Information Sciences Letters
An International Journal

http://dx.doi.org/10.18576/isl/090303

A numerical method based on Legendre differentiation matrices for higher order ODEs

M. Abdelhakem[1,2,*], *M. Biomy*[3,4], *S. A. Kandil*[2], *D. Baleanu*[5,6] *and M. El-Kady*[1,2]

[1]Department of Mathematics, Faculty of Science, Helwan University, Cairo, Egypt
[2]Canadian International College, School of Engineering, Basic Science Department, Cairo, Egypt
[3]Department of Mathematics, Faculty of Science, Suez Canal University, Port-Said, Egypt
[4]Department of Mathematics, College Of Sciences and Arts, Al-Ras, Al-Qassim University, KSA
[5]Department of Mathematics and Computer Sciences, Cankaya University, Ankara, Turkey
[6]Institute of Space Sciences, Magurele-Bucharest, Romania

Received: 17 May 2020, Revised: 21 Aug. 2020, Accepted: 23 Aug. 2020
Published online: 1 Sep. 2020

Abstract: This paper introduces a new method to obtain the spectral accuracy solutions to higher order differential equations and singularly perturbed boundary value problems (BVPs). Legendre polynomials (LPs) $P_n(x)$ have been used and involved in straightforward implementation method. Asymptotic upper bound on the Legendre coefficients for the k^{th} derivatives are presented. Also, We detect the roundoff error effect in the Legendre matrices. The superiority of the suggested method became evident through some examples and applications.

Keywords: Legendre Polynomials, Differentiation Matrices, Roundoff Error Analysis, Higher Order Differential Equations, Singularly Perturbed Boundary Value Problem.

1 Introduction

Through the last years, there are highly increasing for the importance of using LPs in numerical analysis. It has demonstrated to be a very useful tool, from both theoretical and practical points of view. LPs are extensively used for applications especially in spectral methods for ordinary and partial differential equations. It is useful and convenient in various applications to be able to express LPs [1,2,3,4].

They rapidly exponential convergent compared to algebraic convergence rates of both finite difference and element methods. In practice, this means a good accuracy and efficiency can be carry out with fairly coarse discretization [5,6,7,8]. The families of techniques known as the method of weighted residuals have been used broadly to perform approximate solutions of a wide classes of problems. Recently, the higher order differentiation matrices have been calculated and studied by using Chebyshev polynomials in Ref.[9,10,11].

This paper purposes to present new Legendre differentiation matrices formula as discrete approximations to derivatives. This formula is depended on using LP expressed in terms of power of collocation points $x \in [-1,1]$. We focus our consideration to the finite series of the LPs $P_n(x)$ $(n = 1,2,\ldots)$ and the approximation of $f(x)$ is constructing by:

$$(P_N f)(x) = \sum_{n=0}^{N} a_n P_n(x)$$

Gaussian quadrature is one of the numerical integration approximation methods. The essential problem of this method is that it needs computation of LPs to get accurate numerical solution. This method is generally needing to increase the order of polynomial if error bound doesn't satisfies. It is seldom used in practical problems and model as long as it's complicated computational efforts. However, the approximation by using LPs cause huge errors at the interval's boundaries.

The preference of the LPs is to be distinguished by the symmetry and sparsely properties of some types of special matrices. These matrices are the stiffness and mass matrices. The stiffness matrix will be reduced to the

* Corresponding author e-mail: mabdelhakem@yahoo.com

identity matrix. However, the other matrix presents a pentadiagonal profile such that the eigenproblem is most simple. Despite this, the matrices of the obtaining linear systems are generally sparse. So that the stiffness matrix has been turned to the pentadiagonal matrix. While the mass matrix has been turned to upper triangular matrix with nonzero elements for $i = j+2n$, $n \geq 0$ [6].

The paper is prepared as follows. In section 2, the definitions and common properties of LPs were introduced. In section 3, the differentiation Legendre matrices are investigated. In section 4, rounding error analysis is studied. In section 5, results of some test examples were presented. In section 6, applications are used to check the efficiency of the proposed method. Finally, remarks conclusions of the presented work will be introduced in the final section.

2 Preliminaries

LP is a powerful tool for function approximation. In this section, some useful notations and results concerning the LPs are introduced. The resulting interpolation polynomial minimizes the problem of Runge's phenomenon. This provides an approximation that is close to the polynomial of best approximation of a continuous function under the maximum norm.

