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A Unified Transform Approach for Analyzing Fractional Differential Models with Diverse Fractional Operators

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Abstract: This study explores a class of fractional mathematical models through a comprehensive generalized integral transform technique. The models examined include fractional representations of Newton's cooling phenomenon, population dynamic governed by a logistic-type growth law, and a system describing blood alcohol concentration. To formulate these models, various fractional differentiation schemes are utilized, including Caputo, CF, modified ABC, and CPC derivatives. Closed form analytical solutions are derived using the generalized integral transform (GIT) framework, and the corresponding behaviors are illustrated numerically for different fractional orders. The obtained results align with established classical outcomes in limiting cases and further highlight the adaptability and robustness of the proposed transform method when applied to systems involving multiple fractional operators. These findings emphasize the potential of the generalized transform approach in capturing memory-dependent dynamics arising in real-world processes described by fractional differential equations.

Keywords: Integral transform; Caputo derivative; Caputo–Fabrizio derivative; modified ABC derivative; CPC derivative; Newton's law of cooling; Population growth; Logistic equation; Industry; Innovation and Infrastructure; Quality Education; Good Health and Well-Being; Mathematical Innovation for Sustainable Systems; Advanced Fractional Calculus Methods; Modeling of Complex and Multidisciplinary Systems.

1 Introduction

Fractional calculus is a powerful generalization of ordinary differentiation and integration to non-integer orders and has become a crucial tool in describing complex systems that exhibit memory and hereditary effects [1, 2]. Unlike classical derivatives that depend solely on local behaviour, fractional derivatives account for the historical evolution of the process, thereby offering more accurate and realistic models of scientific and engineering phenomena. Over the past few decades, numerous definitions of fractional derivatives have been developed, each characterized by distinct kernel functions and mathematical properties. Among the most widely used are Caputo, Caputo-Fabrizio with an exponential kernel [3], and Caputo sense modified Atangana-Baleanu with Mittag-Leffler kernel [4] and the constant proportional Caputo (CPC) [2, 5] derivatives. These operators extend the classical framework in different ways: the CF and mABC derivatives avoid singular kernels, while the CPC derivative combines proportionality with Caputo, modifying a hybrid structure suitable for modeling proportional processes. This study aligns with the United Nations Sustainable Development Goals by advancing Industry, Innovation, and Infrastructure through unified analytical methods for fractional differential

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models, supporting quality education via improved mathematical frameworks, and contributing to Good Health and Well-Being by strengthening the analysis of models used in real-world scientific, human health, and human disease applications.

The effectiveness of fractional operators is particularly evident in real-world dynamical models, where traditional integer-order combinations fail to capture memory-dependent behaviour [6–8]. One such ordinary model is *Newton's law of cooling*, which describes the rate at which a body exchanges heat with its surroundings:

$$\frac{dx}{dz} = -k(x - x_m), \quad x(0) = X_0, \tag{1}$$

where $x(z)$ represents the temperature of the object at time z, x_m is the ambient temperature, and k is the cooling constant. This model has wide-ranging applications in thermodynamics, engineering and science, where it assists in estimating cooling processes and even determining time of death in forensic investigations. When generalized using fractional derivatives, Newton's law of cooling captures non-local heat transfer effects and complex thermal memory phenomena more accurately.

Another important nonlinear system is the *logistic equation*, which modifies exponential growth by incorporating a carrying capacity:

$$\frac{dy}{dz} = -k(1 - y(z)), \quad y(0) = Y_0. \tag{2}$$

In this context, $y(z)$ represents the population at time z, while k denotes the intrinsic growth rate. This equation has found applications in different medical fields, where population growth slows as resources become limited. Extending the logistic equation to the fractional domain allows for biological memory inclusion and environmental feedback, leading to more realistic growth patterns observed in natural populations.

Similarly, the *blood alcohol model* is a two-compartment system describing the absorption and metabolism of alcohol in the body:

$$\frac{df}{dz} = -\ell_1 f(z), \quad f(0) = f_0 \tag{3}$$

$$\frac{dg}{dz} = \ell_1 f(z) - \ell_2 g(z), \quad g(0) = 0 \tag{4}$$

where $f(z)$ and $g(z)$ denote the alcohol concentrations in the stomach and blood, respectively, and ℓ_1 and ℓ_2 are positive constants representing absorption and elimination rates. Fractional extensions of this model capture the delayed metabolic response and memory effects inherent in physiological processes, thus improving the model's accuracy in biomedical and forensic applications.

In this work several, well known classical dynamical systems such as Newton's law of cooling, the logistic population growth model, and the blood alcohol concentration model are reformulated within a fractional order setting. This extension achieved by employing a generalized integral transform framework under different fractional derivatives, including Caputo, CF, modified ABC, and CPC operators. The developed methodology facilitates the construction of analytical solutions and enables a systematics comparison of system responses across a range of fractional orders. In addition, to recovering classical results as special cases, the proposed approach provides enhanced insights into how fractional parameters influence the evolution of the underlying dynamical systems. Readers can also see these related articles [9–14].

