)}(x)\frac{t^n}{n!}, (k\varepsilon z) \qquad (2.4) $$

Theorem 2.1. For $n \geq 0$, we have

$$ {}_HG_n^{(2)}(x,y) = \sum_{m=0}^{n} \binom{n}{m} \frac{B_m}{m+1} {}_HG_{n-m}(x,y) \qquad (2.5) $$

Proof. Applying Definition (2.1), we have

$$ \sum_{n=0}^{\infty} {}_HG_n^{(k)}(x,y)\frac{t^n}{n!} = \frac{2Li_k(1-e^{-t})}{e^t+1} e^{xt+yt^2} $$

$$ = \frac{2}{e^t+1} e^{xt+yt^2} \int_0^t \frac{1}{e^z-1} \int_0^t \frac{1}{e^z-1} \cdots \frac{1}{e^z-1} \int_0^t \frac{z}{e^z-1} dz \cdots dz $$

In particular $k = 2$, we have

$$ {}_HG_n^{(2)}(x,y) = \frac{2}{e^t+1} e^{xt+yt^2} \int_0^t \frac{z}{e^z-1} dz = \left(\sum_{m=0}^{\infty} \frac{t^m B_m}{m+1} \right) \frac{2t}{e^t+1} e^{xt+yt^2} $$

$$ = \left(\sum_{m=0}^{\infty} \frac{t^m B_m}{m+1} \right) \left(\sum_{n=0}^{\infty} {}_HG_n(x,y)\frac{t^n}{n!} \right) $$

Replacing n by n-m in above equation, we have

$$ = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \binom{n}{m} \frac{B_m}{m+1} {}_HG_{n-m}(x,y)\frac{t^n}{n!} $$

On equating the coefficients of the like powers of t in the above equation, we get the result (2.5).

Remark 1. For $y = 0$ in Theorem (2.1), the result reduces to known result of Kim et al [28.,p. 4777, Theorem (2.1)].

Corollary 1. For $n \geq 0$, we have

$$ G_n^{(2)}(x) = \sum_{m=0}^{n} \binom{n}{m} \frac{B_m}{m+1} G_{n-m}(x) \qquad (2.6) $$

Theorem 2.2. For $n \geq 1$, the degree of ${}_HG_n^{(k)}(x,y)$ is n-1. we have

$$ \frac{{}_HG_n^{(k)}(x,y)}{n} = \sum_{m=0}^{n-1} \binom{n-1}{m} \frac{G_{m+1}^{(k)}}{m+1} H_{n-m-1}(x,y) \qquad (2.7) $$

Proof. By Definition (2.1) of Hermite poly-Genocchi polynomials, we have

$$ \sum_{n=0}^{\infty} {}_HG_n^{(k)}(x,y)\frac{t^n}{n!} = \frac{2Li_k(1-e^{-t})}{1-e^{-t}} e^{xt+yt^2} $$

$$ = \left(\sum_{m=0}^{\infty} G_m^{(k)} \frac{t^m}{m!} \right) \left(\sum_{n=0}^{\infty} H_n(x,y)\frac{t^n}{n!} \right) $$

Replacing n by n-m in above equation and comparing the coefficients of t^n, we get

$$ {}_HG_n^{(k)}(x,y) = \sum_{m=0}^{n} \binom{n}{m} G_m^{(k)} H_{n-m}(x,y), (n \geq 0) \qquad (2.8) $$

From (2.8), we have

$$ \frac{{}_HG_n^{(k)}(x,y)}{n} = \sum_{m=0}^{n-1} \binom{n-1}{m} \frac{G_{m+1}^{(k)}}{m+1} H_{n-m-1}(x,y), (n \geq 1) \qquad (2.9) $$

Therefore by (2.9), we obtain the result (2.7).

Remark 2. For $y = 0$ in Theorem (2.2), the result reduces to known result of Kim et al [28.,p. 4778, Theorem (2.2)].

Corollary 2. For $n \geq 1$, the degree of $G_n^{(k)}(x)$ is n-1. we have

$$ \frac{G_n^{(k)}(x)}{n} = \sum_{m=0}^{n-1} \binom{n-1}{m} \frac{G_{m+1}^{(k)}}{m+1} x^{n-m-1} \qquad (2.10) $$

Theorem 2.3. For $n \geq 0$, we have

$$ {}_HG_n^{(k)}(x,y) = \sum_{p=0}^{n} \sum_{l=1}^{p+1} \frac{(-1)^{l+p+1} l! S_2(p+1,l)}{l^k(p+1)} \binom{n}{p} {}_HG_{n-p}(x,y) \qquad (2.11) $$

Proof. From equation (2.1), we have

$$ \sum_{n=0}^{\infty} {}_HG_n^{(k)}(x,y)\frac{t^n}{n!} = \left(\frac{Li_k(1-e^{-t})}{t} \right) \left(\frac{2t}{e^t+1} e^{xt+yt^2} \right) \qquad (2.12) $$

