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A Certain Generalized Gamma Matrix Functions and Their Properties

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Abstract: The main aim of this paper is to present generalizations of Gamma and Psi matrix functions. Some different properties are established for these new generalizations. By means of the generalized Gamma matrix function, we introduced the generalized Pochhammer symbols and their properties.

Keywords: Generalized Gamma matrix functions, Psi matrix functions, Generalized Pochhammer symbols.

1 Introduction

The Gamma function $\Gamma(z)$, defined for $Re(z) > 0$ by the improper integral

$$\Gamma(z) = \int_0^\infty v^{z-1} e^{-v} dv,$$

was introduced into analysis in the year 1729 by Leonard Euler [7], while seeking a generalization of the factorial $n!$ for non-integral values of n, and subjected to intense study by many eminent mathematicians of the nineteenth and early twentieth centuries and continues to interest the present generation. The logarithmic derivative ψ of the Gamma function is known as the Psi or digamma function, that is, it is given by

$$\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)},$$

for positive real numbers x. The Gamma functions are one of the most important special functions and have many applications in many fields of science, for example, analytic number theory, statistics and physics.

The generalized Gamma functions are widely used in the solution of many problems of wave scattering and diffraction theory [19,20]. For the properties of the analogous extensions of the family of Gamma functions, we refer the reader to [4,5,6], and for a historical profile, we refer the reader to [7]. To date, a few various methods have been developed for the analysis of the generalized Gamma functions [2,13,19,20,21]. Some fundamental properties of these functions were investigated in [19]. In [10,11] a new approach to the computation of the generalized complete and incomplete Gamma functions are proposed, which considerably improved its capabilities during numerical evaluations in significant cases. Using binomial expansion theorem the generalized Gamma and incomplete Gamma functions are expressed through the familiar Gamma and exponential integral functions.

A generalization of a well-known special matrix functions, which extends the domain of that matrix functions, can be expected to be useful provided that the important properties of the special matrix functions are carried over to the generalization in a natural and simple manner. Of course, the original special matrix functions and its properties must be recoverable as a particular case of the generalization. Special functions of a matrix argument appear in the study of spherical functions on certain symmetric spaces and multivariate analysis in statistics, see [14]. Special functions of two diagonal matrix argument have been used in [3]. Some properties of the Gamma and Beta matrix functions have recently been treated in [16,17]. An asymptotic expression of the incomplete Gamma matrix function and integral expressions of Bessel matrix functions are given in [22].

The primary goal of this paper is to consider a generalizations of Gamma and Psi matrix functions.

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Then, difference formula, reflection formula, recurrence relations and differential equation are derived for generalizations of Gamma and Psi matrix functions. Finally, we define generalizations of Pochhammer symbols and some their properties.

Throughout this paper, consider the complex space $\mathbb{C}^{N\times N}$ of complex matrices of common order N. A matrix A is a positive stable matrix in $\mathbb{C}^{N\times N}$ if $Re(\lambda) > 0$ for all $\lambda \in \sigma(A)$ where $\sigma(A)$ is the set of all eigenvalues of A. The two-norm of A, which will be denoted by $\|A\|$, is defined by

$$||A|| = \sup_{x\neq 0} \frac{||Ax||_2}{||x||_2},$$

where for a vector y in $\mathbb{C}^N$, $||y||_2 = (y^H y)^{\frac{1}{2}}$ is the Euclidean norm of y and A^H denotes the Hermitian adjoint of A. We denote by $\mu(A)$ the logarithmic norm of A, defined by [9, 12, 22]

$$\mu(A) = \max\{z : z \text{ eigenvalues of } \frac{A+A^H}{2}\}. \quad (1)$$

We denote by $\widetilde{\mu}(A)$ the number

$$\widetilde{\mu}(A) = \min\{z : z \text{ eigenvalues of } \frac{A+A^H}{2}\}. \quad (2)$$

By [12], it follows that

$$\|e^{At}\| \le e^{t\mu(A)}; \qquad t \ge 0.$$

Hence,

$$\|e^{tA}\| \le e^{t\widetilde{\mu}(A)} \sum_{s=0}^{N-1} \frac{(\|A\| N^{\frac{1}{2}} t)^s}{s!}; \quad t \ge 0,$$

and

$$\|n^A\| \le n^{\widetilde{\mu}(A)} \sum_{s=0}^{N-1} \frac{(\|A\| N^{\frac{1}{2}} \ln n)^s}{s!}; \quad n \ge 1.$$