2 Prelimineries

Definition 1. *We define*

$$W = \{f(z) | \exists M, \tau_1, \tau_2 > 0, |f(z)| < Me^{|z|/\tau_j}, if\ z \in (-1)^j \times [0, \infty)\},$$

where W is a set of functions. The General Integral Transform (GIT) is given as follows:

– *[15] GIT of a function $f(z)$ is defined as*

$$G\{f(z), \hbar\} = F(\hbar) = \phi(\hbar) \int_0^\infty f(z) e^{-\psi(\hbar)z} dz. \tag{5}$$

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– *[15] Convolution property of GIT is*

$$G\{f(z)*g(z),\hbar\} = \frac{1}{\phi(\hbar)}F(\hbar)\cdot G(\hbar), \tag{6}$$

where $*$ is the convolution operation; $F(\hbar)$ and $G(\hbar)$ are the GITs of $f(z)$ and $g(z)$, respectively.

– *[15] GIT of n^{th} derivative of $f(z)$ is*

$$G\{f^n(z),\hbar\} = \psi(\hbar)^n F(\hbar) - \phi(\hbar)\sum_{k=0}^{n-1}\psi(\hbar)^{n-1-k}f^k(0),\ n\geq 1. \tag{7}$$

for $n=1$,

$$G\{f^{'}(z),\hbar\} = \psi(\hbar)F(\hbar) - \phi(\hbar)f(0). \tag{8}$$

– *[15] GIT of $exp(az)$ is*

$$G\{e^{az},\hbar\} = \frac{\phi(\hbar)}{\psi(\hbar)}\cdot\frac{1}{1-a\cdot\frac{1}{\psi(\hbar)}}, \tag{9}$$

therefore,

$$G\{1-e^{az},\hbar\} = \frac{\phi(\hbar)}{\psi(\hbar)}\cdot\frac{-a\cdot\frac{1}{\psi(\hbar)}}{1-a\cdot\frac{1}{\psi(\hbar)}} \tag{10}$$

– *[16] GIT of z^α is*

$$G\{z^\alpha,\hbar\} = \frac{\Gamma(\alpha+1)\phi(\hbar)}{\psi(\hbar)^{\alpha+1}}. \tag{11}$$

Definition 2. *[17] The Mittag–Leffler functions play a crucial role in fractional calculus and are introduced as*

–*The single-parameter Mittag–Leffler function is given by*

$$E_\alpha(z) = \sum_{\vartheta=0}^{\infty}\frac{z^\vartheta}{\Gamma(\alpha\vartheta+1)}, \tag{12}$$

where α is a complex number and $Re(\alpha)>0$.

–*The double-parameter Mittag–Leffler function is given by*

$$E_{\alpha,\beta}(z) = \sum_{\vartheta=0}^{\infty}\frac{z^\vartheta}{\Gamma(\alpha\vartheta+\beta)}, \tag{13}$$

where α and β are the complex numbers and $Re(\alpha)>0$.

Definition 3. *[18] The Caputo fractional derivative for $n=1$ of order α is*

$${}_0^C D_z^\alpha f(z) = \frac{1}{\Gamma(1-\alpha)}f^{'}(z)*z^{-\alpha},\ 0<\alpha<1. \tag{14}$$

Definition 4. *[17] The Caputo-Fabrizio (CF) fractional order derivative is defined as*

$${}_0^{CF} D_z^\alpha f(z) = \frac{\mathscr{L}(\alpha)}{1-\alpha}f^{'}(z)*exp(-\lambda z),\ 0<\alpha<1, \tag{15}$$

where $\lambda = \frac{\alpha}{1-\alpha}$.

Definition 5. *[4] Modified Atangana-Baleanu (mABC) fractional derivative of order α is*

$${}^{mABC}{}_0 D_z^\alpha f(z) = \frac{\mathscr{AB}(\alpha)}{1-\alpha}\left[f(z) - E_\alpha(-\lambda z^\alpha)f(0) - \lambda z^{\alpha-1}*E_{\alpha,\alpha}(-\lambda z^\alpha)f(z)\right], \tag{16}$$

where $\lambda = \frac{\alpha}{1-\alpha}$.

Definition 6. *[19] Constant proportional Caputo (CPC) fractional derivative of order α is*

$${}^{CPC}{}_0 D_z^\alpha f(z) = \frac{1}{\Gamma(1-\alpha)}\Big(R_1(\alpha)f(z) + R_0(\alpha)f^{'}(z)\Big)*z^{-\alpha}, \tag{17}$$

where $R_1(\alpha)$ and $R_0(\alpha)$ are the constants with respect to t, and depending upon α only.

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3 Generalized Integral Transform Involving Fractional Operators

Lemma 1. *[20] The GIT of a one-parameter (12) type ML function is*

$$G\{E_{\alpha}(-\lambda z^{\alpha})\} = \frac{\phi(\hbar)}{\psi(\hbar)} \frac{1}{1+\frac{\lambda}{\psi(\hbar)^{\alpha}}}. \tag{18}$$

Therefore,

$$G\{1-E_{\alpha}(-\lambda z^{\alpha})\} = \frac{\phi(\hbar)}{\psi(\hbar)} \frac{\lambda}{\psi(\hbar)^{\alpha}+\lambda}. \tag{19}$$

Lemma 2. *[16] The GIT of the Caputo fractional derivative is*

$$G\{{}_{0}^{C}D_{z}^{\alpha} f(z)\} = \psi(\hbar)^{\alpha}\left[F(\hbar) - \frac{\phi(\hbar)}{\psi(\hbar)} f(0)\right]. \tag{20}$$

Corollary 1.