Now

$$ \frac{1}{t} Li_k(1-e^{-t}) = \frac{1}{t} \sum_{l=1}^{\infty} \frac{(1-e^{-t})^l}{l^k} = \frac{1}{t} \sum_{l=1}^{\infty} \frac{(-1)^l}{l^k} (1-e^{-t})^l $$

$$ = \frac{1}{t} \sum_{l=1}^{\infty} \frac{(-1)^l}{l^k} l! \sum_{p=l}^{\infty} (-1)^p S_2(p,l) \frac{t^p}{p!} $$

$$ = \frac{1}{t} \sum_{p=1}^{\infty} \sum_{l=1}^{p} \frac{(-1)^{l+p}}{l^k} l! S_2(p,l) \frac{t^p}{p!} $$

$$ = \sum_{p=0}^{\infty} \left(\sum_{l=1}^{p+1} \frac{(-1)^{l+p+1}}{l^k} l! \frac{S_2(p+1,l)}{p+1} \right) \frac{t^p}{p!} \qquad (2.13) $$

From equations (2.12) and (2.13), we get

$$ \sum_{n=0}^{\infty} {}_HG_n^{(k)}(x,y)\frac{t^n}{n!} = \sum_{p=0}^{\infty} \left(\sum_{l=1}^{p+1} \frac{(-1)^{l+p+1}}{l^k} l! \frac{S_2(p+1,l)}{p+1} \right) \frac{t^p}{p!} \left(\sum_{n=0}^{\infty} {}_HG_n(x,y)\frac{t^n}{n!} \right) $$

Replacing n by n-p in the r.h.s of above equation and comparing the coefficients of t^n, we get the result (2.11).

Remark 3. For $y=0$ in Theorem (2.3), the result reduces to known result of Kim et al [28.,p. 4779, Theorem (2.3)].

Corollary 3. For $n\geq 0$, we have

$$G_n^{(k)}(x)=\sum_{p=0}^{n}\sum_{l=1}^{p+1}\frac{(-1)^{l+p+1}l!S_2(p+1,l)}{l^k(p+1)}\binom{n}{p}G_{n-p}(x) \tag{2.14}$$

Theorem 2.4. For $n\geq 1$, we have

$$_HG_n^{(k)}(x+1,y)+{}_HG_n^{(k)}(x,y)=2\sum_{p=1}^{n}\sum_{l=1}^{p}\frac{(-1)^{l+p}}{l^k}l!S_2(p,l)\binom{n}{p}H_{n-p}(x,y) \tag{2.15}$$

Proof. Using the Definition (2.1), we have

$$\sum_{n=0}^{\infty}{}_HG_n^{(k)}(x+1,y)\frac{t^n}{n!}+\sum_{n=0}^{\infty}{}_HG_n^{(k)}(x,y)\frac{t^n}{n!}$$

$$=\frac{2Li_k(1-e^{-t})}{e^t+1}e^{(x+1)t+yt^2}+\frac{2Li_k(1-e^{-t})}{e^t+1}e^{xt+yt^2}$$

$$=2Li_k(1-e^{-t})e^{xt+yt^2}$$

$$=\sum_{p=1}^{\infty}\left(2\sum_{l=1}^{p}\frac{(-1)^{l+p}}{l^k}l!S_2(p,l)\right)\frac{t^p}{p!}e^{xt+yt^2}$$

$$=\left(\sum_{p=1}^{\infty}\left(2\sum_{l=1}^{p}\frac{(-1)^{l+p}}{l^k}l!S_2(p,l)\right)\frac{t^p}{p!}\right)\left(\sum_{n=0}^{\infty}H_n(x,y)\frac{t^n}{n!}\right)$$

Replacing n by n-p in the above equation and comparing the coefficients of t^n, we get the result (2.15).

Remark 4. For $y=0$ in Theorem (2.4), the result reduces to known result of Kim et al [28.,p. 4780, Theorem (2.4)].

Corollary 4. For $n\geq 1$, we have

$$G_n^{(k)}(x+1)+G_n^{(k)}(x)=2\sum_{p=1}^{n}\sum_{l=1}^{p}\frac{(-1)^{l+p}}{l^k}l!S_2(p,l)\binom{n}{p}x^{n-p} \tag{2.16}$$

Theorem 2.5. For $d\varepsilon N$ with $d\equiv 1(mod 2)$, we have

$$_HG_n^{(k)}(x,y)=\sum_{p=0}^{n}\binom{n}{p}d^{n-p-1}\sum_{l=0}^{p+1}\sum_{a=0}^{d-1}\frac{(-1)^{l+p+1}l!S_2(p+1,l)}{l^k}(-1)^a{}_HG_{n-p}\left(\frac{a+x}{d},y\right) \tag{2.17}$$

Proof. From equation (2.1), we can be written as

$$\sum_{n=0}^{\infty}{}_HG_n^{(k)}(x,y)\frac{t^n}{n!}=\frac{2Li_k(1-e^{-t})}{e^t+1}e^{xt+yt^2}$$

$$=\left(\frac{2Li_k(1-e^{-t})}{t}\right)\left(\frac{2t}{e^{bt}+1}\sum_{a=0}^{d-1}(-1)^a e^{(a+x)t+yt^2}\right)$$

$$=\left(\sum_{p=0}^{\infty}\left(\sum_{l=1}^{p+1}\frac{(-1)^{l+p+1}}{l^k}l!\frac{S_2(p+1,l)}{p+1}\right)\frac{t^p}{p!}\right)\left(\sum_{m=0}^{\infty}d^{m-1}\sum_{a=0}^{d-1}(-1)^a{}_HG_n\left(\frac{a+x}{d},y\right)\frac{t^n}{n!}\right)$$

Replacing n by n-p in above equation and comparing the coefficient of t^n, we get the result (2.17).