Let $N \geq 1$ be an integer, and let $x_0, x_1, \cdots, x_N$ be a set of points in $[-1,1]$, where $-1 = x_0 < x_1 < \cdots < x_N = 1$. Takes these points as equal spaced over $[-1,1]$ defined as:

$$x_i = -1 + \frac{2i}{N}, \quad i = 0,1,\ldots,N \tag{1}$$

Throughout this work, we consider the LPs in terms of power of x as follows [1]:

$$P_n(x) = \sum_{k=0}^{[n/2]} \frac{(-1)^k (2n-2k)!}{2^n (n-k)!(n-2k)!k!} x^{n-2k} \tag{2}$$

where $[n/2] = \begin{cases} n/2 & n \text{ is even} \\ (n-1)/2 & n \text{ is odd} \end{cases}$

An important property of the LPs is that they are orthogonal with respect to the L_2 inner product on the interval $-1 \leq x \leq 1$ with the weight function $w(x) = 1$, i.e., [12]

$$\int_{-1}^{1} P_n(x) P_m(x) dx = \begin{cases} 0 & \text{if } n \neq m \\ \dfrac{2}{2n+1} & \text{if } n = m \end{cases} \tag{3}$$

The recurrence relation is [12]:

$$(n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x) = 0, \tag{4}$$

with the initial conditions: $P_0(x) = 1$ and $P_1(x) = x$. From equation (2), consider the LPs:

$$P_n(x) = \sum_{k=0}^{[n/2]} c_k^{(n)} x^{n-2k} \tag{5}$$

where $c_k^{(n)} = \dfrac{(-1)^k (2n-2k)!}{2^n (n-k)!(n-2k)!k!}$. The recurrence formulae which link pairs of coefficients are often more useful than explicit formulae for the coefficients of equation (5):

$$c_{k+1}^{(n)} = -\frac{(n-k)(n-2k)(n-2k-1)}{(k+1)(2n-2k)(2n-2k-1)} c_k^{(n)}, \tag{6}$$

where $c_0^{(n)} = \dfrac{(2n)!}{2^n (n!)^2}$ for $n \geq 0$ and $k \geq 1$.

Theorem 2.1. The m^{th} derivatives of the LPs are given by:

$$P_n^{(m)}(x) = \sum_{k=0}^{[(n-m)/2]} a_{k,m}^{(n)} x^{n-2k-m}, \quad m \geq 1 \tag{7}$$

where
$a_{k,m}^{(n)} = \frac{(-1)^k (2n-2k)!(n-2k)(n-2k-1)\cdots(n-2k-m+1)}{2^n (n-k)!(n-2k)!k!}$

The recurrence formulae of coefficients are found by:

$$a_{k+1,m}^{(n)} = -\frac{(n-2k-m)(n-2k-m-1)}{2(k+1)(2n-2k-1)} a_{k,m}^{(n)}, \tag{8}$$

with $a_{0,m}^{(n)} = \dfrac{(2n)!}{2^n n!(n-m)!}$ for $k \geq 0$ and $m, n \geq 1$.

Proof. We shall prove the theorem by induction. For $m = 1$ and from equation (5) we have,

$$\frac{d}{dx} P_n(x) = \sum_{k=0}^{[(n-1)/2]} a_{k,1}^{(n)} x^{n-2k-1}, \tag{9}$$

where $a_{k,1}^{(n)} = \dfrac{(-1)^k (2n-2k)!(n-2k)}{2^n (n-k)!(n-2k)!k!} = (n-2k) c_k^{(n)}$.

This relation is true because it is the same as first derivative of equation (7). Now, assume that equations (7) and (8) are true for arbitrary $m = l$. Thus,

$$P_n^{(l)}(x) = \sum_{k=0}^{[(n-l)/2]} a_{k,l}^{(n)} x^{n-2k-l} \tag{10}$$

The relation (7) at $m = l+1$ can be proved by differentiate equation (10). Hence, the proof is completed.