– *When $\phi(\hbar) = 1$ and $\phi(\hbar) = \hbar$, equation (20) reduced to [21]*

$$G\{{}_{0}^{C}D_{z}^{\alpha} f(z)\} = \hbar^{\alpha}\left[L(\hbar) - \frac{1}{\hbar} f(0)\right], \tag{21}$$

where $L(\hbar)$ is the Laplace transform of $f(z)$.

– *When $\phi(\hbar) = \frac{\hbar}{\nu}$ and $\phi(\hbar) = \frac{\hbar}{\nu}$, equation (20) reduced to [10]*

$$G\{{}_{0}^{C}D_{z}^{\alpha} f(z)\} = \left(\frac{\hbar}{\nu}\right)^{\alpha}[F(\hbar,\nu) - f(0)], \tag{22}$$

where $F(\hbar,\nu)$ is the Formable transform of $f(z)$.

– *When $\phi(\hbar) = \frac{1}{\hbar}$ and $\phi(\hbar) = \frac{1}{\hbar}$, equation (20) reduced to [22]*

$$G\{{}_{0}^{C}D_{z}^{\alpha} f(z)\} = \frac{1}{\hbar^{\alpha}}[Su(\hbar) - f(0)], \tag{23}$$

where $Su(\hbar)$ is the Sumudu transform of $f(z)$.

– *When $\phi(\hbar) = \frac{1}{\hbar}$ and $\phi(\hbar) = \hbar$, equation (20) reduced to*

$$G\{{}_{0}^{C}D_{z}^{\alpha} f(z)\} = \hbar^{\alpha}\left[E(\hbar) - \frac{1}{\hbar^{2}} f(0)\right], \tag{24}$$

where $E(\hbar)$ is the Elzaki transform of $f(z)$.

– *When $\phi(\hbar) = \frac{1}{\hbar}$ and $\phi(\hbar) = \frac{1}{\hbar^{2}}$, equation (20) reduced to*

$$G\{{}_{0}^{C}D_{z}^{\alpha} f(z)\} = \frac{1}{\hbar^{2\alpha}}[Sa(\hbar) - \hbar f(0)], \tag{25}$$

where $Sa(\hbar)$ is the Sawi transform of $f(z)$.

Lemma 3. *The GIT of the CF fractional derivative is*

$$G\{{}_{0}^{CF}D_{z}^{\alpha} f(z)\} = \frac{\mathscr{L}(\alpha)}{1-\alpha+\alpha\frac{1}{\psi(\hbar)}}\left[F(\hbar) - \frac{\phi(\hbar)}{\psi(\hbar)} f(0)\right]. \tag{26}$$

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Proof. Applying the GIT (5) on the CF derivative (15), we get

$$\begin{aligned} G\{{}^{CF}_{0}D^{\alpha}_{z}f(z)\} &= \frac{\mathscr{L}(\alpha)}{1-\alpha}G\left[f'(z)*exp(-\lambda z)\right] \\ &= \frac{\mathscr{L}(\alpha)}{1-\alpha}\frac{1}{\phi(\hbar)}[\psi(\hbar)F(\hbar)-\phi(\hbar)f(0)]\cdot\frac{\phi(\hbar)}{\psi(\hbar)+\frac{\alpha}{1-\alpha}} \\ &= \frac{\mathscr{L}(\alpha)}{1-\alpha}\frac{1}{\phi(\hbar)}[\psi(\hbar)F(\hbar)-\phi(\hbar)f(0)]\cdot\frac{\phi(\hbar)}{\psi(\hbar)}\frac{1-\alpha}{1-\alpha+\frac{\alpha}{\psi(\hbar)}} \\ &= \frac{\mathscr{L}(\alpha)}{1-\alpha+\alpha\frac{1}{\psi(\hbar)}}\left[F(\hbar)-\frac{\phi(\hbar)}{\psi(\hbar)}f(0)\right]. \end{aligned}$$

Corollary 2.

–*When $\phi(\hbar)=1$ and $\phi(\hbar)=\hbar$, equation (26) reduced to [21]*

$$G\{{}^{CF}_{0}D^{\alpha}_{z}f(z)\} = \frac{\mathscr{L}(\alpha)}{1-\alpha+\alpha\frac{1}{\hbar}}\left[L(\hbar)-\frac{1}{\hbar}f(0)\right], \tag{27}$$

where $L(\hbar)$ is the Laplace transform of $f(z)$.

