Sufficient Condition Starlikeness and Convexity of Integral Operators Related to Multivalent Functions

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1 Introduction

Let $A_p(n)$ denote the class of all functions of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \quad (p, n \in \{1, 2, 3, \ldots \}).$$

which is analytic in open unit disc $U = \{z \in \mathbb{C} | |z| < 1 \}$. In particular, we set

$$A_p(1) = A_p, A_1(1) = A_1 := A.$$

If $f \in A_p(n)$ is given by (1) and $g \in A_p(n)$ is given by

$$g(z) = z^p + \sum_{k=p+1}^{\infty} b_k z^k \quad (p, n \in \{1, 2, 3, \ldots \}),$$

then the Hadamard product (or convolution) $f \ast g$ of $f$ and $g$ is given by

$$(f \ast g)(z) = z^p + \sum_{k=p+1}^{\infty} a_k b_k z^k = (g \ast f)(z).$$

We observe that several known operators are deducible from the convolutions. That is, for various choices of $g$ in (3), we obtain some interesting operators. For example, for functions $f \in A_p(n)$ and the function $g$ is defined by

$$g(z) = z^p + \sum_{k=p+1}^{\infty} \psi_{k,m}(\alpha, \lambda, l, p) z^k \quad (m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\})$$

where

$$\psi_{k,m}(\alpha, \lambda, l, p) = \left[ \frac{\Gamma(k+1)\Gamma(p-\alpha+1)}{\Gamma(p+1)\Gamma(k-\alpha+1)} \right]^m \frac{\Gamma(p+\lambda(k-p)+l)}{\Gamma(p-l+1)}.$$
(ii) For $\delta = 0$ and $\beta = 0$

$$\mathcal{U}\mathcal{H}^p_{\delta, \lambda, \mu, \alpha, m, a}(0, 0, b) = \mathcal{H}^p_{\delta, \lambda, \mu, \alpha, m, a}(b)$$

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(iii) For $\delta = 0$, $\beta = 0$ and $b = 1$

$$\mathcal{U}\mathcal{H}^p_{\delta, \lambda, \mu, \alpha, m, a}(0, 0, b) = \mathcal{H}^p_{\delta, \lambda, \mu, \alpha, m, a}$$

$$\mathcal{U}\mathcal{H}^p_{\delta, \lambda, \mu, \alpha, m, a}(0, 0, b) = \mathcal{H}^p_{\delta, \lambda, \mu, \alpha, m, a}$$

(iv) For $g(z) = z^p/(1 - z)$, we have two classes

$$\mathcal{U}\mathcal{H}^{m, p, n}_{\delta, \lambda, \mu, \alpha, m, a}(\delta, \beta, b)$$

and

$$\mathcal{U}\mathcal{H}^{m, p, n}_{\delta, \lambda, \mu, \alpha, m, a}(\delta, \beta, b)$$

which is introduced by Guney and Bulut [1].

Now we define two integral operators

**Definition 3.** Let $\eta \in N, m = (m_1, \ldots, m_\eta) \in N^\eta_0$ and $k = (k_1, \ldots, k_\eta) \in R^\eta_+$. One defines the following general integral operators:

$$\mathcal{F}_p^{\eta, \lambda, \alpha, m, a, k}(z) = p z^{p-1} \eta \left( \frac{D_{\lambda, \alpha, m, a} f(z)}{(p - 1)!!} \right) \, dz,$$

$$\mathcal{F}_p^{\eta, \lambda, \alpha, m, a, k}(z) = p z^{p-1} \eta \left( \frac{D_{\lambda, \alpha, m, a} f(z)}{(p - 1)!!} \right) \, dz,$$

where $z \in \mathcal{U}$, $f, g \in \mathcal{A}_p(n), 1 \leq j \leq \eta$.

**Remark.** (i) For $\eta = 1, m_1 = m, k_1 = k$, and $f_1 = f$, we have the new two new integral operators

$$\mathcal{F}_p^{\eta, \lambda, \alpha, m, a, k}(z) = p z^{p-1} \eta \left( \frac{D_{\lambda, \alpha, m, a} f(z)}{(p - 1)!!} \right) \, dz,$$

$$\mathcal{F}_p^{\eta, \lambda, \alpha, m, a, k}(z) = p z^{p-1} \eta \left( \frac{D_{\lambda, \alpha, m, a} f(z)}{(p - 1)!!} \right) \, dz,$$

(ii) For $g(z) = z^p/(1 - z)$, we have

$$\mathcal{F}_p^{\eta, \lambda, \alpha, m, a, k}(z) = p z^{p-1} \eta \left( \frac{D_{\lambda, \alpha, m, a} f(z)}{(p - 1)!!} \right) \, dz,$$

$$\mathcal{F}_p^{\eta, \lambda, \alpha, m, a, k}(z) = p z^{p-1} \eta \left( \frac{D_{\lambda, \alpha, m, a} f(z)}{(p - 1)!!} \right) \, dz,$$

These operator were introduced by Bulut [1].