If $f(z)$ and $g(z)$ are holomorphic functions of the complex variable z which are defined in an open set Ω of the complex plane and A is a matrix in $\mathbb{C}^{N\times N}$ such that $\sigma(A) \subset \Omega$, then from the properties of the matrix functional calculus [8], it follows that

$$f(A)g(A) = g(A)f(A). \quad (3)$$

Hence, if B in $\mathbb{C}^{N\times N}$ is a matrix for which $\sigma(B) \subset \Omega$ and if $AB = BA$, then

$$f(A)g(B) = g(B)f(A). \quad (4)$$

Let A in $\mathbb{C}^{N\times N}$, $F(z)$ a matrix function, and let $g(z)$ be a positive scalar function. We say that $F(z)$ behaves $O(g(z),A)$ in a domain Ω, if $||F(z)|| \le M(A)g(z)$, for some positive constant $M(A), z \in \Omega$, and $F(z)$ commutes with A.

2 Preliminaries

In this section, we shall adopt in this work a somewhat different notation and facts from that used by previous results as follows:

The reciprocal Gamma function denoted by $\Gamma^{-1}(z) = \frac{1}{\Gamma(z)}$ is an entire function of the complex variable z. Then for any matrix A in $\mathbb{C}^{N\times N}$, the image of $\Gamma^{-1}(z)$ acting on A denoted by $\Gamma^{-1}(A)$ is a well defined matrix. Furthermore, if

$$A + nI \quad \text{is invertible for all integer } n \ge 0, \quad (5)$$

where I is the identity matrix in $\mathbb{C}^{N\times N}$, then $\Gamma(A)$ is invertible, its inverse coincides with $\Gamma^{-1}(A)$ and one gets the formula [16]

$$(A)_n = \Gamma(A+nI)\Gamma^{-1}(A); \quad n \ge 1; (A)_0 = I. \quad (6)$$

Jódar and Cortés have proved in [16] that

$$\Gamma(A) = \lim_{n\to\infty} (n-1)! [(A)_n]^{-1} n^A. \quad (7)$$

For any matrix A and in $C^{N\times N}$ we use the following relation due to [17], as follows

$$(1-z)^{-A} = \sum_{n=0}^{\infty} \frac{(A)_n}{n!} z^n; \qquad |z| < 1.$$

Let A be two positive stable matrices in $\mathbb{C}^{N\times N}$. The Gamma matrix function $\Gamma(A)$ and the Digamma matrix function $\psi(A)$ have been defined in [17, 18] as follows

$$\Gamma(A) = \int_0^\infty e^{-t} t^{A-I} dt, \quad t^{A-I} = \exp((A-I)\ln t) \quad (8)$$

and

$$\psi(A) = \frac{d}{dA} \ln \Gamma(A) = \Gamma^{-1}(A)\Gamma'(A). \quad (9)$$

Jodar and Sastre have proved asymptotic behavior of Gamma matrix function in [15] by using

$$A\Gamma(A) = e^{-\gamma A} \Big[\Pi_{n=1}^{\infty} (I + \frac{A}{n}) e^{-\frac{A}{n}}\Big]^{-1}, \quad (10)$$

where γ is the Euler-Mascheroni constant.

Some integral forms of the Bessel matrix functions and the modified Bessel matrix functions proved in [22] are the following

$$J_A(z) = (\frac{z}{2})^A \frac{\Gamma^{-1}(A+\frac{I}{2})}{\sqrt{\pi}} \int_{-1}^{1} (1-t^2)^{A-\frac{I}{2}} \cos(zt) dt, \quad (11)$$

$$I_A(z) = (\frac{z}{2})^A \frac{\Gamma^{-1}(A+\frac{I}{2})}{\sqrt{\pi}} \int_{-1}^{1} (1-t^2)^{A-\frac{I}{2}} \cosh(zt) dt \quad (12)$$

and

$$K_A(z) = \frac{1}{2}(\frac{z}{2})^A \int_0^\infty \exp(-t - \frac{z^2}{4t})\, t^{-(A+I)} dt, \quad (13)$$

where $K_A(z)$, also known as modified Hankel matrix function or Macdonald matrix function.