–*When $\phi(\hbar)=\frac{\hbar}{\nu}$ and $\phi(\hbar)=\frac{\hbar}{\nu}$, equation (26) reduced to [10]*

$$G\{{}^{CF}_{0}D^{\alpha}_{z}f(z)\} = \frac{\mathscr{L}(\alpha)}{1-\alpha+\alpha\frac{\nu}{\hbar}}[F(\hbar,\nu)-f(0)], \tag{28}$$

where $F(\hbar,\nu)$ is the Formable transform of $f(z)$.

–*When $\phi(\hbar)=\frac{1}{\hbar}$ and $\phi(\hbar)=\frac{1}{\hbar}$, equation (26) reduced to [22]*

$$G\{{}^{CF}_{0}D^{\alpha}_{z}f(z)\} = \frac{\mathscr{L}(\alpha)}{1-\alpha+\alpha\hbar}[Su(\hbar)-f(0)], \tag{29}$$

where $Su(\hbar)$ is the Sumudu transform of $f(z)$.

–*When $\phi(\hbar)=\frac{1}{\hbar}$ and $\phi(\hbar)=\hbar$, equation (26) reduced to [23]*

$$G\{{}^{CF}_{0}D^{\alpha}_{z}f(z)\} = \frac{\mathscr{L}(\alpha)}{1-\alpha+\alpha\frac{1}{\hbar}}\left[E(\hbar)-\frac{1}{\hbar^2}f(0)\right], \tag{30}$$

where $E(\hbar)$ is the Elzaki transform of $f(z)$.

–*When $\phi(\hbar)=\frac{1}{\hbar}$ and $\phi(\hbar)=\frac{1}{\hbar^2}$, equation (26) reduced to*

$$G\{{}^{CF}_{0}D^{\alpha}_{z}f(z)\} = \frac{\mathscr{L}(\alpha)}{1-\alpha+\alpha\hbar^2}[Sa(\hbar)-\hbar f(0)], \tag{31}$$

where $Sa(\hbar)$ is the Sawi transform of $f(z)$.

Lemma 4. *The GIT of the modified ABC fractional derivative is*

$$G\{{}^{mABC}{}_{0}D^{\alpha}_{z}f(z)\} = \frac{\mathscr{AB}(\alpha)}{1-\alpha+\alpha\psi(\hbar)^{-\alpha}}\left[F(\hbar)-\frac{\phi(\hbar)}{\psi(\hbar)}f(0)\right]. \tag{32}$$

Proof. Applying the GIT (5) on the modified ABC derivative (16), we get

$$\begin{aligned} &G\{{}^{mABC}{}_{0}D^{\alpha}_{z}f(z)\} \\ &= \frac{\mathscr{AB}(\alpha)}{1-\alpha}G[f(z)-E_{\alpha}(-\lambda z^{\alpha})f(0)] - \frac{\mathscr{AB}(\alpha)}{1-\alpha}G\left[\lambda z^{\alpha-1}E_{\alpha,\alpha}(-\lambda z^{\alpha})*f(z)\right] \\ &= \frac{\mathscr{AB}(\alpha)}{1-\alpha}\left[F(\hbar)-\frac{\phi(\hbar)}{\psi(\hbar)}\frac{1}{1+\frac{\lambda}{\psi(\hbar)^{\alpha}}}f(0)-\lambda G\sum_{k=0}^{\infty}(-\lambda)^k(I_0^{\alpha k+\alpha}f)(z)\right] \\ &= \frac{\mathscr{AB}(\alpha)}{1-\alpha}\left[F(\hbar)-\frac{\phi(\hbar)}{\psi(\hbar)}\frac{1}{1+\frac{\alpha}{1-\alpha}\frac{1}{\psi(\hbar)^{\alpha}}}f(0)+\sum_{k=0}^{\infty}(-\lambda)^{k+1}\frac{F(\hbar)}{\psi(\hbar)^{\alpha k+\alpha}}\right] \end{aligned}$$

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$$= \frac{\mathscr{A}\mathscr{B}(\alpha)}{1-\alpha}\left[F(\hbar) - \frac{\phi(\hbar)}{\psi(\hbar)}\frac{1}{1+\frac{\alpha}{1-\alpha}\frac{1}{\psi(\hbar)^{\alpha}}}f(0) + \sum_{k=0}^{\infty}\left(\frac{-\lambda}{\psi(\hbar)^{\alpha}}\right)^{k+1}F(\hbar)\right]$$

$$= \frac{\mathscr{A}\mathscr{B}(\alpha)}{1-\alpha}\left[F(\hbar) - \frac{\phi(\hbar)}{\psi(\hbar)}\frac{1}{1+\frac{\alpha}{1-\alpha}\frac{1}{\psi(\hbar)^{\alpha}}}f(0) - \frac{\lambda}{\psi(\hbar)^{\alpha}+\lambda}F(\hbar)\right],$$

on simplifying by taking $\lambda = \frac{\alpha}{1-\alpha}$, we reach

$$G\{^{mABC}{}_{0}D_{z}^{\alpha}f(z)\} = \frac{\mathscr{A}\mathscr{B}(\alpha)}{1-\alpha}\left(\frac{F(\hbar)-\frac{\phi(\hbar)}{\psi(\hbar)}f(0)}{1-\alpha+\alpha\psi(\hbar)^{-\alpha}}\right)(1-\alpha).$$

This completes the proof.