3 Higher Order Legendre Pseudospectral Differentiation Matrices

Let $f(x) \in C^{\infty}[-1,1]$ be approximated by Legendre finite expansion $P_j(x)$ at $(N+1)$ points define in (1), i.e.:

$$f_(x) = \sum_{j=0}^{N} \theta_j a_j P_j(x) \tag{11}$$

The coefficients in the continuous least-squares fitting are given by:

$$a_j = \frac{2j+1}{2}\int_{-1}^{1} P_j(x)f(x)dx, \quad j = 0,1,\cdots,N. \quad (12)$$

While the coefficients in the discrete least-squares fitting are given by:

$$a_j = \frac{2j+1}{2}\sum_{i=0}^{N}\theta_i P_j(x_i)f(x_i), \quad j = 0,1,\cdots,N. \quad (13)$$

where $\theta_0 = \theta_N = 1/2$ and $\theta_k, \quad k = 1,2,\cdots,N-1$.

We approximate the derivatives of a function $f(x)$ by interpolating the function with a polynomial at a set of $N+1$ equal spaced points. In the next theorem, we construct a global m^{th} derivative Legendre interpolating polynomial of the function $f(x)$.

Theorem 3.1. Let the derivative of $f(x)$ be approximated by LPs at the given point in equation (1), then the m^{th} differentiation matrix $D^m = [d^{(m)}_{i,j}]$, where $i,j = 0,1,\cdots,N$, satisfy:

$$\frac{d^m}{dx^m} f(x)\Big|_{x=x_i} = \sum_{j=0}^{N} d^{(m)}_{i,j} f(x_j) \quad (14)$$

where

$$d^{(m)}_{i,j} = \sum_{l=m}^{N}\sum_{k=0}^{[(j-m)/2]} \frac{2l+1}{N} P_l(x_i) a^{(l)}_{k,m} x_j^{l-2k-m}. \quad (15)$$

Proof. If $f(x)$ is approximated by (11) and (13), then

$$f(x) = \sum_{n=0}^{N}\sum_{j=0}^{N}\theta_j\theta_n \frac{2n+1}{N} P_n(x_j)P_n(x)f(x_j)$$

The m^{th} derivative of $f(x)$ can be approximated at $x = x_i$ as follow:

$$\frac{d^m}{dx^m} f(x)\Big|_{x=x_i} = \sum_{j=0}^{N}\left\{\sum_{n=0}^{N}\theta_j\theta_n\frac{2n+1}{N}P_n(x_j)P_n^{(m)}(x_i)\right\}f(x_j) = \sum_{j=0}^{N} d^{(m)}_{i,j} f(x_j)$$

where

$$d^{(m)}_{i,j} = \sum_{n=0}^{N}\theta_j\theta_n\frac{2n+1}{N}P_n(x_j)P_n^{(m)}(x_i) \quad (16)$$

The proof is completed by using theorem (2.1).

The matrix form is:

$$\left[\frac{d^m}{dx^m} f\right] = D^m [f] \quad (17)$$

where D^m is $(N+1)\times(N+1)$ squared matrix and their elements given by (15). The elements of the column matrix $[f]$ are given by $f(x_i)$, $i = 0,1,\cdots,N$. This method works with the roots of LPs and finding coefficients for using these roots. The main advantage of this formula is that it depends on exact values of power of x_i, $i = 0,1,\cdots,N$. Therefore, the formula (17) gives us an improvement of the results as shown in the numerical examples.

As we will see in section (6), the applications of this method to differential equations construct to a system of algebraic equations. Then, we form the coefficient matrix of this system. The first row and last row of the generated matrix are replaced by an appropriate formulation of the boundary conditions. By making a comparison with familiar methods, like the finite difference and finite elements, the presented found to be more accurate. This due to the approximation of the derivatives is defined over the whole domain.

4 Rounding Error Analysis

In this section we focus our attention on formula (15) since it turns out to be very useful. This formula can be rewritten simply to represent some derivatives as follow:

$$d^{(1)}_{i,l} = \frac{1}{N}\sum_{j=1}^{N}\sum_{k=0}^{[(j-1)/2]}\sum_{s=0}^{[j/2]}\gamma_s^{(j)} a^{(j)}_{k,1} x_i^{j-2s} x_l^{j-2k-1}. \quad (18)$$

where

$$a^{(j)}_{k,1} = \frac{(-1)^k(2j-2k)!(j-2k)}{2^j(j-k)!(j-2k)!k!},$$

and

$$\gamma_s^{(j)} = \frac{(-1)^s(2j+1)(2j-2s)!}{2^j(j-s)!(j-2s)!s!}$$

$$d^{(2)}_{i,l} = \frac{1}{N}\sum_{j=2}^{N}\sum_{k=0}^{[(j-2)/2]}\sum_{s=0}^{[j/2]}\gamma_s^{(j)} a^{(j)}_{k,2} x_i^{j-2s} x_l^{j-2k-2}. \quad (19)$$

