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On the oscillation of higher–order half–linear delay difference equations

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Abstract: In this paper, sufficient conditions are established for the oscillatory and asymptotic behavior of higher–order half–linear delay difference equation of the form

$$\Delta(p_n(\Delta^{m-1}(x_n + q_n x_{\tau_n}))^{\alpha}) + r_n x^{\beta}_{\sigma_n} = 0, \; n \geq n_0,$$

where it is assumed that $\sum_{s=n_0}^{\infty} 1/p_s^{1/\alpha} < \infty$. The main theorem improves some existing results in the literature. An example is provided to demonstrate the effectiveness of the main result.

Keywords: Oscillation; Delay difference equation; Higher–order half–linear difference equation.

1. Introduction

Due to its numerous applications in fields such as economics and mathematical biology, the oscillation theory of difference equations has been receiving intensive attention in the last few decades; we refer the reader to the monographs [1–3] and the references cited therein. In particular, the study of oscillatory and asymptotic behavior of second and third order difference equations has occupied a great part of interest among researchers [4–14]. Although it is considered as natural generalization, higher–order difference equations has received considerably less attention [15–20].

In view of the above quoted papers, one can conclude that most of their results have investigated various forms of the following difference equation

$$\Delta(p_n(\Delta^{m-1}(x_n)^{\alpha}) + r_n f(x_{\sigma_n}) = 0, \;\; n \geq n_0, \qquad (1)$$

where $m \geq 2$ and under the assumptions

$$\sum_{s=n_0}^{\infty} \frac{1}{p_s^{1/\alpha}} = \infty \;\text{ and }\; \triangle p_n \geq 0.$$

The purpose of this paper is to relax these conditions and derive some oscillation and asymptotic criteria for higher–order half–linear delay difference equation of the form

$$\Delta(p_n(\Delta^{m-1}(x_n+q_n x_{\tau_n}))^{\alpha})+r_n x^{\beta}_{\sigma_n} = 0, \; n \geq n_0, \; (2)$$

where

$$\sum_{s=n_0}^{\infty} \frac{1}{p_s^{1/\alpha}} < \infty \qquad (3)$$

and without using that $\triangle p_n \geq 0$

Throughout the paper, we assume that α, β are the ratio of odd positive integers, $\beta \leq \alpha$, $p_n > 0$ for $n \geq n_0$, q_n is an oscillating sequence satisfying $\lim\limits_{n\to\infty} q_n = 0$, $r_n > 0$ with $\Delta r_n > 0$ for $n \geq n_0$, $\tau_n < n$ with $\tau_n \to +\infty$ as $n \to +\infty$ and $\sigma_n < n$ with $\sigma_n \to +\infty$ as $n \to +\infty$.

Let $\mathbb{Z}$ and $\mathbb{R}$ be the sets of integer and real numbers, respectively. By a solution of equation (2), we mean a nontrivial sequence $x_n : \mathbb{Z} \to \mathbb{R}$ which is defined for all $n \geq \min\limits_{i\geq 0}\{\tau_i, \sigma_i\}$ and satisfies equation (2) for sufficiently large n. We restrict our attention to those solutions of (2) which satisfy $\sup\{|x_n| : n \geq N\}$ for all $N \geq N_x$. For our

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purpose, we assume that equation (2) possesses such a solution. A solution of (2) is called oscillatory if it is neither eventually positive nor negative and otherwise it is called non–oscillatory.

2. Main results

To obtain our main results, we need the following lemmas. The first of these is the discrete analog of the well–known Kiguradze's lemma.

Lemma 1.[1] *Let x_n be defined for $n \geq n_0 \in$, and $x_n > 0$ with $\Delta^m x_n$ of constant sign for $n \geq n_0$ and not identically zero. Then, there exists an integer k, $0 \leq k \leq m$ with $(m+k)$ odd for $\Delta^m x_n \leq 0$ and $(m+k)$ even for $\Delta^m x_n \geq 0$ such that*

(i) $k \leq m-1$ *implies* $(-1)^{m+i}\Delta^i x_n > 0$ *for all* $n \geq n_0, k \leq i \leq m-1$,
(ii) $k \geq 1$ *implies* $\Delta^i x_n > 0$ *for all large* $n \geq n_0$, $1 \leq i \leq k-1$.