Corollary 3.

–*When* $\phi(\hbar) = 1$ *and* $\phi(\hbar) = \hbar$, *equation* (32) *reduced to* [24]

$$G\{^{mABC}{}_{0}D_{z}^{\alpha}f(z)\} = \frac{\mathscr{A}\mathscr{B}(\alpha)}{1-\alpha+\alpha\frac{1}{\hbar^{\alpha}}}\left[L(\hbar) - \frac{1}{\hbar}f(0)\right], \tag{33}$$

where $L(\hbar)$ *is the Laplace transform of* $f(z)$.

–*When* $\phi(\hbar) = \frac{\hbar}{\nu}$ *and* $\phi(\hbar) = \frac{\hbar}{\nu}$, *equation* (32) *reduced to* [10]

$$G\{^{mABC}{}_{0}D_{z}^{\alpha}f(z)\} = \frac{\mathscr{A}\mathscr{B}(\alpha)}{1-\alpha+\alpha\left(\frac{\nu}{\hbar}\right)^{\alpha}}[F(\hbar,\nu) - f(0)], \tag{34}$$

where $F(\hbar,\nu)$ *is the Formable transform of* $f(z)$.

–*When* $\phi(\hbar) = \frac{1}{\hbar}$ *and* $\phi(\hbar) = \frac{1}{\hbar}$, *equation* (32) *reduced to* [22]

$$G\{^{mABC}{}_{0}D_{z}^{\alpha}f(z)\} = \frac{\mathscr{A}\mathscr{B}(\alpha)}{1-\alpha+\alpha\hbar^{\alpha}}[Su(\hbar) - f(0)], \tag{35}$$

where $Su(\hbar)$ *is the Sumudu transform of* $f(z)$.

–*When* $\phi(\hbar) = \frac{1}{\hbar}$ *and* $\phi(\hbar) = \hbar$, *equation* (32) *reduced to*

$$G\{^{mABC}{}_{0}D_{z}^{\alpha}f(z)\} = \frac{\mathscr{A}\mathscr{B}(\alpha)}{1-\alpha+\alpha\frac{1}{\hbar^{\alpha}}}\left[E(\hbar) - \frac{1}{\hbar^{2}}f(0)\right], \tag{36}$$

where $E(\hbar)$ *is the Elzaki transform of* $f(z)$.

–*When* $\phi(\hbar) = \frac{1}{\hbar}$ *and* $\phi(\hbar) = \frac{1}{\hbar^{2}}$, *equation* (32) *reduced to*

$$G\{^{mABC}{}_{0}D_{z}^{\alpha}f(z)\} = \frac{\mathscr{A}\mathscr{B}(\alpha)}{1-\alpha+\alpha\hbar^{2\alpha}}[Sa(\hbar) - \hbar f(0)], \tag{37}$$

where $Sa(\hbar)$ *is the Sawi transform of* $f(z)$.

Lemma 5. *[16] The GIT of the CPC fractional derivative is*

$$G\{^{CPC}{}_{0}D_{z}^{\alpha}f(z)\} = \left[R_{1}(\alpha)\psi(\hbar)^{\alpha-1} + R_{0}(\alpha)\psi(\hbar)^{\alpha}\right]F(\hbar) - R_{0}(\alpha)\phi(\hbar)^{\alpha-1}\psi(\hbar)f(0). \tag{38}$$

Corollary 4.

When $\phi(\hbar) = 1$ *and* $\phi(\hbar) = \hbar$, *equation* (38) *reduced to* [16]

$$G\{^{CPC}{}_{0}D_{z}^{\alpha}f(z)\} = \left[\frac{R_{1}(\alpha)}{\hbar} + R_{0}(\alpha)\right]\frac{L(\hbar)}{\hbar^{-\alpha}} - \frac{R_{0}(\alpha)}{\hbar^{1-\alpha}}f_{0}, \tag{39}$$

where $L(\hbar)$ *is the Laplace transform of* $f(z)$.

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–*When* $\phi(\hbar) = \frac{\hbar}{\nu}$ *and* $\phi(\hbar) = \frac{\hbar}{\nu}$, *equation* (38) *reduced to* [10]

$$G\{{}^{CPC}_{0}D^{\alpha}_{z} f(z)\} = \left[\frac{R_1(\alpha)}{\hbar/\nu} + R_0(\alpha)\right] \frac{F(\hbar,\nu)}{(\hbar/\nu)^{-\alpha}} - \frac{R_0(\alpha)}{(\hbar/\nu)^{-\alpha}} f_0, \tag{40}$$

where $F(\hbar,\nu)$ *is the Formable transform of* $f(z)$.

–*When* $\phi(\hbar) = \frac{1}{\hbar}$ *and* $\phi(\hbar) = \frac{1}{\hbar}$, *equation* (38) *reduced to* [22]

$$G\{{}^{CPC}_{0}D^{\alpha}_{z} f(z)\} = \left[\frac{R_1(\alpha)}{\hbar^{-1}} + R_0(\alpha)\right] \frac{Su(\hbar)}{\hbar^{\alpha}} - \frac{R_0(\alpha)}{\hbar^{\alpha}} f_0, \tag{41}$$

where $Su(\hbar)$ *is the Sumudu transform of* $f(z)$.