Remark 5. For $y=0$ in Theorem (2.5), the result reduces to known result of Kim et al [28.,p. 4780].

Corollary 5. For $d\varepsilon N$ with $d\equiv 1(mod 2)$, we have

$$G_n^{(k)}(x)=\sum_{p=0}^{n}\binom{n}{p}d^{n-p-1}\sum_{l=0}^{p+1}\sum_{a=0}^{d-1}\frac{(-1)^{l+p+1}l!S_2(p+1,l)}{l^k}(-1)^aG_{n-p}\left(\frac{a+x}{d}\right) \tag{2.18}$$

3 Implicit summation formulae involving Hermite poly-Genocchi polynomials

For the derivation of implicit formulae involving poly-Genocchi polynomials $G_n^{(k)}(x)$ and Hermite poly-Genocchi polynomials $_HG_n^{(k)}(x,y)$ the same considerations as developed for the ordinary Hermite and related polynomials in Khan et al [24] and Hermite-Bernoulli polynomials in Pathan and Khan [30 to 36] holds as well. First we prove the following results involving Hermite poly-Genocchi polynomials $_HG_n^{(k)}(x,y)$.

Theorem 3.1. For $x,y\varepsilon R$ and $n\geq 0$, The following implicit summation formulae for Hermite poly-Genocchi polynomials $_HG_n^{(k)}(x,y)$ holds true:

$$_HG_{l+p}^{(k)}(z,y)=\sum_{m,n=0}^{l,p}\binom{l}{m}\binom{p}{n}(z-x)^{m+n}{}_HG_{l+p-m-n}^{(k)}(x,y) \tag{3.1}$$

Proof. We replace t by $t+u$ and rewrite the generating function (2.1) as

$$\frac{2Li_k(1-(e)^{-(t+u)})}{e^{t+u}+1}e^{y(t+u)^2}=e^{-x(t+u)}\sum_{l,p=0}^{\infty}{}_HG_{l+p}^{(k)}(x,y)\frac{t^l}{l!}\frac{u^p}{p!} \tag{3.2}$$

Replacing x by z in the above equation and equating the resulting equation to the above equation, we get

$$e^{(z-x)(t+u)}\sum_{m,l=0}^{\infty}{}_HG_{l+p}^{(k)}(x,y)\frac{t^l}{l!}\frac{u^p}{p!}=\sum_{l,p=0}^{\infty}{}_HG_{l+p}^{(k)}(z,y)\frac{t^l}{l!}\frac{u^p}{p!} \tag{3.3}$$

On expanding exponential function (3.3) gives

$$\sum_{N=0}^{\infty}\frac{[(z-x)(t+u)]^N}{N!}\sum_{l,p=0}^{\infty}{}_HG_{l+p}^{(k)}(x,y)\frac{t^l}{l!}\frac{u^p}{p!}=\sum_{l,p=0}^{\infty}{}_HG_{l+p}^{(k)}(z,y)\frac{t^l}{l!}\frac{u^p}{p!} \tag{3.4}$$

which on using formula [[37], p.52(2)]

$$\sum_{N=0}^{\infty}f(N)\frac{(x+y)^N}{N!}=\sum_{n,m=0}^{\infty}f(n+m)\frac{x^n}{n!}\frac{y^m}{m!} \tag{3.5}$$

in the left hand side becomes

$$\sum_{m,n=0}^{\infty}\frac{(z-x)^{m+n}t^mu^n}{m!n!}\sum_{l,p=0}^{\infty}{}_HG_{l+p}^{(k)}(x,y)\frac{t^l}{l!}\frac{u^p}{p!}=\sum_{l,p=0}^{\infty}{}_HG_{l+p}^{(k)}(z,y)\frac{t^l}{l!}\frac{u^p}{p!} \tag{3.6}$$

Now replacing l by l-m, p by p-n and using the lemma [[37], p.100(1)] in the left hand side of (3.6), we get

$$\sum_{m,n=0}^{\infty}\sum_{l,p=0}^{\infty}\frac{(z-x)^{m+n}}{m!n!}{}_HG_{l+p-m-n}^{(k)}(x,y)\frac{t^l}{(l-m)!}\frac{u^p}{(p-n)!}$$

$$=\sum_{l,p=0}^{\infty}{}_HG_{l+p}^{(k)}(z,y)\frac{t^l}{l!}\frac{u^p}{p!} \qquad (3.7)$$

Finally on equating the coefficients of the like powers of t and u in the above equation, we get the required result.