where

$$a^{(j)}_{k,2} = \frac{(-1)^k(2j-2k)!(j-2k)(j-2k-1)}{2^j(j-k)!(j-2k)!k!}.$$

$$d^{(3)}_{i,l} = \frac{1}{N}\sum_{j=3}^{N}\sum_{k=0}^{[(j-3)/2]}\sum_{s=0}^{[j/2]}\gamma_s^{(j)} a^{(j)}_{k,3} x_i^{j-2s} x_l^{j-2k-3}. \quad (20)$$

where

$$a^{(j)}_{k,3} = \frac{(-1)^k(2j-2k)!(j-2k)(j-2k-1)(j-2k-2)}{2^j(j-k)!(j-2k)!k!}.$$

Now, proceeding to the investigation of the roundoff error effect on the elements $d_{i,l}$ in equations (18), (19) and (20). In finite precision arithmetic, we got ([13], [14]):

$$x_n^* = x_n + \delta_n$$

where δ_n presents a Infinitesimal error, with $|\delta_n|$ almost tends to machine precision ε and

$$\delta = \max_n |\{\delta_n\}|$$

we use the notation x_n^* for the exact value whereas x_n for the computed value. Since,

$$\left|x_n^* x_k^* - x_n x_k\right| = (\delta_n - \delta_k) - O\left(\frac{1}{N^2}\delta_n\right) - O\left(\frac{1}{N^2}\delta_k\right)$$

So, the absolute errors (AE) of the quantities $x_n x_k$ still as the order that shown in Ref. [10]

4.1 Bounds of the 1st Order Derivative Approximation

By using equation (18) to evaluate the errors in elements of the 1^{st} order differentiation matrix. The roundoff error found to be:

$$\begin{aligned} d_{i,l}^{(1)^*} - d_{i,l}^{(1)} &= \frac{1}{N}\sum_{j=1}^{N}\sum_{k=0}^{[(j-1)/2]}\sum_{s=0}^{[j/2]} \gamma_s^{(j)} a_{k,1}^{(j)} \{(\delta_i - \delta_l) \\ &\quad - O\left(\frac{1}{N^2}\delta_i\right) - O\left(\frac{1}{N^2}\delta_l\right)\} \\ &\le \frac{2}{N}\left(\delta - O\left(\frac{1}{N^2}\delta\right)\right)\sum_{j=1}^{N}\sum_{k=0}^{[(j-1)/2]}\sum_{s=0}^{[j/2]} \gamma_s^{(j)} a_{k,1}^{(j)} \\ &\le \frac{2}{N}\left(\delta - O\left(\frac{1}{N^2}\delta\right)\right)\left\{\frac{N}{6}(4N^2+9N+5)\right\} \\ &\le \left(\delta - O\left(\frac{1}{N^2}\delta\right)\right)\left\{\frac{4N^2+9N+5}{3}\right\} \end{aligned}$$

According to [13], the most important element is the element $d_{0,1}^{(1)}$. By comparing with the other elements, this element carries the major error responsibility. So, for equation (18) we have:

$$d_{0,1}^{(1)} = \frac{1}{N}\sum_{j=1}^{N}\sum_{k=0}^{[(j-1)/2]}\sum_{s=0}^{[j/2]} (-1)^{j-2s}\gamma_s^{(j)} a_{k,1}^{(j)} x_1^{j-2k-1}.$$

with error upper bound given by:

$$\begin{aligned} d_{0,1}^{(1)^*} - d_{0,1}^{(1)} &= \frac{1}{N}\sum_{j=1}^{N}\sum_{k=0}^{[(j-1)/2]}\sum_{s=0}^{[j/2]} (-1)^{j-2s}\gamma_s^{(j)} a_{k,1}^{(j)} \\ &\quad \times \left(x_1^{j-2k-1} - x_1^{(j-2k-1)^*}\right) \\ &= \frac{1}{N}\sum_{j=1}^{N}\sum_{k=0}^{[(j-1)/2]}\sum_{s=0}^{[j/2]} (-1)^{j-2s}\gamma_s^{(j)} a_{k,1}^{(j)} \delta_1 \\ &\le (-1)^N \delta\left\{N^2 + \frac{3}{2} + \frac{1}{4N}\right\} - \frac{\delta}{4N} \end{aligned}$$

The error in $d_{0,1}^{(1)}$ is clearly of order $\left(N^2\delta\right)$ whereas the error given in [13] is $\left(N^4\delta\right)$.