3 A generalized Gamma matrix functions

It is possible to extend the classical Gamma function in infinitely many ways. some of these extensions could be useful in certain types of problems. We define the generalized Gamma matrix function (GGMF) and the generalized Psi matrix function (GPMF) in Definition 3.1 and Definition 3.2. The generalized Gamma and Psi matrix function have several interesting properties. Some useful properties are listed below in Theorem 3.1 and Theorem 3.2 of this section.

Definition 3.1. Let A and B be positive stable matrices in $C^{N\times N}$, then, the generalized Gamma matrix function $\Gamma(A,B)$ is defined by

$$\Gamma(A,B) := \int_0^\infty t^{A-I}e^{-(It+\frac{B}{t})}dt, \quad t^{A-I} = \exp((A-I)\ln t). \tag{14}$$

The factor $\exp(-\frac{B}{t})$ in the integral (14) plays the role of a regularizer. For this generalized we present difference formula, reflection formula, differential equation and various particular cases.

Let us suppose that B is a matrix such that

$$Re(z) > 0, \quad for\ every\ eigenvalye\ z \in \sigma(B),$$

and let us denote $\sqrt{B} = B^{\frac{1}{2}} = \exp(\frac{1}{2}\log B)$ the image of the function $z^{\frac{1}{2}} = \exp(\frac{1}{2}\log z)$ by the Riesz-Dunford functional calculus, acting on the matrix B, where $\log z$ denotes the principal branch of the complex logarithm (see [8]). Then by the integral in (14) can be simplified in terms of the Macdonald matrix function to give

$$\Gamma(A,B) = 2\exp(\frac{A}{2}\ln B)K_A(2\sqrt{B}); \quad \mu(B) > 0, \tag{15}$$

is also useful in the evaluation of certain Mellin and Laplace transforms.

Theorem 3.1. Let A and B be positive stable matrices in $C^{N\times N}$, then each of the following properties holds true:

(i)$\Gamma(A+I,B) = A\Gamma(A,B) + B\Gamma(A-I,B)$.
(ii)$\Gamma(-A,B) = \exp(-A\ln B)\Gamma(A,B)$.
(iii)$\Gamma(I-A,B) = \exp(-A\ln B)\Big(\Gamma(A+I,B) - A\Gamma(A,B)\Big)$.
(iv)$B\frac{\partial^2 U}{\partial B^2} + (I-A)\frac{\partial U}{\partial B} - U = 0; \ U = \Gamma(A,B)$.

Proof.(i)-Let M be Mellin transform operator as defined by

$$M\{f(t);A\} := \int_0^\infty f(t)t^{A-I}dt. \tag{16}$$

Then $\Gamma(A,B)$ is simply the Mellin transform of $f(t) := \exp[-It - Bt^{-1}]$ in A, thus

$$\Gamma(A,B) := M\{e^{-It-Bt^{-1}};A\}.$$

Exploiting the relationship

$$M\{f'(t);A\} = -(A-I)M(f(t)(A-I)), \tag{17}$$

between the Mellin transform of a function and derivative we see that

$$-(A-I)\Gamma(A-I,B) = M\{(-I+Bt^{-2})e^{(-It-Bt^{-1})};A\}. \tag{18}$$

Hence,

$$-(A-I)\Gamma(A-I,B) = -\Gamma(A,B) + B\Gamma(A-2I;B). \tag{19}$$

Replacing A by $A = A+I$ in (19), we get the proof of(i).

(ii)-Putting $t = B\xi^{-1}$, in (14) yield

$$\Gamma(A,B) = \exp(A\ln B)\int_0^\infty \xi^{-(A+I)}e^{-(B\xi^{-1}+\xi)}d\xi, \tag{20}$$

which is exactly

$$\Gamma(-A,B) = \exp(-A\ln B)\Gamma(A,B), \tag{21}$$

as desired.

(iii)- This follows from (i) and (ii) when we replace A by $-A$ and use the Eq.(21).