Remark 1. By taking $l = 0$ in equation (3.1), we immediately deduce the following result.

Corollary 3.1. The following implicit summation formula for Hermite poly-Genocchi polynomials ${}_HG_n^{(k)}(z,y)$ holds true:

$${}_HG_p^{(k)}(z,y)=\sum_{n=0}^{p}\binom{p}{n}(z-x)^n{}_HG_{p-n}^{(k)}(x,y) \qquad (3.8)$$

Remark 2. On replacing z by z+x and setting $y=0$ in Theorem (3.1), we get the following result involving poly-Genocchi polynomials of one variable

$$G_{l+p}^{(k)}(z+x)=\sum_{m,n=0}^{l,p}\binom{l}{m}\binom{p}{n}(z)^{m+n}G_{l+p-m-n}^{(k)}(x) \qquad (3.9)$$

whereas by setting z=0 in Theorem 3.1, we get another result involving poly-Genocchi polynomials of one and two variables

$$G_{l+p}^{(k)}(y)=\sum_{m,n=0}^{l,p}\binom{l}{m}\binom{p}{n}(-x)^{m+n}{}_HG_{l+p-m-n}^{(k)}(x,y) \qquad (3.10)$$

Remark 3. Along with the above results we will exploit extended forms of poly-Genocchi polynomials $G_{l+p}^{(k)}(z)$ by setting y=0 in the Theorem (3.1) to get

$$G_{l+p}^{(k)}(z)=\sum_{m,n=0}^{l,p}\binom{l}{m}\binom{p}{n}(z-x)^{n+m}G_{l+p-m-n}^{(k)}(x) \qquad (3.11)$$

Theorem 3.2. For $x,y\varepsilon R$ and $n\geq 0$. Then

$${}_HG_n^{(k)}(x+u,y)=\sum_{j=0}^{n}\binom{n}{j}u^j{}_HG_{n-j}^{(k)}(x,y) \qquad (3.14)$$

Proof. Since

$$\sum_{n=0}^{\infty}{}_HG_n^{(k)}(x+u,y)\frac{t^n}{n!}=\frac{Li_k(1-(e)^{-t})}{e^t+1}e^{(x+u)t+yt^2}=\left(\sum_{n=0}^{\infty}{}_HG_n^{(k)}(x,y)\frac{t^n}{n!}\right)\left(\sum_{j=0}^{\infty}u^j\frac{t^j}{j!}\right)$$

Now replacing n by n-j and comparing the coefficients of t^n, we get the result (3.14).

Theorem 3.3. For $x,y\varepsilon R$ and $n\geq 0$. Then

$${}_HG_n^{(k)}(x+u,y+w)=\sum_{m=0}^{n}\binom{n}{m}{}_HG_{n-m}^{(k)}(x,y)H_m(u,w) \qquad (3.15)$$

Proof. By the definition of poly-Genocchi polynomials and the definition (1.2), we have

$$\frac{Li_k(1-(e)^{-t})}{e^t+1}e^{(x+u)t+y(t+w)^2}=\left(\sum_{n=0}^{\infty}{}_HG_n^{(k)}(x,y)\frac{t^n}{n!}\right)\left(\sum_{m=0}^{\infty}H_m(u,w)\frac{t^m}{m!}\right)$$

Now replacing n by n-m and comparing the coefficients of t^n, we get the result (3.15).

Theorem 3.4. For $x,y\varepsilon R$ and $n\geq 0$. Then

$${}_HG_n^{(k)}(x,y)=\sum_{m=0}^{n-2j}\sum_{j=0}^{[\frac{n}{2}]}y^jx^{n-m-2j}G_m^{(k)}\frac{n!}{m!j!(n-2j-m)!} \qquad (3.16)$$

Proof. Applying the definition (2.1) to the term $\frac{Li_k(1-(e)^{-t})}{e^t+1}$ and expanding the exponential function e^{xt+yt^2} at $t=0$ yields

$$\frac{Li_k(1-(e)^{-t})}{e^t+1}e^{xt+yt^2}=\left(\sum_{m=0}^{\infty}G_m^{(k)}\frac{t^m}{m!}\right)\left(\sum_{n=0}^{\infty}x^n\frac{t^n}{n!}\right)\left(\sum_{j=0}^{\infty}y^j\frac{t^{2j}}{j!}\right)$$

$$=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}G_m^{(k)}x^{n-m}\right)\frac{t^n}{n!}\left(\sum_{j=0}^{\infty}y^j\frac{t^{2j}}{j!}\right)$$

Replacing n by n-2j, we have

$$\sum_{n=0}^{\infty}{}_HG_n^{(k)}(x,y)\frac{t^n}{n!}$$

$$=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n-2j}\sum_{j=0}^{[\frac{n}{2}]}\binom{n-2j}{m}G_m^{(k)}x^{n-m-2j}y^j\right)\frac{t^n}{(n-2j)!j!} \qquad (3.17)$$

Equating their coefficients of t^n, we get the result (3.16).