(iii) If we take $g(z) = z^p/(1 - z)$, the we have

$$\mathcal{F}_p^{\eta, \lambda, \alpha, m, a, k}(z) = p z^{p-1} \eta \left( \frac{f(z)}{(p - 1)!!} \right) \, dz,$$

$$\mathcal{F}_p^{\eta, \lambda, \alpha, m, a, k}(z) = p z^{p-1} \eta \left( \frac{f(z)}{(p - 1)!!} \right) \, dz,$$

These two operators were introduced by Frasin [3].

### 2 Sufficient Conditions for $\mathcal{F}_p^{\eta, \lambda, \alpha, m, a, k}(z)$

**Theorem 1.** Let $\eta \in N, m = (m_1, \ldots, m_\eta) \in N^\eta_0$ and $k = (k_1, \ldots, k_\eta) \in R^\eta_+$. Also let $b \in C - \{0\}, \delta \geq 0, 0 \leq \beta < p$, and $f_j \in \mathcal{U}\mathcal{H}^{m, p, n}_{\delta, \lambda, \alpha, m, a}(\delta, \beta, b)$ for $1 \leq j \leq \eta$. If

$$0 \leq p + \sum_{j=1}^{\eta} k_j (\beta_j - p) < p,$$

then the integral operator $\mathcal{F}_p^{\eta, \lambda, \alpha, m, a, k}(z)$, defined by (8), is in the class $\mathcal{H}^{m, p, n}_{\delta, \lambda, \alpha, m, a}(\delta, \beta, b)$ where

$$\tau = p + \sum_{j=1}^{\eta} k_j (\beta_j - p).$$

**Proof.** From the definition (8), we observe that

$$\left( \mathcal{F}_p^{\eta, \lambda, \alpha, m, a, k}(z) \right)' = \frac{p z^{p-1} \eta \left( \frac{D_{\lambda, \alpha, m, a} f(z)}{(p - 1)!!} \right)}{z^p} \, dz,$$

Differentiating (2) logarithmically and multiplying by ‘z’, we obtain

$$1 + \frac{z (\mathcal{F}_p^{\eta, \lambda, \alpha, m, a, k}(z))''}{\mathcal{F}_p^{\eta, \lambda, \alpha, m, a, k}(z)} = p - 1 + \sum_{j=1}^{\eta} k_j \left( \frac{D_{\lambda, \alpha, m, a} f_j(z)}{(p - 1)!!} \right) - p$$

or equivalently

$$1 + \frac{z (\mathcal{F}_p^{\eta, \lambda, \alpha, m, a, k}(z))''}{\mathcal{F}_p^{\eta, \lambda, \alpha, m, a, k}(z)} = p - \sum_{j=1}^{\eta} k_j \left( \frac{D_{\lambda, \alpha, m, a} f_j(z)}{(p - 1)!!} \right) - p$$

Then, by multiplying (4) with ‘1/b’, we have

$$\frac{1}{b} \left( 1 + \frac{z (\mathcal{F}_p^{\eta, \lambda, \alpha, m, a, k}(z))''}{\mathcal{F}_p^{\eta, \lambda, \alpha, m, a, k}(z)} \right) = p - \sum_{j=1}^{\eta} k_j \left( \frac{D_{\lambda, \alpha, m, a} f_j(z)}{(p - 1)!!} \right) - p$$

or

$$p + \frac{1}{b} \left( 1 + \frac{z (\mathcal{F}_p^{\eta, \lambda, \alpha, m, a, k}(z))''}{\mathcal{F}_p^{\eta, \lambda, \alpha, m, a, k}(z)} \right) - p = p + \sum_{j=1}^{\eta} k_j \left( \frac{D_{\lambda, \alpha, m, a} f_j(z)}{(p - 1)!!} \right) - p$$