4.2 Bounds of the Second Order Derivative Approximation

By using equation (19) to evaluate the errors in elements of the 2^{nd} order differentiation matrix. The roundoff error found to be:

$$\begin{aligned} d_{i,l}^{(2)^*} - d_{i,l}^{(2)} &= \frac{1}{N}\sum_{j=2}^{N}\sum_{k=0}^{[(j-2)/2]}\sum_{s=0}^{[j/2]} \gamma_s^{(j)} a_{k,2}^{(j)} \{(\delta_i - \delta_l) \\ &\quad - O\left(\frac{1}{N^2}\delta_i\right) - O\left(\frac{1}{N^2}\delta_l\right)\} \\ &\le \frac{2}{N}\left(\delta - O\left(\frac{1}{N^2}\delta\right)\right)\sum_{j=2}^{N}\sum_{k=0}^{[(j-2)/2]}\sum_{s=0}^{[j/2]} \gamma_s^{(j)} a_{k,2}^{(j)} \\ &\le \left(\delta - O\left(\frac{1}{N^2}\delta\right)\right)\left\{\frac{3N^3+4N^2-3N-4}{3}\right\} \end{aligned}$$

Again, according to [13], the most important element is the element $d_{0,1}^{(2)}$. By comparing with the other elements, this element carries the major error responsibility. So, for equation (19) we have:

$$d_{0,1}^{(2)} = \frac{1}{N}\sum_{j=2}^{N}\sum_{k=0}^{[(j-2)/2]}\sum_{s=0}^{[j/2]} (-1)^{j-2s}\gamma_s^{(j)} a_{k,2}^{(j)} x_1^{j-2k-2}.$$

with error upper bound given by:

$$\begin{aligned} d_{0,1}^{(2)^*} - d_{0,1}^{(2)} &= \frac{1}{N}\sum_{j=2}^{N}\sum_{k=0}^{[(j-2)/2]}\sum_{s=0}^{[j/2]} (-1)^{j-2s}\gamma_s^{(j)} a_{k,2}^{(j)} \\ &\quad \times \left(x_1^{j-2k-2} - x_1^{(j-2k-2)^*}\right) \\ &= \frac{1}{N}\sum_{j=2}^{N}\sum_{k=0}^{[(j-2)/2]}\sum_{s=0}^{[j/2]} (-1)^{j-2s}\gamma_s^{(j)} a_{k,2}^{(j)} \delta_1 \\ &\le \delta\left\{\frac{3N^3+4N^2-3N-9}{6}\right\} \end{aligned}$$

The error in $d_{0,1}^{(2)}$ is clearly of order $\left(N^3\delta\right)$.

4.3 Bounds of the third Order Derivative Approximation

Similarly as the previous to sections and by using equation (20), the roundoff error of the third order differentiation matrix is:

$$\begin{aligned} d_{i,l}^{(3)^*} - d_{i,l}^{(3)} &\le \left(\delta - O\left(\frac{1}{N^2}\delta\right)\right)\frac{1}{10}\{8N^4 - 5N^3 \\ &\quad - 30N^2 + 5N + 22\} \end{aligned}$$

and

$$d_{0,1}^{(3)} = \frac{1}{N}\sum_{j=3}^{N}\sum_{k=0}^{[(j-3)/2]}\sum_{s=0}^{[j/2]} (-1)^{j-2s}\gamma_s^{(j)} a_{k,3}^{(j)} x_1^{j-2k-3}.$$

with error upper bound given by:

$$d_{0,1}^{(3)^*} - d_{0,1}^{(3)} \leq \delta \left\{ N^4 \right\}$$

The error in $d_{0,1}^{(3)}$ is clearly of order $\left(N^4\delta\right)$.

5 Numerical Tests

In this section, a straightforward implementation of the Legendre pseudospectral differentiation matrices is presented. Two test functions have been examined. Then, some ODEs examples have presented acorrding to formulae (18), (19) and (20) of the first three derivatives for test functions $y(x) = e^x$ and $y(x) = \sin \pi x$.