Theorem 3.5. For $x,y\varepsilon R$ and $n\geq 0$. Then

$${}_HG_n^{(k)}(x+1,y)=\sum_{j=0}^{[\frac{n}{2}]}\sum_{m=0}^{n-2j}\binom{n-2j}{m}y^jG_m^{(k)}(x) \qquad (3.18)$$

Proof. By the definition of Hermite poly-Genocchi polynomials, we have

$$\frac{Li_k(1-(e)^{-t})}{e^t+1}e^{(x+1)t+yt^2}=\sum_{n=0}^{\infty}{}_HG_n^{(k)}(x+1,y)\frac{t^n}{n!} \qquad (3.19)$$

$$=\left(\sum_{m=0}^{\infty}G_m^{(k)}(x)\frac{t^m}{m!}\right)\left(\sum_{n=0}^{\infty}\frac{t^n}{n!}\right)\left(\sum_{j=0}^{\infty}y^j\frac{t^{2j}}{j!}\right)$$

$$=\sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m}G_m^{(k)}(x)\frac{t^n}{n!}\left(\sum_{j=0}^{\infty}y^j\frac{t^{2j}}{j!}\right)=\sum_{n=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m}y^jG_m^{(k)}(x)\frac{t^{n+2j}}{n!j!}$$

Replacing n by n-2j, we have

$$\sum_{n=0}^{\infty} {}_HG_n^{(k)}(x+1,y)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\left(\sum_{j=0}^{[\frac{n}{2}]}\sum_{m=0}^{n-2j}\binom{n-2j}{m} y^j G_m^{(k)}(x)\right)\frac{t^n}{n!} \tag{3.20}$$

Combining (3.19) and (3.20) and equating their coefficients of t^n leads to formula (3.18).

Theorem 3.6. The following implicit summation formula involving Hermite poly-Genocchi polynomials ${}_HG_n^{(k)}(x,y)$ holds true:

$$ {}_HG_n^{(k)}(x+1,y) = \sum_{m=0}^{n}\binom{n}{m} {}_HG_{n-m}^{(k)}(x,y) \tag{3.21}$$

Proof. By the definition of Hermite poly-Genocchi polynomials, we have

$$\sum_{n=0}^{\infty} {}_HG_n^{(k)}(x+1,y)\frac{t^n}{n!} - \sum_{n=0}^{\infty} {}_HG_n^{(k)}(x,y)\frac{t^n}{n!} = \frac{2Li_k(1-e^{-t})}{e^t+1}e^{xt+yt^2}(e^t-1)$$

$$= \left(\sum_{n=0}^{\infty} {}_HG_n^{(k)}(x,y)\frac{t^n}{n!}\right)\left(\sum_{m=0}^{\infty}\frac{t^m}{m!}\right) - \sum_{n=0}^{\infty} {}_HG_n^{(k)}(x,y)\frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{n} {}_HG_{n-m}^{(k)}(x,y)\frac{t^n}{(n-m)!m!} - \sum_{n=0}^{\infty} {}_HG_n^{(k)}(x,y)\frac{t^n}{n!}$$

Finally, equating the coefficients of the like powers of t^n, we get (3.21).

Theorem 3.7. The following implicit summation formula involving Hermite poly-Genocchi polynomials ${}_HG_n^{(k)}(x,y)$ holds true:

$$ {}_HG_n^{(k)}(-x,y) = (-1)^n {}_HG_n^{(k)}(x,y) \tag{3.22}$$

Proof. We replace t by -t in (2.1) and then subtract the result from (2.1) itself finding

$$e^{yt^2}\left[\frac{2Li_k(1-e^{-t})}{e^t+1}(e^{xt}-e^{-xt})\right] = \sum_{n=0}^{\infty}[1-(-1)^n]{}_HG_n^{(k)}(x,y)\frac{t^n}{n!}$$

which is equivalent to

$$\sum_{n=0}^{\infty} {}_HG_n^{(k)}(x,y)\frac{t^n}{n!} - \sum_{n=0}^{\infty} {}_HG_n^{(k)}(-x,y)\frac{t^n}{n!} = \sum_{n=0}^{\infty}[1-(-1)^n]{}_HG_n^{(k)}(x,y)\frac{t^n}{n!}$$

$$\sum_{n=0}^{\infty} {}_HG_n^{(k)}(x,y)\frac{t^n}{n!} - {}_HG_{n-m}^{(k)}(-x,y)\frac{t^n}{n!} = \sum_{n=0}^{\infty}[1-(-1)^n]{}_HG_n^{(k)}(x,y)\frac{t^n}{n!}$$

and thus by equating coefficients of like powers of t^n, we get (3.22).

4 General symmetry identities for Hermite poly-Genocchi polynomials

In this section, we give general symmetry identities for the poly-Genocchi polynomials $G_n^{(k)}(x)$ and the Hermite poly-Genocchi polynomials ${}_HG_n^{(k)}(x,y)$ by applying the generating function(2.1) and (2.4). The results extend some known identities of Zhang and Yang [43], Yang [42,Eqs.(9)], Khan [26,27] and Pathan and Pathan et al [[30] to [35]].