Lemma 2.[1] *Let x_n be defined for $n \geq n_0$, and $x_n > 0$ with $\Delta^m x_n \leq 0$ for $n \geq n_0$ and not identically zero. Then, there exists a large integer $n_1 \geq n_0$ such that*

$$x_n \geq \frac{1}{(m-1)!}(n-n_1)^{m-1}\Delta^{m-1}x_{2^{m-k-1}n}, \quad n \geq n_1,$$

where k is defined as in Lemma 1. Further, if x_n is increasing, then

$$x_n \geq \frac{1}{(m-1)!}\left(\frac{n}{2^{m-1}}\right)^{m-1}\Delta^{m-1}x_n, \quad n \geq 2^{m-1}n_1.$$

For the sake of convenience, the function z is defined as

$$z_n = x_n + q_n x_{\tau_n}. \tag{4}$$

Theorem 1. *Let $m \geq 2$. Assume that (3) is satisfied. Further, assume that the difference equation*

$$\Delta y_n + r_n\left(\frac{\sigma_n^{m-1}}{(m-1)!p_{\sigma_n}^{1/\alpha}}\right)^{\beta} y_{\sigma_n}^{\beta/\alpha} = 0 \tag{5}$$

is oscillatory. If

$$\limsup_{n\to\infty}\sum_{s=n_0}^{n-1}\left[M^{\beta-\alpha}r_s\frac{2^{(4-2m)\beta}\sigma_s^{\beta(m-2)}}{2\big((m-2)!\big)^{\beta}}\delta_s^{\alpha}+\frac{\Delta\delta_s^{\alpha}}{\delta_s^{\alpha}}\right]=\infty \tag{6}$$

holds for every constant $M > 0$ where $\delta_s := \sum_{s=n_0}^{\infty}\frac{1}{p_s^{1/\alpha}}$, then every solution of equation (2) either oscillates or tends to zero.

Proof. Assume, on the contrary, that equation (2) has a bounded non–oscillatory solution x_n. Without loss of generality, we assume that x_n is eventually positive (the proof is similar when x_n is eventually negative). That is, $x_n > 0$, $x_{\tau_n} > 0$ and $x_{\sigma_n} > 0$ for all $n \geq n_1 \geq n_0$. Further, suppose that x_n does not tend to zero as $n \to \infty$. By (2) and (4), we have

$$\Delta(p_n\Delta^{m-1}z_n)^{\alpha} = -r_n x_{\sigma_n}^{\beta} \leq 0, \quad n \geq n_1. \tag{7}$$

Since x_n is bounded and does not tend to zero as $n \to \infty$, we have $\lim_{n\to\infty} q_n x_{\tau_n} = 0$. Then, we can find a $n_2 \geq n_1$ such that $z_n = x_n + q_n x_{\tau_n} > 0$ eventually and z_n is also bounded for sufficiently large $n \geq n_2$. In virtue of Lemma 1, it follows from equation (2) that there exist two possible cases:

(i) $z_n > 0,\ \Delta^{m-1}z_n > 0,\ \Delta^m z_n < 0$, and $\Delta(p_n(\Delta^{m-1}z_n)^{\alpha}) < 0$,
(ii) $z_n > 0,\ \Delta^{m-2}z_n > 0,\ \ \Delta^{m-1}z_n < 0$, and $\Delta(p_n(\Delta^{m-1}z_n)^{\alpha}) < 0$,

for all $n \geq n_2$. Then there exists a large enough $n_3 \geq n_2$ so that

$$x_n = z_n - q_n x_{\tau_n} \geq \frac{1}{2}z_n > 0$$

for all $n \geq n_3$. We may find a $n_4 \geq n_3$ such that for $n \geq n_4$ we have

$$x_{\sigma_n} \geq \frac{1}{2}z_{\sigma_n} > 0. \tag{8}$$

In view of (7) and (8), we obtain

$$\Delta(p_n\Delta^{m-1}z_n)^{\alpha} + \frac{1}{2}r_n z_{\sigma_n}^{\beta} \leq 0 \tag{9}$$

for $n \geq n_4$.

Assume that case (i) holds. From Lemma 2, we have

$$y_n \geq \frac{1}{(m-1)!p_n^{1/\alpha}}\left(\frac{n}{2^{m-1}}\right)^{m-1}(p_n^{1/\alpha}\Delta^{m-1}y_n), \tag{10}$$

where $n \geq n_5 = 2^{m-1}n_4$. Hence by (2), we see that $y_n := p_n(\Delta^{m-1}y_n)^{\alpha}$ is a positive solution of the difference inequality

$$\Delta y_n + r_n\left(\frac{\sigma_n^{m-1}}{(m-1)!p_{\sigma_n}^{1/\alpha}}\right)^{\beta} y_{\sigma_n}^{\beta/\alpha} \leq 0, \ n \geq n_5.$$

Therefore, by Lemma 5 of Section 2 in [15], the difference equation

$$\Delta y_n + r_n\left(\frac{\sigma_n^{m-1}}{(m-1)!p_{\sigma_n}^{1/\alpha}}\right)^{\beta} y_{\sigma_n}^{\beta/\alpha} = 0$$

has an eventually positive solution for $n \geq n_5$. This contradicts the fact that (5) is oscillatory.