Since $f_j \in \mathcal{U}\mathcal{H}^{m, p, n}_{\delta, \lambda, \alpha, m, a}(\delta, \beta, b)$ for $1 \leq j \leq \eta$, we get

$$\sum_{j=1}^{\eta} k_j \delta_j \left( \frac{z (\mathcal{F}_p^{\eta, \lambda, \alpha, m, a, k}(z))'}{D_{\lambda, \alpha, m, a} f_j(z)} - p \right) + p - \sum_{j=1}^{\eta} k_j \delta_j$$

or

$$\sum_{j=1}^{\eta} k_j \delta_j \left( \frac{z (\mathcal{F}_p^{\eta, \lambda, \alpha, m, a, k}(z))'}{D_{\lambda, \alpha, m, a} f_j(z)} - p \right) > 0$$

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because the integral operator $\mathcal{J}^{p,\eta,\lambda,\alpha,k}(z)$, defined by (8), is in the class $\mathcal{X}_{g}^{p,\eta,\lambda,\alpha,k}(\tau, b)$ with

$$\tau = p + \sum_{j=1}^{\eta} k_j (\beta_j - p).$$

3 Sufficient Conditions for $\mathcal{F}_{g}^{p,\eta,\lambda,\alpha,k}(z)$

**Theorem 2.** Let $\eta \in N, m = (m_1, \ldots, m_\eta) \in N_0^n$ and $k = (k_1, \ldots, k_\eta) \in R_+^n$. Also let $b \in C - \{0\}, \delta \geq 0, 0 \leq \beta < p$, and $f_j \in \mathcal{U} \mathcal{J}^{p,\eta,\lambda,\alpha}(\delta_j, \beta_j, b)$ for $1 \leq j \leq \eta$. If

$$0 \leq p + \sum_{j=1}^{\eta} k_j (\beta_j - p) < p,$$

then the integral operator $\mathcal{F}_{g}^{p,\eta,\lambda,\alpha,k}(z)$, defined by (8), is in the class $\mathcal{X}_{g}^{p,\lambda,\alpha,k}(\tau, b)$ where

$$\tau = p + \sum_{j=1}^{\eta} k_j (\beta_j - p).$$

**Proof.** From the definition (8), we observe that $\mathcal{F}_{g}^{p,\eta,\lambda,\alpha,k}(z) \in \mathcal{A}_p(n)$. We can easy to see that

$$\left(\mathcal{F}_{g}^{p,\eta,\alpha,k}(z)\right)' = p^{2p-1} \sum_{j=1}^{\eta} \left( D_{A,F,g}^{m_\alpha,n,j}(f_j)^{g_j}(z) \right)^{k_j}.$$ (2)

Differentiating (2) logarithmically and multiplying by 'z', we obtain

$$\frac{z}{(\mathcal{F}_{g}^{p,\eta,\alpha,k}(z))'} = p - 1 + \sum_{j=1}^{\eta} k_j \left( \frac{z}{D_{A,F,g}^{m_\alpha,n,j}(f_j)^{g_j}(z)} + 1 - p \right)$$

or equivalently

$$1 + \frac{z}{(\mathcal{F}_{g}^{p,\eta,\alpha,k}(z))'} = p - 1 + \sum_{j=1}^{\eta} k_j \left( \frac{z}{D_{A,F,g}^{m_\alpha,n,j}(f_j)^{g_j}(z)} + 1 - p \right)$$

Then, by multiplying (4) with '1/b', we have

$$\frac{1}{b} \left( 1 + \frac{z}{(\mathcal{F}_{g}^{p,\eta,\alpha,k}(z))'}\right) - p = \sum_{j=1}^{\eta} k_j \left( \frac{z}{D_{A,F,g}^{m_\alpha,n,j}(f_j)^{g_j}(z)} + 1 - p \right)$$

or

$$p + \frac{1}{b} \left( 1 + \frac{z}{(\mathcal{F}_{g}^{p,\eta,\alpha,k}(z))'}\right) - p = \sum_{j=1}^{\eta} k_j \left( \frac{z}{D_{A,F,g}^{m_\alpha,n,j}(f_j)^{g_j}(z)} + 1 - p + p + p \sum_{j=1}^{\eta} k_j \right)$$

Since $f_j \in \mathcal{U} \mathcal{J}^{p,\eta}(\delta_j, \beta_j, b)$ for $1 \leq j \leq \eta$, we get

$$\text{Re} \left\{ \frac{p}{b} + \frac{1}{b} \left( 1 + \frac{z}{(\mathcal{F}_{g}^{p,\eta,\alpha,k}(z))'}\right) - p \right\} = p + \sum_{j=1}^{\eta} k_j \left( \frac{z}{D_{A,F,g}^{m_\alpha,n,j}(f_j)^{g_j}(z)} + 1 - p \right) + p + \sum_{j=1}^{\eta} k_j (\beta_j - p).$$