Theorem 4.1. Let $a,b>0$ and $a\neq b$. For $x,y\varepsilon R$ and $n\geq 0$. Then the following identity holds true:

$$\sum_{m=0}^{n}\binom{n}{m} b^m a^{n-m} {}_HG_{n-m}^{(k)}(bx,b^2y){}_HG_m^{(k)}(ax,a^2y)$$

$$= \sum_{m=0}^{n}\binom{n}{m} a^m b^{n-m} {}_HG_{n-m}^{(k)}(ax,a^2y){}_HG_m^{(k)}(bx,b^2y) \tag{4.1}$$

Proof. Start with

$$g(t) = \left(\frac{(2Li_k(1-e^{-t}))^2}{(e^{at}+1)(e^{bt}+1)}\right)e^{abxt+a^2b^2yt^2} \tag{4.2}$$

Then the expression for g(t) is symmetric in a and b and we can expand g(t) into series in two ways to obtain

$$g(t) = \frac{1}{ab}\sum_{n=0}^{\infty} {}_HG_n^{(k)}(bx,b^2y)\frac{(at)^n}{n!}\sum_{m=0}^{\infty} {}_HG_m^{(k)}(ax,a^2y)\frac{(bt)^m}{m!}$$

$$= \frac{1}{ab}\sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m} a^{n-m}b^m {}_HG_{n-m}^{(k)}(bx,b^2y){}_HG_m^{(k)}(ax,a^2y)t^n$$

On the similar lines we can show that

$$g(t) = \frac{1}{ab}\sum_{n=0}^{\infty} {}_HG_n^{(k)}(ax,a^2y)\frac{(bt)^n}{n!}\sum_{m=0}^{\infty} {}_HG_m^{(k)}(bx,b^2y)\frac{(at)^m}{m!}$$

$$= \frac{1}{ab}\sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m} a^m b^{n-m} {}_HG_{n-m}^{(k)}(ax,a^2y){}_HG_m^{(k)}(bx,b^2y)t^n$$

Comparing the coefficients of t^n on the right hand sides of the last two equations we arrive the desired result.

Remark 1. By setting $b = 1$ in Theorem 4.1, we immediately following result

$$\sum_{m=0}^{n}\binom{n}{m} a^{n-m} {}_HG_{n-m}^{(k)}(x,y){}_HG_m^{(k)}(ax,a^2y)$$

$$= \sum_{m=0}^{n}\binom{n}{m} a^m {}_HG_{n-m}^{(k)}(ax,a^2y){}_HG_m^{(k)}(x,y) \tag{4.3}$$

Theorem 4.2. Let $a,b>0$ and $a\neq b$. For $x,y\varepsilon R$ and $n\geq 0$. Then the following identity holds true:

$$\sum_{m=0}^{n}\binom{n}{m}\sum_{i=0}^{a-1}\sum_{j=0}^{b-1} {}_HG_{n-m}^{(k)}\left(bx+\frac{b}{a}i+j,b^2z\right)G_m^{(k)}(ay)b^m a^{n-m}$$

$$= \sum_{m=0}^{n}\binom{n}{m}\sum_{i=0}^{b-1}\sum_{j=0}^{a-1} {}_HG_{n-m}^{(k)}\left(ax+\frac{a}{b}i+j,a^2z\right)G_m^{(k)}(by)a^m b^{n-m} \tag{4.4}$$

Proof. Let

$$g(t)=\left(\frac{(2Li_k(1-e^{-t}))^2}{(e^{at}+1)(e^{bt}+1)}\right)\frac{(e^{abt}-1)^2e^{ab(x+y)t+a^2b^2zt^2}}{(e^{at}-1)(e^{bt}-1)}$$

$$g(t)=\left(\frac{2Li_k(1-e^{-t})}{(e^{at}+1}\right)e^{abxt+a^2b^2zt^2}\left(\frac{e^{abt}-1}{e^{bt}-1}\right)\left(\frac{2Li_k(1-e^{-t})}{e^{bt}+1}\right)e^{abyt}\left(\frac{e^{abt}-1}{e^{at}-1}\right)$$

$$=\left(\frac{2Li_k(1-e^{-t})}{(e^{at}+1}\right)e^{abxt+a^2b^2zt^2}\sum_{i=0}^{a-1}e^{bti}\left(\frac{2Li_k(1-e^{-t})}{e^{bt}+1}\right)e^{abyt}\sum_{j=0}^{b-1}e^{atj} \tag{4.5}$$

$$=\left(\frac{2Li_k(1-e^{-t})}{e^{at}+1}\right)e^{a^2b^2zt^2}\sum_{i=0}^{a-1}\sum_{j=0}^{b-1}e^{(bx+\frac{b}{a}i+j)at}\sum_{m=0}^{\infty}G_m^{(k)}(ay)\frac{(bt)^m}{m!}$$