–*When* $\phi(\hbar) = \frac{1}{\hbar}$ *and* $\phi(\hbar) = \hbar$, *equation* (38) *reduced to*

$$G\{{}^{CPC}_{0}D^{\alpha}_{z} f(z)\} = \left[\frac{R_1(\alpha)}{\hbar} + R_0(\alpha)\right] \frac{E(\hbar)}{\hbar^{-\alpha}} - \frac{R_0(\alpha)}{\hbar^{2-\alpha}} f_0, \tag{42}$$

where $E(\hbar)$ *is the Elzaki transform of* $f(z)$.

–*When* $\phi(\hbar) = \frac{1}{\hbar}$ *and* $\phi(\hbar) = \frac{1}{\hbar^2}$, *equation* (38) *reduced to*

$$G\{{}^{CPC}_{0}D^{\alpha}_{z} f(z)\} = \left[\frac{R_1(\alpha)}{\hbar^{-2}} + R_0(\alpha)\right] \frac{Sa(\hbar)}{\hbar^{2\alpha}} - \frac{R_0(\alpha)}{\hbar^{2\alpha-1}} f_0, \tag{43}$$

where $Sa(\hbar)$ *is the Sawi transform of* $f(z)$.

4 Fractional Newton's Law of Cooling

This section analyze the Newton's cooling model by utilizing several types of fractional derivatives and present their corresponding solutions.

Problem 1. We investigate the fractional form of Newton's cooling model (1), expressed by the Caputo operator

$${}^{C}_{0}D^{\alpha}_{t} x(z) = -k(x - x_m), \tag{44}$$

which has the following solution:

$$x(z) = 1 - E_{\alpha}(-kz^{\alpha})x_m + E_{\alpha}(-kz^{\alpha})X_0. \tag{45}$$

Proof. Applying the GIT (5) on (44), we get

$$\psi(\hbar)^{\alpha}\left[X(\hbar) - \frac{\phi(\hbar)}{\psi(\hbar)X_0}\right] = -k[X(\hbar) - x_m]$$
$$[\psi(\hbar)^{\alpha} + k]X(s) = \frac{\phi(\hbar)}{\psi(\hbar)} kx_m + \frac{\phi(\hbar)}{\psi(\hbar)^{1-\alpha}} X_0$$

on simplifying

$$X(\hbar) = \frac{\frac{\phi(\hbar)}{\psi(\hbar)} kx_m + \frac{\phi(\hbar)}{\psi(\hbar)^{1-\alpha}} X_0}{\psi(\hbar)^{\alpha} + k}$$
$$= \frac{\phi(\hbar)}{\psi(\hbar)} \frac{kx_m}{\psi(\hbar)^{\alpha} + k} + \frac{\psi(\hbar)}{\psi(\hbar)} \frac{\psi(\hbar)^{\alpha}}{\psi(\hbar)^{\alpha} + k} X_0.$$

On applying the inverse of GIT, we arrive at (45).

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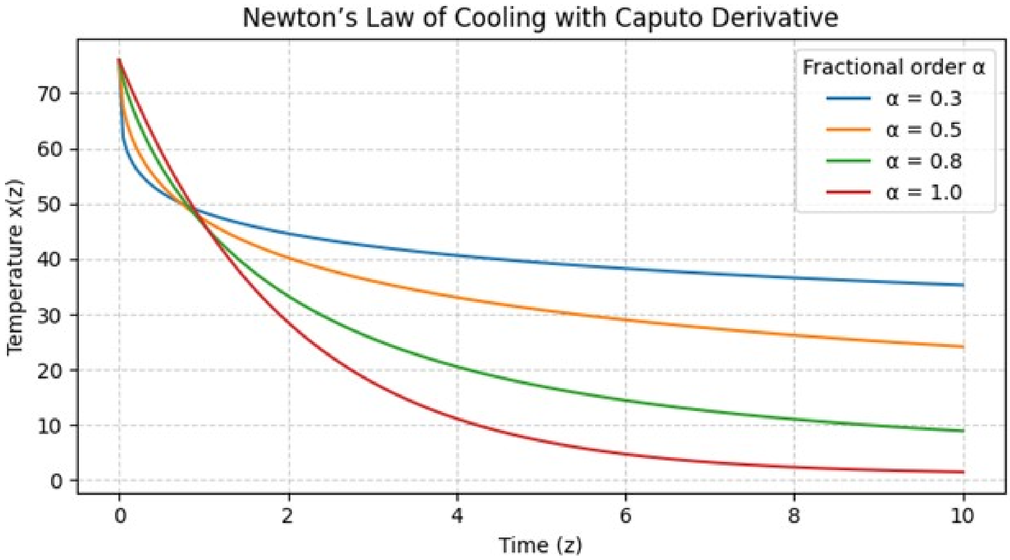