(iv)- Form Definition 3. 1. We have

$$\frac{\partial^n}{\partial B^n}\{\Gamma(A,B)\} = (-1)^n\Gamma(A-nI,B) \quad \mu(B) > 0, \quad n = 0,1,2,... \tag{22}$$

Replacing A by $A-I$ in (i), we find

$$B\Gamma(A-2I,B) - (I-A)\Gamma(A-I,B) - \Gamma(A,B) = 0, \tag{23}$$

form (22) and (23), we obtain the required second-order differential equation.

Definition 3.2. Let A and B be positive stable matrices in $\mathbb{C}^{N\times N}$ such that for all integral $n \geq 0$ satisfies the condition

$$A+nI \text{ and } B+nI \text{ are invertible.} \tag{24}$$

Then $\Gamma(A,B)$ is invertible, its inverse coincides with $\Gamma^{-1}(A,B)$ and the generalized Psi (Digamma) matrix function $\psi(A,B)$ given in the formula

$$\psi(A,B) := \frac{\partial}{\partial A}\{\ln(\Gamma(A,B))\} = \Gamma^{-1}(A,B)\frac{\partial}{\partial A}\{\Gamma(A,B)\}. \tag{25}$$

It follows from the integral representation (14) of the generalized Psi matrix function that

$$\psi(A,B) = \Gamma^{-1}(A,B)\int_0^\infty t^{A-I}(\ln t)e^{-It-Bt^{-1}}dt \quad \mu(B) \geq 0, \mu(A) \geq 0. \tag{26}$$

Remark 3.1. (14) and (26) are matrix versions of generalized Gamma and Psi functions given in [6].

Remark 3.2. (14) and (26) for $B = 0$ reduces Gamma and Psi matrix functions in [16,18], respectively.

Theorem 3.2. Let A and B be positive stable matrices in $C^{N\times N}$ satisfying the condition (24) for all integral $n \geq 0$. Then each of the following properties holds true:

(i) $\psi(A,B)=\int_0^\infty \Gamma^{-1}(A,B) \times \left\{e^{-s}\Gamma(A,B)-(1+s)^{-A}\,\Gamma(A,B(1+s))\right\}\frac{ds}{s}$; $\mu(B)\geq 0,\ \mu(A)>0$.

(ii) $\psi(A,B)=\int_0^\infty \Gamma^{-1}(A,B)\left\{\frac{e^{-t}\Gamma(A,B)}{t}-\frac{e^{-At}\Gamma(A,Be^t)}{1-e^{-t}}\right\}dt$; $\mu(B)\geq 0,\ \mu(A)>0$.

(iii) $\psi(A,B)=\ln A+\int_0^\infty \Gamma^{-1}(A,B) \times\left\{\frac{\Gamma(A,B)}{t}-\frac{\Gamma(A,Be^t)}{1-e^{-t}}\right\}e^{-At}dt$; $\mu(B)\geq 0,;\mu(A)>0$.

(iv) $\psi(-A,B)=\ln B+\psi(A,B)$; $\mu(B)>0$.

(v) $\psi(A+I,B)\Gamma(A+I,B)-B\psi(A-I,B)\Gamma(A+I,B) =\Gamma(A,B)+A\psi(A,B)\Gamma(A,B)$; $\mu(B)\geq 0,\ \mu(A)>0$.

proof. (i)-Let us consider the double integral

$$\Lambda=\int_0^\infty\int_0^\infty t^{A-I}e^{-Bt^{-1}}\left\{\frac{e^{-t-s}-e^{-t(1+s)}}{s}\right\}dt\,ds. \tag{27}$$

If we integrate the double integral with respect to t, becomes

$$\Lambda=\int_0^\infty\left\{e^{-s}\int_0^\infty t^{A-I}e^{-(It+Bt^{-1})}dt -\int_0^\infty t^{A-I}e^{-(t(1+s)I+Bt^{-1})}dt\right\}\frac{ds}{s}. \tag{28}$$

From Definition 3.1. We obtain

$$\Lambda=\int_0^\infty\left\{e^{-s}\Gamma(A,B)-(1+s)^{-A}\Gamma(A,B(1+s))\right\}\frac{ds}{s}. \tag{29}$$