$$=\frac{1}{ab}\sum_{n=0}^{\infty}\sum_{i=0}^{a-1}\sum_{j=0}^{b-1}{}_HG_n^{(k)}\left(bx+\frac{b}{a}i+j,b^2z\right)\frac{(at)^n}{n!}\sum_{m=0}^{\infty}G_m^{(k)}(ay)\frac{(bt)^m}{(m)!}$$

$$=\frac{1}{ab}\sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m}\sum_{i=0}^{a-1}\sum_{j=0}^{b-1}{}_HG_{n-m}^{(k)}\left(bx+\frac{b}{a}i+j,b^2z\right)G_m^{(k)}(ay)b^ma^{n-m}t^n \tag{4.6}$$

On the other hand

$$g(t)=\frac{1}{ab}\sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m}\sum_{i=0}^{b-1}\sum_{j=0}^{a-1}{}_HG_{n-m}^{(k)}\left(ax+\frac{a}{b}i+j,a^2z\right)G_m^{(k)}(by)a^mb^{n-m}t^n \tag{4.7}$$

By comparing the coefficients of t^n on the right hand sides of the last two equations, we arrive at the desired result.

5 Conclusion

Based on the definition of Hermite polynomials and polylogarithmic function, we introduced a new class of Hermite poly-Genocchi polynomials. By using Jolany's methods introduced in [20] and [21], we gave Hermite poly-Genocchi polynomials with two variable, and also we analysed its behaviours including general symmeric properties.

References

[1] Andrews, L.C: Special functions for engineers and mathematicians, Macmillan.Co.New York, 1985.

[2] Apostol, T.M: On the Lerch zeta function, Pacific J.Math. 1(1951), 161-167.

[3] Araci, S: Novel identities for q-Genocchi numbers and polynomials, J. Funct. Spaces Appl. 2012 (2012) 13p (Article ID 214961).

[4] Araci, S: Novel identities involving Genocchi numbers and polynomials arising from applications of umbral calculus, Appl.Math. and Comput. 233(2014), 599-607.

[5] Araci, S, M. Acikgoz, M, Seo, J.J: Explicit formulas involving q-Euler numbers and polynomials, Abstr. Appl. Anal. 2012 (2012) 11p (Article ID 298531).

[6] Araci, S, Erdal, D and Seo, J.J: A study on the fermionic p-adic q-integral representation on Zp associated with weighted q-Bernstein and q-Genocchi polynomials, Abstr. Appl. Anal. 2011 (2011) 10p (Article ID 649248).

[7] Araci, S, Acikgoz, M, Jolany, H, Seo, J.J: A unified generating function of the q-Genocchi polynomials with their interpolation functions, Proc. Jangjeon Math. Soc. 15 (2) (2012), 227-233.

[8] Araci, Sen, E, Acikgoz, M: A note on the modified q-Dedekind sums, Notes Number Theory Discrete Math. 19 (3) (2013), 60-65.

[9] Araci, A, Acikgoz, S M, Bagdasaryan, A, Sen, E: The Legendre polynomials associated with Bernoulli, Euler, Hermite and Bernstein polynomials, Turkish J.Anal. Number Theory (1) (2013) 13. http://dx.doi.org/10.12691/tjant-1-1-1.

[10] Acikgoz, M, Araci, S, Cangul, I.N: A note on the modified q-Bernstein polynomials for functions of several variables and their reflections on q-Volkenborn integration, Appl. Math. Comput. 218 (3) (2011), 707-712.

[11] Araci, Sen, E, Acikgoz, M: Theorems on Genocchi polynomials of higher order arising from Genocchi basis, Taiwanese.J.Math.18(2014), 473-482.

[12] Araci, Acikgoz, M, Sen, E: On the von Staudt-Clausen's theorem associated with q-Genocchi numbers, Appl.Math. and Comput. 247(2014), 780-785.

[13] Bell, E.T: Exponential polynomials, Ann. of Math. 35(1934), 258-277.

[14] Cangul, I.N, Kurt, V, Ozden, H, Simsek, Y: On the higher-order w-q-Genocchi numbers, Adv. Stud. Contemp. Math. 19 (1) (2009), 39-57.

[15] Dattoli, G, Lorenzutta, S and Cesarano, C: Finite sums and generalized forms of Bernoulli polynomials Rendiconti di Mathematica, 19(1999), 385-391.

[16] Dere, R, Simsek, Y: Genocchi polynomials associated with the Umbral algebra, Appl. Math. Comput. 218 (3) (2011), 756-761.

[17] He, Y, Araci, S, Srivastava, H.M, Acikgoz, M: Some new identities for the Apostol-Bernoulli polynomials and Apostol-Genocchi polynomials, Appl.Math. and Comput. 262(2015), 31-41.

[18] Hamahata, Y, Masubuch, H: Special Multi-Poly- Bernoulli numbers, Journal of Integer sequences,10 (2007), 1-6.