Figure 1 Caputo type Newton's cooling model for different values of α

Problem 2. We investigate the fractional form of Newton's cooling model (1), expressed by the CF operator

$$ {}^{CF}_{0}D^{\alpha}_{t}x(z) = -k(x - x_m), \tag{46} $$

which has the forllowing solution:

$$ \begin{aligned} x(z) = &\frac{\mathscr{Z}(\alpha)X_0}{\mathscr{Z}(\alpha)+k(1-\alpha)} exp\left(\frac{-k\alpha}{\mathscr{Z}(\alpha)+k(1-\alpha)}z\right) \\ &+ \frac{k(1-\alpha)x_m}{\mathscr{Z}(\alpha)+k(1-\alpha)} exp\left(\frac{-k\alpha}{\mathscr{Z}(\alpha)+k(1-\alpha)}z\right) \\ &+ x_m\left[1 - exp\left(\frac{-k\alpha}{\mathscr{Z}(\alpha)+k(1-\alpha)}z\right)\right]. \end{aligned} \tag{47} $$

Proof. Applying the GIT (5) on (46), we get

$$ \frac{\mathscr{Z}(\alpha)}{1-\alpha+\alpha\frac{1}{\psi(\hbar)}}X(\hbar) - \frac{\psi(\hbar)}{\psi(\hbar)}\frac{\mathscr{Z}(\alpha)}{1-\alpha+\alpha\frac{1}{\psi(\hbar)}}X_0 = -kX(\hbar) + k\frac{\phi(\hbar)}{\psi(\hbar)}x_m $$

$$ \left[\frac{\mathscr{Z}(\alpha)}{1-\alpha+\alpha\frac{1}{\psi(\hbar)}} + k\right]X(\hbar) = \frac{\psi(\hbar)}{\psi(\hbar)}\frac{\mathscr{Z}(\alpha)}{1-\alpha+\alpha\frac{1}{\psi(\hbar)}}X_0 + k\frac{\phi(\hbar)}{\psi(\hbar)}x_m $$

$$ \frac{\mathscr{Z}(\alpha)+k(1-\alpha)+\frac{k\alpha}{\psi(\hbar)}}{1-\alpha+\alpha\frac{1}{\psi(\hbar)}}X(\hbar) = \frac{\phi(\hbar)}{\psi(\hbar)}\left[\frac{\mathscr{Z}(\alpha)X_0 + k(1-\alpha)x_m + \frac{k\alpha}{\psi(\hbar)}x_m}{1-\alpha+\alpha\frac{1}{\psi(\hbar)}}\right], $$

on simplifying

$$ X(\hbar) = \frac{\phi(\hbar)}{\psi(\hbar)}\left[\frac{\mathscr{Z}(\alpha)X_0 + k(1-\alpha)x_m + \frac{k\alpha}{\psi(\hbar)}x_m}{\mathscr{Z}(\alpha)+k(1-\alpha)+\frac{k\alpha}{\psi(\hbar)}}\right] $$

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$$X(\hbar)=\frac{\phi(\hbar)}{\psi(\hbar)}\frac{\mathscr{Z}(\alpha)X_0}{\mathscr{Z}(\alpha)+k(1-\alpha)}\frac{1}{1+\frac{k\alpha}{\mathscr{Z}(\alpha)+k(1-\alpha)}\frac{1}{\psi(\hbar)}}
+\frac{\phi(\hbar)}{\psi(\hbar)}\frac{k(1-\alpha)x_m}{\mathscr{Z}(\alpha)+k(1-\alpha)}\frac{1}{1+\frac{k\alpha}{\mathscr{Z}(\alpha)+k(1-\alpha)}\frac{1}{\psi(\hbar)}}
+x_m\frac{\phi(\hbar)}{\psi(\hbar)}\frac{\frac{k\alpha}{\mathscr{Z}(\alpha)+k(1-\alpha)}\frac{1}{\psi(\hbar)}}{1+\frac{k\alpha}{\mathscr{Z}(\alpha)+k(1-\alpha)}\frac{1}{\psi(\hbar)}}.$$

Now, by applying the inverse of GIT (5), we arrive at (47).