4 Corollaries and Consequences

For $\eta = 1, m_1 = m, k_1 = k$, and $f_1 = f$, we have

**Corollary 1.** Let $\eta \in N, m \in N_0^n$ and $k \in R_+^n$. Also let $b \in C - \{0\}, \delta \geq 0, 0 \leq \beta < p$, and $f \in \mathcal{U} \mathcal{J}^{p,\lambda,\alpha}(\delta_j, \beta_j, b)$ for $1 \leq j \leq \eta$. If

$$0 \leq p + k (\beta - p) < p,$$

then the integral operator $\mathcal{J}^{p,\eta,\lambda,\alpha}(z)$ is in the class $\mathcal{X}_{g}^{p,\lambda,\alpha}(\tau, b)$ where

$$\tau = p + \sum_{j=1}^{\eta} k_j (\beta_j - p).$$

**Corollary 2.** Let $\eta \in N, m \in N_0^n$ and $k \in R_+^n$. Also let $b \in C - \{0\}, \delta \geq 0, 0 \leq \beta < p$, and $f \in \mathcal{U} \mathcal{J}^{p,\lambda,\alpha}(\delta_j, \beta_j, b)$ for $1 \leq j \leq \eta$. If

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$$\tau = p + \sum_{j=1}^{\eta} k_j (\beta_j - p).$$

**Corollary 3.** Let $\eta \in N, m = (m_1, \ldots, m_\eta) \in N_0^n$ and $k = (k_1, \ldots, k_\eta) \in R_+^n$. Also let $b \in C - \{0\}, \delta \geq 0, 0 \leq \beta < p$, and $f_j \in \mathcal{U} \mathcal{J}^{p,\lambda,\alpha}(\delta_j, \beta_j, b)$ for $1 \leq j \leq \eta$. If

$$0 \leq p + \sum_{j=1}^{\eta} k_j (\beta_j - p) < p,$$

then the integral operator $\mathcal{J}^{p,\eta,m,k}(z)$ is in the class $\mathcal{X}_{g}^{p,\eta,m,k}(\tau, b)$ where

$$\tau = p + \sum_{j=1}^{\eta} k_j (\beta_j - p).$$
Corollary 4. Let \( \eta \in N, m = (m_1, \ldots, m_\eta) \in N_0^\eta \) and \( k = (k_1, \ldots, k_\eta) \in R_0^\eta \). Also let \( b \in C - \{0\}, \delta \geq 0, 0 \leq \beta < p \), and \( \mathcal{A} \mathcal{K}^{m,p,n}_\alpha(\delta_j, \beta_j, b) \) for \( 1 \leq j \leq \eta \). If
\[
0 \leq p + \sum_{j=1}^\eta k_j(\beta_j - p) < p,
\]
then the integral operator \( \mathcal{A} \mathcal{K}^{p,n}(\tau, b) \) where
\[
\tau = p + \sum_{j=1}^\eta k_j(\beta_j - p),
\]
which are known results obtained by Guney and Bulut [2]. Further, if put \( p = 1 \), we have

Corollary 5. Let \( \eta \in N, m = (m_1, \ldots, m_\eta) \in N_0^\eta \) and \( k = (k_1, \ldots, k_\eta) \in R_0^\eta \). Also let \( b \in C - \{0\}, \delta \geq 0, 0 \leq \beta < 1 \), and \( f_j \in \mathcal{A} \mathcal{K}^{p,\lambda,1,m,\alpha}_\eta(\delta_j, \beta_j, b) \) for \( 1 \leq j \leq \eta \). If
\[
0 \leq 1 + \sum_{j=1}^\eta k_j(\beta_j - 1) < 1,
\]
then the integral operator \( \mathcal{A} \mathcal{K}^{1,\lambda,1,m,\alpha}_\eta(z) \) is in the class \( \mathcal{K}^{1,\lambda,1,m,\alpha}_\eta(\tau, b) \) where
\[
\tau = 1 + \sum_{j=1}^\eta k_j(\beta_j - 1).
\]