Assume that case (ii) holds. Define the function w by

$$w_n := \frac{p_n(\Delta^{m-1}z_n)^{\alpha}}{(\Delta^{m-2}z_n)^{\alpha}}, \quad n \geq n_1. \tag{11}$$

One can easily figure out that $w_n < 0$ for $n \geq n_1$. Taking into consideration that $p_n(\Delta^{m-1}z_n)^\alpha$ is decreasing, we have

$$p_s^{1/\alpha}\Delta^{m-1}z_s \leq p_n^{1/\alpha}\Delta^{m-1}z_n, \quad s \geq n \geq n_1.$$

Dividing the above inequality by $p_s^{1/\alpha}$ and summing up from n to $l-1$, we obtain

$$\Delta^{m-2}z_l \leq \Delta^{m-2}z_n + p_n^{1/\alpha}\Delta^{m-1}z_n \sum_{s=n}^{l-1} \frac{1}{p_s^{1/\alpha}}.$$

Letting $l \to \infty$, we have

$$0 \leq \Delta^{m-2}z_n + p_n^{1/\alpha}\Delta^{m-1}z_n\delta_n,$$

which yields

$$-\frac{p_n^{1/\alpha}\Delta^{m-1}z_n}{\Delta^{m-2}z_n}\delta_n \leq 1.$$

Thus, by (11) we obtain

$$-\delta_n^\alpha w_n \leq 1. \tag{12}$$

In view of (11), we have

$$\begin{aligned}
\Delta w_n &= \frac{\Delta(p_n(\Delta^{m-1}z_n)^\alpha) - p_n(\Delta^{m-1}z_n)^\alpha\Delta(\Delta^{m-2}z_n)^\alpha}{(\Delta^{m-2}z_n)^\alpha(\Delta^{m-2}z_{n+1})^\alpha} \\
&= \frac{\Delta(p_n(\Delta^{m-1}z_n)^\alpha)}{(\Delta^{m-2}z_{n+1})^\alpha} \\
&\quad - \frac{p_n(\Delta^{m-1}z_n)^\alpha((\Delta^{m-2}z_{n+1})^\alpha - (\Delta^{m-2}z_n)^\alpha)}{(\Delta^{m-2}z_n)^\alpha(\Delta^{m-2}z_{n+1})^\alpha} \\
&= \frac{\Delta(p_n(\Delta^{m-1}z_n)^\alpha)}{(\Delta^{m-2}z_{n+1})^\alpha} \\
&\quad - \frac{p_n(\Delta^{m-1}z_n)^\alpha}{(\Delta^{m-2}z_n)^\alpha} + \frac{p_n(\Delta^{m-1}z_n)^\alpha(\Delta^{m-2}z_n)^\alpha}{(\Delta^{m-2}z_n)^\alpha(\Delta^{m-2}z_{n+1})^\alpha} \\
&\leq -\frac{1}{2}r_n\frac{z_{\sigma_n}^\beta}{(\Delta^{m-2}z_{n+1})^\alpha} - w_n \\
&\quad + w_n\frac{(\Delta^{m-2}z_n)^\alpha}{(\Delta^{m-2}z_{n+1})^\alpha}.
\end{aligned} \tag{13}$$

We observe that since $\Delta^{m-1}z_n < 0$, we deduce that $\Delta^{m-2}z_n$ is decreasing. Therefore $\Delta^{m-2}z_n \geq \Delta^{m-2}z_{n+1} > 0$ and $w_n\frac{(\Delta^{m-2}z_n)^\alpha}{(\Delta^{m-2}z_{n+1})^\alpha} \leq w_n$ for all $n \geq n_1$.

Hence, (13) becomes

$$\Delta w_n \leq -\frac{1}{2}r_n\frac{z_{\sigma_n}^\beta}{(\Delta^{m-2}z_n)^\alpha}.$$

On the other hand, by Lemma 2 we get

$$x_n \geq \frac{1}{(m-2)!}\frac{n^{m-2}}{2^{2m-4}}\Delta^{m-2}x_n, \; n \geq n_2 = 2^{m-2}n_1.$$

Thus, we have

$$z_{\sigma_n} \geq \frac{2^{4-2m}}{(m-2)!}\sigma_n^{m-2}\Delta^{m-2}z_{\sigma_n},$$

for sufficiently large $n \geq n_3 \geq n_2$. Then, there exists a constant $M > 0$ such that

$$\begin{aligned}
\Delta w_n &\leq -\frac{1}{2}r_n\left(\frac{2^{4-2m}}{(m-2)!}\sigma_n^{m-2}\right)^\beta \frac{(\Delta^{m-2}z_{\sigma_n})^\beta}{(\Delta^{m-2}z_n)^\alpha} \\
&\leq -\frac{1}{2}r_n\left(\frac{2^{4-2m}}{(m-2)!}\sigma_n^{m-2}\right)^\beta M^{\beta-\alpha}, \; n \geq n_3.
\end{aligned}$$