Integrating (27) with respect to s, we have

$$\Lambda=\int_0^\infty t^{A-I}e^{-(It+Bt^{-1})}\left\{\int_0^\infty\frac{e^{-s}-e^{-ts}}{s}ds\right\}dt. \tag{30}$$

The inner integral in (30) is the integral representation of $\ln t$ ([6], p. 24), we see that

$$\Lambda=\int_0^\infty t^{A-I}\ln t\, e^{-(It+Bt^{-1})}dt=\frac{\partial}{\partial A}\Gamma(A,B). \tag{31}$$

Form (29) and (31), it follows that

$$\frac{\partial}{\partial A}(\Gamma(A,B)) =\int_0^\infty\left\{e^{-s}\Gamma(A,B)-(1+s)^{-A}\Gamma(A,B(1+s))\right\}\frac{ds}{s}. \tag{32}$$

Multiplying both sides in (32) by $\Gamma^{-1}(A,B)$. Thus, the proof of relation (i) is completed.

(ii)-From result (i), we see that

$$\psi(A,B) =\lim_{\delta\to 0}\left[\int_\delta^\infty\frac{e^{-s}}{s}ds -\int_\delta^\infty\frac{(1+s)^{-A}\Gamma(A,B(1+s))\Gamma^{-1}(A,B)}{s}ds\right]. \tag{33}$$

The transformation $s=e^t-1$ in the second integral on the right -hand side yields

$$\int_\delta^\infty\frac{(1+s)^{-A}\Gamma(A,B(1+s))\Gamma^{-1}(A,B)}{s}ds =\int_{\ln(1+\delta)}^\infty\frac{\Gamma(A,Be^t)\Gamma^{-1}(A,B)e^{-tA}}{1-e^{-t}}dt. \tag{34}$$

From (33) and (34), we obtain the required relationship

$$\psi(A,B)=\int_0^\infty\left(\frac{e^{-t}}{t}-\frac{\Gamma(A,Be^t)\Gamma^{-1}(A,B)e^{-tA}}{1-e^{-t}}\right)dt. \tag{35}$$

(iii)- Adding and subtracting the factor $\frac{e^{-At}}{t}$ in the first term of the integrand in (ii), we find that

$$\psi(A,B) =\int_0^\infty\frac{e^{-t}-e^{-At}}{t}dt+\int_0^\infty\left\{\frac{1}{t} -\frac{\Gamma(A,Be^t)\Gamma^{-1}(A,B)e^{-tA}}{1-e^{-t}}\right\}e^{-At}dt. \tag{36}$$

The first on the right- hand side of (34) is the standard integral representation of $\ln(A)$ (see [6]). This proves (iii).

(iv)-Replacing A by $-A$ in (26), it follows that

$$\psi(-A,B)=\Gamma^{-1}(-A,B)\int_0^\infty t^{-(A+I)}\ln t\, e^{-(tI+Bt^{-1})}dt. \tag{37}$$

The substitutions $t=B\tau$,$dt=-B\tau^{-2}d\tau$ in (37) yield

$$\psi(-A,B) =B^{-A}\Gamma^{-1}(-A,B)\int_0^\infty(\ln B-\ln\tau)\tau^{A-I}e^{-\tau I-B\tau^{-1}}d\tau. \tag{38}$$

Using result (ii) in Theorem 3.1. We have

$$\psi(-A,B) =\Gamma^{-1}(A,B)\int_0^\infty(\ln B-\ln\tau)\tau^{A-I}e^{-\tau I-B\tau^{-1}}d\tau, \tag{39}$$

which is exactly the right-hand side of (iv)(v)- According to the relation (15) with straightforward computation shows that

$$\int_0^\infty \ln t\left(t^A-At^{A-I}-Bt^{A-2I}\right)e^{-tI-Bt^{-1}}dt =2B^{A/2}K_A(2\sqrt{B}), \quad \mu(B)>0, \tag{40}$$

that is,

$$\Gamma(A+I,B)\psi(A+I,B)-A\Gamma(A,B)\psi(A,B) -B\Gamma(A-I,B).\psi(A,B)=\Gamma(A,B). \tag{41}$$

Multiplying the above equation by $A^{-1}\,\Gamma^{-1}(A,B)$ and rearranging the terms completes the proof of (v).