Corollary 6. Let \( \eta \in N, m = (m_1, \ldots, m_\eta) \in N_0^\eta \) and \( k = (k_1, \ldots, k_\eta) \in R_0^\eta \). Also let \( b \in C - \{0\}, \delta \geq 0, 0 \leq \beta < 1 \), and \( f_j \in \mathcal{A} \mathcal{K}^{1,\lambda,1,m,\alpha}_\eta(\delta_j, \beta_j, b) \) for \( 1 \leq j \leq \eta \). If
\[
0 \leq 1 + \sum_{j=1}^\eta k_j(\beta_j - 1) < 1,
\]
then the integral operator \( \mathcal{A} \mathcal{K}^{1,\eta,m,\alpha}_\eta(z) \) is in the class \( \mathcal{K}^{1,\eta,m,\alpha}_\eta(\tau, b) \) where
\[
\tau = 1 + \sum_{j=1}^\eta k_j(\beta_j - 1).
\]

Upon setting \( g(z) = z^\eta/(1 - z) \), we have

Corollary 7. Let \( \eta \in N, m = (m_1, \ldots, m_\eta) \in N_0^\eta \) and \( k = (k_1, \ldots, k_\eta) \in R_0^\eta \). Also let \( b \in C - \{0\}, \delta \geq 0, 0 \leq \beta < p \), and \( f_j \in \mathcal{A} \mathcal{K}^{p,\lambda,1,m,\alpha}_\eta(\delta_j, \beta_j, b) \) for \( 1 \leq j \leq \eta \). If
\[
0 \leq p + \sum_{j=1}^\eta k_j(\beta_j - p) < p,
\]
then the integral operator \( \mathcal{A} \mathcal{K}^{p,\eta,m,\alpha}_\eta(z) \) is in the class \( \mathcal{K}^{p,\lambda,1,m,\alpha}_\eta(\tau, b) \) where
\[
\tau = p + \sum_{j=1}^\eta k_j(\beta_j - p).
\]

Corollary 8. Let \( \eta \in N, m = (m_1, \ldots, m_\eta) \in N_0^\eta \) and \( k = (k_1, \ldots, k_\eta) \in R_0^\eta \). Also let \( b \in C - \{0\}, \delta \geq 0, 0 \leq \beta < p \), and \( f_j \in \mathcal{A} \mathcal{K}^{p,\lambda,1,m,\alpha}_\eta(\delta_j, \beta_j, b) \) for \( 1 \leq j \leq \eta \). If
\[
0 \leq p + \sum_{j=1}^\eta k_j(\beta_j - p) < p,
\]
then the integral operator \( \mathcal{A} \mathcal{K}^{p,\eta,m,\alpha}_\eta(z) \) is in the class \( \mathcal{K}^{p,\lambda,1,m,\alpha}_\eta(\tau, b) \) where
\[
\tau = p + \sum_{j=1}^\eta k_j(\beta_j - p).
\]

Upon setting \( g(z) = z^\eta/(1 - z) \) and \( \delta = 0 \), we have

Corollary 9. Let \( \eta \in N, m = (m_1, \ldots, m_\eta) \in N_0^\eta \) and \( k = (k_1, \ldots, k_\eta) \in R_0^\eta \). Also let \( b \in C - \{0\}, 0 \leq \beta < p \), and \( f_j \in \mathcal{A} \mathcal{K}^{p,\lambda,1,m,\alpha}_\eta(0, \beta, b) \) for \( 1 \leq j \leq \eta \). If
\[
0 \leq p + \sum_{j=1}^\eta k_j(\beta_j < p),
\]
then the integral operator \( \mathcal{A} \mathcal{K}^{p,\eta,m,\alpha}_\eta(z) \) is in the class \( \mathcal{K}^{p,\lambda,1,m,\alpha}_\eta(\tau, b) \) where
\[
\tau = p + \sum_{j=1}^\eta k_j(\beta_j - p).
\]

Corollary 10. Let \( \eta \in N, m = (m_1, \ldots, m_\eta) \in N_0^\eta \) and \( k = (k_1, \ldots, k_\eta) \in R_0^\eta \). Also let \( b \in C - \{0\}, \delta \geq 0, 0 \leq \beta < p \), and \( f_j \in \mathcal{A} \mathcal{K}^{p,\lambda,1,m,\alpha}_\eta(0, \beta, b) \) for \( 1 \leq j \leq \eta \). If
\[
0 \leq p + \sum_{j=1}^\eta k_j(\beta_j - p) < p,
\]
then the integral operator \( \mathcal{A} \mathcal{K}^{p,\eta,m,\alpha}_\eta(z) \) is in the class \( \mathcal{K}^{p,\lambda,1,m,\alpha}_\eta(\tau, b) \) where
\[
\tau = p + \sum_{j=1}^\eta k_j(\beta_j - p).
\]

References