Multiplying the above inequality by δ_n^α and summing up from n_3 to $n-1$, we obtain

$$\begin{aligned}
&\delta_n^\alpha w_n - \delta_{n_3}^\alpha w_{n_3} - \sum_{s=n_3}^{n-1} w_s\Delta\delta_s^\alpha \\
&\quad + \sum_{s=n_3}^{n-1} M^{\beta-\alpha}r_s\frac{2^{(4-2m)\beta}\sigma_s^{\beta(m-2)}}{2((m-2)!)^\beta}\delta_s^\alpha \leq 0.
\end{aligned} \tag{14}$$

From (14), we have

$$\sum_{s=n_3}^{n-1}\left(M^{\beta-\alpha}r_s\frac{2^{(4-2m)\beta}\sigma_s^{\beta(m-2)}}{2\big((m-2)!\big)^\beta}\delta_s^\alpha + \frac{\Delta\delta_s^\alpha}{\delta_s^\alpha}\right) \leq \delta_{n_3}^\alpha w_{n_3} + 1.$$

By using (11) and the fact that $\Delta\delta_s^\alpha < 0$, we arrive at a contradiction to (6). This completes the proof.

Corollary 1. *Let $m \geq 2$. Assume that (3) is satisfied. Further, assume that $\alpha = \beta$. If*

$$\liminf_{n\to\infty}\sum_{s=\sigma_n}^{n-1} r_s\frac{(\sigma_s^{m-1})^\alpha}{p_{\sigma_s}} > \frac{\big((m-1)!\big)^\alpha}{e} \tag{15}$$

and

$$\limsup_{n\to\infty}\sum_{s=n_0}^{n-1}\left[r_s\frac{2^{(4-2m)\beta}\sigma_s^{\beta(m-2)}}{2\big((m-2)!\big)^\beta}\delta_s^\alpha + \frac{\Delta\delta_s^\alpha}{\delta_s^\alpha}\right] = \infty \tag{16}$$

hold, then every solution of equation (2) either oscillates or tends to zero.

Corollary 2. *Let $m \geq 2$. Assume that (3) is satisfied. Further, assume that $\alpha > \beta$, σ_n is a strictly increasing sequence and*

$$\limsup_{n\to\infty}\sum_{s=\sigma_n}^{n-1} r_s\frac{(\tau_s^{m-1})^\beta}{(p_{\tau_s})^{\frac{\beta}{\alpha}}} > 0.$$

If (6) holds for every constant $M > 0$, then every solution of equation (2) either oscillates or tends to zero.

Remark. Let $m = 3$ and $\alpha = \beta$, then equation (2) reduces to equation (1.1) studied in [14]. Let m be even number, $p_n = 1$ and $\alpha = \beta = 1$, then (2) reduces to equation (1) studied in [20].

Example 1. Consider the fourth order delay difference equation

$$\Delta\big(e^n\Delta^3(x_n+\frac{1}{n}x_{n-2})\big)+(n+1)e^{n-1}x_{n-1}=0,\, n\geq 3, \tag{17}$$

where $m=4$, $\alpha=\beta=1$, $p_n=e^n$, $q_n=\frac{1}{n}$, $r_n=(n+1)e^{n-1}$, $\tau_n=n-2$, $\sigma_n=n-1$ and $n_0=3$. Then, one can easily see that the assumptions on equation (2) are satisfied. Moreover, $\sum_{s=3}^{\infty}\frac{1}{e^s}=\frac{1}{e^2(e-1)}$ and thus condition (3) holds as well. It remains to check conditions (15) and (16) of Corollary 1. We observe that

$$\liminf_{n\to\infty}\sum_{s=n-1}^{n-1}(s+1)e^{s-1}\frac{(s-1)^3}{e^{s-1}}=\liminf_{n\to\infty}n(n-2)^3>\frac{6}{e}$$

and

$$\limsup_{n\to\infty}\sum_{s=3}^{n-1}\big[\frac{(s+1)(s-1)e^{s-1}}{64e^2(e-1)}+\frac{1-e}{e}\big]=\infty.$$

Therefore, every solution of equation (17) either oscillates or tend to zero.

Concluding remark

In this paper, we have studied higher–order half–linear delay difference equation of the form (2) by establishing new sufficient conditions to show that every solution of this equation either oscillates or tends to zero. To the best of authors' observation, most existing results in the literature regarding second, third and higher order equations have been obtained under the assumptions $\sum_{s=n_0}^{\infty}1/p_s^{1/\alpha}=\infty$, $\triangle p_n\geq 0$ and $\alpha=\beta$; see in particular [13, 14, 18, 20]. In this paper, however, one can easily see that these assumptions have been bypassed and new results have been established. Therefore, the main theorem of this paper improves some previously obtained results and thus presents a new approach.

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