4 A generalized Pochhammer symbol

The introduction of the generalized Gamma matrix function $\Gamma(A,B)$ in Section 3 is useful in defining generalization of the Pochhammer symbol $(A)_n$, $n \geq 1$, (cf., e.g., [16,17,18]). The introduction of the concept and notion of this symbol is destined to lead to the analytic study of a class of problems in engineering and other sciences. As a matter of fact, several known properties of the Pochhammer symbol are recovered from those of the generalized Pochhammer symbol introduced here. We begin by introducing the generalized Pochhammer symbol under condition (24) can be written in the form

$$(A,B)_n = \Gamma(A+nI,B)\Gamma^{-1}(A); \quad \mu(B) > 0. \tag{42}$$

Which leads us easily to integral representation for the generalized Pochhammer symbol $(A;B)_n$:

$$\begin{aligned}(A;B)_n =&\Gamma^{-1}(A)\int_0^\infty t^{A+(n-1)I}\exp-(tI+Bt^{-1})dt;\\ &\mu(B)>0,\quad \mu(A+nI)>0.\end{aligned} \tag{43}$$

Since the generalized Pochhammer symbol $(A;B)_n$ is related to the modified Bassel function of the third kind from the relation (15), can be written as follows:

$$(A;B)_n = 2\Gamma^{-1}(A)\ \exp(\frac{A+nI}{2}\ln B)\ K_{A+nI}(2\sqrt{B}) \quad , \quad \mu(B)>0. \tag{44}$$

Theorem 4.1. Let A and B be positive stable matrices in $C^{N\times N}$, such that satisfying the condition (24) and $m,n \in \mathbb{N}_0 := \mathbb{N}\cup 0$. Then

$$(A;B)_{n+m} = (A)_n(A+nI;B)_m. \tag{45}$$

proof. Form the relations (6) and (42), we find that

$$\begin{aligned}(A;B)_{n+m} =&\Gamma^{-1}(A)\Gamma(A+I(n+m),B)\\ =&\Gamma(A+I(m+n),B)\Gamma(A+nI)\Gamma^{-1}(A)\\ \times\Gamma^{-1}&(A+nI)\\ =&(A)_n(A+nI;B)_m.\end{aligned} \tag{46}$$

Thus, by applying the well-known properties of the Pochhammer symbol $(A)_n$ in (46) (see, for example, [16, 17,18,22]), it is fairly straightforward to derive the corresponding properties of the generalized Pochhammer symbol $(A;B)_n$ as follows:

Corollary 4.1 Let $k,l,m,n \in \mathbb{N}_0$. Then each of the following identities holds true:

(i)$(A;B)_{n+m+l} = (A)_n(A+nI)_m(A+(n+m)I;B)_l$,

(ii)$(A;B)_{2m+l} = 2^{(2m)}(\frac{A}{2})_m(\frac{A+I}{2})_m(A+2mI,B)_l$,

(iii)$(A+nI;B)_{n+l} = (A+nI)_n(A+2nI;B)_l = (A)_{2n}[(A)_n]^{-1}(A+2nI,B)_l$,

(iv)$(A+kmI;B)_{kn+l} = (A)_{k(m+n)}[(A)_{km}]^{-1}(A+Ik(n+m);B)_l$,

(v)$(A-kmI)_{kn+l} = (-1)^{km}(A)_{k(n-m)}(I-A)_{km}(A+k(n-m);B)_l$.

5 Concluding comments

The material developed in Sections $3-4$ provides several important properties of the generalized Gamma matrix function (GGMF) $\Gamma(A,B)$, introduced in (14), under a certain conditions on the matrices A and B. By using the GGMF, we have defined generalizations of Psi matrix function and Pochhammer symbol. We have investigated some properties of these generalized functions, most of which are analogous with the original matrix functions. Most of the special matrix functions of mathematical physics and engineering, such as hypergeometric, Whittaker and modified Bessel matrix functions can be expressed in terms of generalized Pochhammer symbol. Therefore, the corresponding extensions of several other familiar special matrix functions are expected to be useful and need to be investigated.

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