5.1 Test (1)

Consider the function $y(x) = e^x$. The Legendre pseudospectral differentiation matrices will be used to approximate the test function as follows:

$$D^m y(x_i) = \sum_{j=0}^{N} d_{i,j}^{(m)} y(x_j), m = 1,2,3 \text{ and } i = 0,1,\cdots,N.$$

Table 1: Errors $\left\|D^m y - y^{(m)}\right\|$, $m = 1,2,3$ for different implementations of $y(x) = e^x$.

N	$\|Dy-y'\|$	$\|D^2y-y''\|$	$\|D^3y-y'''\|$
6	8.72e-05	2.15e-03	3.35e-02
8	3.91-e07	1.69e-05	3.26e-04
10	1.09e-09	7.30e-08	2.19e-06
12	2.01e-12	1.99e-10	8.55e-09
14	3.79e-14	1.73e-12	4.27e-11

5.2 Test (2)

Consider the function $y(x) = \sin \pi x$. The same Legendre pseudospectral approximations will be used to obtain the following results.

Errors $\left\|D^m y - y^{(m)}\right\|$, $m = 1,2,3$ for tow implementations of Legendre pseudospectral differentiation matrices are presented in tables (1) and (2). the m^{th} derivative computed by multiplying the one column matrix $[f]$ by the derivatives matrix D^m. Table (1) represents the function $y(x) = e^x$,. It is clear that, the maximum absolute error (MAE) observed are due to roundoff. The second function $y(x) = \sin \pi x$ is more typical in Table (2). As in the first observe, the pseudospectral approximation roundoff error is again dominant at exponential convergence rate. We demonstrate that methods for setup those matrices will be effected from severe lack of accuracy. This lack done because the roundoff errors. Intuitively, one would suspect that computing the spectral differentiation matrix D in the most accurate way should lead to the best numerical results. It therefore comes as a surprise that simply using the Legendre pseudospectral differentiation matrices gives consistently the best results.

Table 2: Errors $\left\|D^m y - y^{(m)}\right\|$, $m = 1,2,3$ for different implementations of $y(x) = \sin \pi x$.

N	$\|Dy-y'\|$	$\|D^2y-y''\|$	$\|D^3y-y'''\|$
6	1.47e-01	3.49e+00	3.48e+01
8	7.45e-03	3.18e-01	5.92D+00
10	2.23e-04	1.49e-02	4.40e-01
12	4.41e-06	4.24e-04	1.82e-02
14	6.24e-08	8.16e-06	4.78e-04
16	6.63e-10	1.13e-07	8.66e-06
18	6.51e-12	1.23e-09	1.15e-07
20	5.99e-13	1.78e-10	1.77e-08

6 Applications

In this section, generally, the next examples show the accuracy of the spectral differentiation matrix.

6.1 Singularly Perturbed BVP

We show the efficiency of the proposed method using the following singularly perturbed second order BVP:

$$\left.\begin{array}{l} \varepsilon y'' - y' = 0.5, \quad -1 \leq x \leq 1, \\ y(-1) = 0, \\ y(1) = 0, \end{array}\right\} \tag{21}$$

with the exact solution is:

$$y = -\frac{1+x}{2} - \frac{e^{-2/\varepsilon} - e^{-(x-1)/\varepsilon}}{1 - e^{-2/\varepsilon}}.$$

Table 3: Observed maximum absolute error with $\varepsilon = 0.01$.

N	40	64	80	100	120
Errors	7.3e-05	2.4e-10	8.6e-12	1.1e-10	1.5e-09

In table (3), the problem(21) has been approximated for $\varepsilon = 0.01$ by a Legendre pseudospectral collocation method up to $N = 120$. Figure (1) shows the exact and numerical solutions.