[19] Hamahata, Y, Masubuchi, H: Recurrence formulae for Multi-poly-Bernoulli numbers, Elec. J.Comb.Num.Theo. 7(2007), A-46.

[20] Jolany, H, Darafsheh, M.R, Alikelaye,R.E: Generalizations of Poly-Bernoulli Numbers and Polynomials, Int. J. Math. Comb. 2 (2010), A07-14.

[21] Jolany, H, Corcino, R.B: Explicit formula for generalization of Poly-Bernoulli numbers and polynomials with a,b,c parameters, Journal of Classical Analysis, 6(2015), 119-135.

[22] Jolany, H, Aliabadi, M, Corcino, R. B and Darafsheh, M.R: A Note on Multi Poly-Euler Numbers and Bernoulli Polynomials, General Mathematics, 20(2-3) (2012), 122-134.

[23] Jolany, H and Corcino, R.B: More properties on Multi-Euler polynomials, arXiv;1401.627IvI[math NT] 24 Jan 2014.

[24] Khan, S, Pathan, M.A, Hassan, Nader Ali Makhboul , Yasmin, G: Implicit summation formula for Hermite and related polynomials,J.Math.Anal.Appl. 344(2008), 408-416.

[25] Kaneko, M: Poly-Bernoulli numbers, J.de Theorie de Nombres 9 (1997), 221-228.

[26] Khan W.A: Some properties of the generalized Apostol type Hermite-Based polynomials, Kyungpook Math. J. 55(2015), 597-614.
[27] Khan W.A: A note on Hermite-based poly-Euler and multi poly-Euler polynomials, Palestine J. Math. 2015. In Reprint.
[28] Kim, T, Jang, Y.S and Seo, J.J: A note on poly-Genocchi numbers and polynomials, Appl.Math.Sci. 8(2014), 4475-4781.
[29] Kim, T: Some identities for the Bernoulli, the Euler and the Genocchi numbers and polynomials, Adv. Stud. Contemp. Math. 20 (1) (2010), 23-28.
[30] Pathan, M.A and Khan, W.A: Some implicit summation formulas and symmetric identities for the generalized Hermite based- polynomials, Acta Universitatis Apulensis, 39(2014), 113-136.
[31] Pathan, M.A and Khan, W.A: Some implicit summation formulas and symmetric identities for the generalized Hermite-Bernoulli polynomials, Mediterr. J. Math. 12 (2015), 679-695.
[32] Pathan, M.A and Khan, W.A: A new class of generalized polynomials associated with Hermite and Euler polynomials , To appear in Mediterr. J. Math. DOI 10.1007/s00009-015-0551-1, Springer Basel 2015.
[33] Pathan, M.A and Khan, W.A:Some implicit summation formulas and symmetric identities for the generalized Hermite-Euler polynomials,East-West J.Maths. 16(1) (2014), 92-109.
[34] Pathan, M.A and Khan, W.A: A new class of generalized polynomials associated with Hermite and Bernoulli polynomials, LE MATEMATICHE, Vol. LXX (2015), 5370
[35] Pathan, M.A and Khan, W.A: Some new classes of generalized Hermite-based Apostol-Euler and Apostol-Genocchi polynomials, Fasciculli.Math. Vol.55 (2015), In Press.
[36] Sandor, J and Crisci: Handbook of Number Theory, Vol.II. Kluwer Academic Publishers, Dordrecht Boston and London, 2004.
[37] Srivastava, H.M and Manocha, H.L: A treatise on generating functions Ellis Horwood Limited, New York, 1984.
[38] Srivastava, H.M and Choi, J: Series associated with the Zeta and related functions, Kluwer Academic Publishers, Dordrecht, Boston and London, 2001.
[39] Srivastava, H.M : Some generalizations and basic (or q-) extensions of the Bernoulli, Euler and Genocchi polynomials, Appl. Math. Inf. Sci. 5 (2011), 390-444.
[40] Srivastava, H.M, Kurt, B, Simsek, B: Some families of Genocchi type polynomials and their interpolation functions, Integral Transforms Spec. Funct. 23(2012) 919938. see also Corrigendum, Integral Transforms Spec. Funct. 23 (2012), 939-940.
[41] Srivastava, H.M and Pinter, A: Remarks on some relationships between the Bernoulli and Euler polynomials, Appl.Math.Lett. 17(2004), 375-380.
[42] Yang, S: An identity of symmetry for the Bernoulli polynomials, Discrete Math. Vol. 308 (2008), 550-554.
[43] Zhang, Z and Yang, H: Several identities for the generalized Apostol-Bernoulli polynomials, Computers and Mathematics with Applications, 56(2008), 2993-2999.

Waseem A. Khan has received M.Phil and Ph.D Degree in 2008 and 2011 from Department of Applied Mathematics, Aligarh Muslim University, Aligarh, India. He is an Assistant Professor in the Department of Mathematics, Integral University, Lucknow India. He has published more than 17 research papers in referred National and International journals. He has also attended and delivered talks in many National and International Conferences, Symposiums. He is a life member of Society for Special functions and their Applications (SSFA).