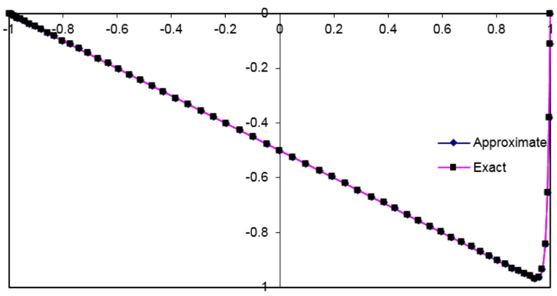

Fig. 1: Exact and Numerical solutions, N=64

6.2 Higher order BVP

Consider the following higher order BVP:

$$\left.\begin{array}{l} y^{(5)} = e^{-x}y^2, \quad 0 \leq x \leq 1 \\ y(0) = y^{(1)}(0) = y^{(2)}(0) = 1 \\ y(1) = y^{(1)}(1) = e \end{array}\right\} \quad (22)$$

The exact solution of the above BVP is given by $y = e^x$. We use the presented method and obtain the results as shown in table (4) for different selected values of N. Table (5) presents a comparison between the MAE obtained by using the presented method with the results of [15], [16] and [17]. It has been observed that our method is more efficient.

Table 4: Observed MAE for the BVP (22).

N	8	16	32	64	128
Errors	1.1e-07	3.7e-09	8.2e-11	1.0e-12	4.4e-13

Table 5: Comparison with other methods for BVP (22).

Methods	Errors
Decomposition method [17]	e-08
B-spline method [17]	e-04
Homotopy perturbation method [15]	e-08
Variational iteration method [16]	e-08
[18]	e-06

The results show that the Legendre pseudospectral collocation method is still efficient in both of singularly perturbed and higher order case.

7 Conclusion

Legendre pseudospectral collocation method haven seated up. Then, the roundoff errors of this method have been studied. To avoid these errors, some rows of the matrices must be replaced. We have pointed out that, however, the inaccurate standard formulas of the spectral differentiation matrices gave the most accurate way to approximate the derivative of a function. The shown examples proved the accuracy and efficiency of the presented matrix. Also, additional privileges as that our method is easy and reliable are shown.

Acknowledgment

The authors are grateful to the anonymous referee for a careful checking of the details and for helpful comments that improved this paper.

Conflict of Interest

The authors declare that there is no conflict of interest regarding the publication of this article.

References

[1] M. El-Kady and M. Biomy, Commun. Nonlinear Sci. Numer. Simul. **15(7)**, 1724-1739 (2010).
[2] J. Tang and H. Ma, Appl. Numer. Math. **57(1)**, 1-11 (2007).
[3] M. M.A. Khater, R. A. M. Attia, A. Abdel-Aty, W. Alharbi, D. Lu, Chaos, Solitons & Fractals, **136**, 10982 (2020).
[4] S. Owyed, M. A. Abdou, A. Abdel-Aty, A. A. Ibraheem, R. Nekhili, D. Baleanu, Journal of Intelligent and Fuzzy Systems, **38**, 2859-2866 (2020).
[5] J. Shen, SIAM J. Sci. Comput. **15(6)**, 1489-1505 (1994).
[6] J. Shen, SIAM J. Sci. Comput. **16(1)**, 74-87 (1995).
[7] J. Shen, SIAM J. Numer. Anal. **41(5)**, 1595-1619 (2003).
[8] Y. Chen and J. Zhou, Appl. Math. Comput. **268**, 1217-1226 (2015).
[9] M. Abdelhakem, A. Ahmed, and M. El-kady, Math. Sci. Lett. **8(2)**, 11-17 (2019).
[10] E. M. E. Elbarbary and S. M. El-Sayed, Appl. Numer. Math. **55(4)**, 425-438 (2005).
[11] M. El-Kady, H. Bakheet and M. Khalil, Int. J. Math. Comput. **1(8)**, 95-105 (2009).
[12] X. Tang and H. Xu, Appl. Numer. Math. **125**, 51-67 (2018).
[13] R. Baltensperger and M. R. Trummer, SIAM J. Sci. Comput. **24(5)**, 1465-1487 (2003).
[14] J. Shen, T. Tang, and L.-L. Wang, Spectral Methods: Algorithms, Analysis and Applications, Springer-Verlag, Berlin Heidelberg, 2011.
[15] M. Aslam Noor and S. T. Mohyud-Din, Math. Comput. Model. **54(7)**, 954-964 (2007).
[16] M. Aslam Noor and S. T. Mohyud-Din, Appl. Math. Comput. **189(2)**, 1929-1942 (2007).
[17] M. Aslam Noor and S. T. Mohyud-Din, J. Comput. Appl. Math. **136(1)**, 259-270 (2001).
[18] A. Aslanov, Math. Comput. Model. **55(3)**, 785-790 (2012).