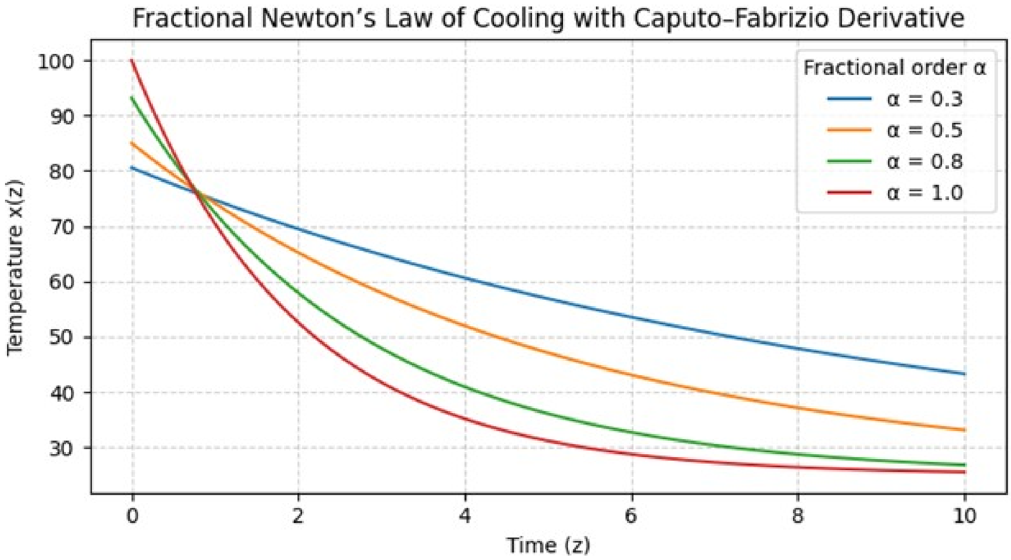

Figure 2 CF type Newton's cooling model for different values of α

Problem 3. We investigate the fractional form of Newton's cooling model (1), expressed by the modified ABC derivative

$$^{mABC}{}_0D_t^{\alpha}x(z)=-k(x-x_m), \tag{48}$$

which has the following solution:

$$x(z)=\frac{\mathscr{AB}(\alpha)X_0}{\mathscr{AB}(\alpha)+k(1-\alpha)}E_\alpha\left(\frac{-k\alpha}{\mathscr{AB}(\alpha)+k(1-\alpha)}z^\alpha\right)
+\frac{k(1-\alpha)x_m}{\mathscr{AB}(\alpha)+k(1-\alpha)}E_\alpha\left(\frac{-k\alpha}{\mathscr{AB}(\alpha)+k(1-\alpha)}z^\alpha\right)
+x_m\left[1-E_\alpha\left(\frac{-k\alpha}{\mathscr{AB}(\alpha)+k(1-\alpha)}z^\alpha\right)\right]. \tag{49}$$

Proof. Applying GIT (5) on both sides of (48), we get

$$\frac{\mathscr{AB}(\alpha)}{1-\alpha+\alpha\psi(\hbar)^{-\alpha}}\left[X(\hbar)-\frac{\phi(\hbar)}{\psi(\hbar)}X_0\right]=-kX(\hbar)+k\frac{\phi(\hbar)}{\psi(\hbar)}x_m$$

$$\left[\frac{\mathscr{AB}(\alpha)}{1-\alpha+\alpha\psi(\hbar)^{-\alpha}}+k\right]X(\hbar)=\frac{\phi(\hbar)}{\psi(\hbar)}\frac{\mathscr{AB}(\alpha)X_0}{1-\alpha+\alpha\psi(\hbar)^{-\alpha}}+k\frac{\phi(\hbar)}{\psi(\hbar)}x_m$$

$$\left[\frac{\mathscr{AB}(\alpha)+k(1-\alpha)+k\alpha\psi(\hbar)^{-\alpha}}{1-\alpha+\alpha\psi(\hbar)^{-\alpha}}\right]X(\hbar)=\frac{\phi(\hbar)}{\psi(\hbar)}\frac{\mathscr{AB}(\alpha)X_0}{1-\alpha+\alpha\psi(\hbar)^{-\alpha}}+k\frac{\phi(\hbar)}{\psi(\hbar)}x_m$$

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$$X(\hbar)=\frac{\phi(\hbar)}{\psi(\hbar)}\frac{\mathscr{AB}(\alpha)X_0}{\mathscr{AB}(\alpha)+k(1-\alpha)+k\alpha\psi(\hbar)^{-\alpha}}+\frac{\phi(\hbar)}{\psi(\hbar)}\frac{kx_m(1-\alpha+\alpha\psi(\hbar)^{-\alpha})}{\mathscr{AB}(\alpha)+k(1-\alpha)+k\alpha\psi(\hbar)^{-\alpha}}.$$

On simplification, this implies

$$\begin{aligned}X(\hbar)=&\frac{\phi(\hbar)}{\psi(\hbar)}\frac{\mathscr{AB}(\alpha)X_0}{\mathscr{AB}(\alpha)+k(1-\alpha)}\frac{1}{1+\frac{k\alpha}{\mathscr{AB}(\alpha)+k(1-\alpha)}\psi(\hbar)^{-\alpha}}\\&+\frac{\phi(\hbar)}{\psi(\hbar)}\frac{kx_m(1-\alpha)}{\mathscr{AB}(\alpha)+k(1-\alpha)}\frac{1}{1+\frac{k\alpha}{\mathscr{AB}(\alpha)+k(1-\alpha)}\psi(\hbar)^{-\alpha}}\\&+\frac{\phi(\hbar)}{\psi(\hbar)}x_m\frac{\frac{k\alpha}{\mathscr{AB}(\alpha)+k(1-\alpha)}\psi(\hbar)^{-\alpha}}{1+\frac{k\alpha}{\mathscr{AB}(\alpha)+k(1-\alpha)}\psi(\hbar)^{-\alpha}}.\end{aligned}$$

Now, by using the inverse of GIT (5), we arrive at (49).

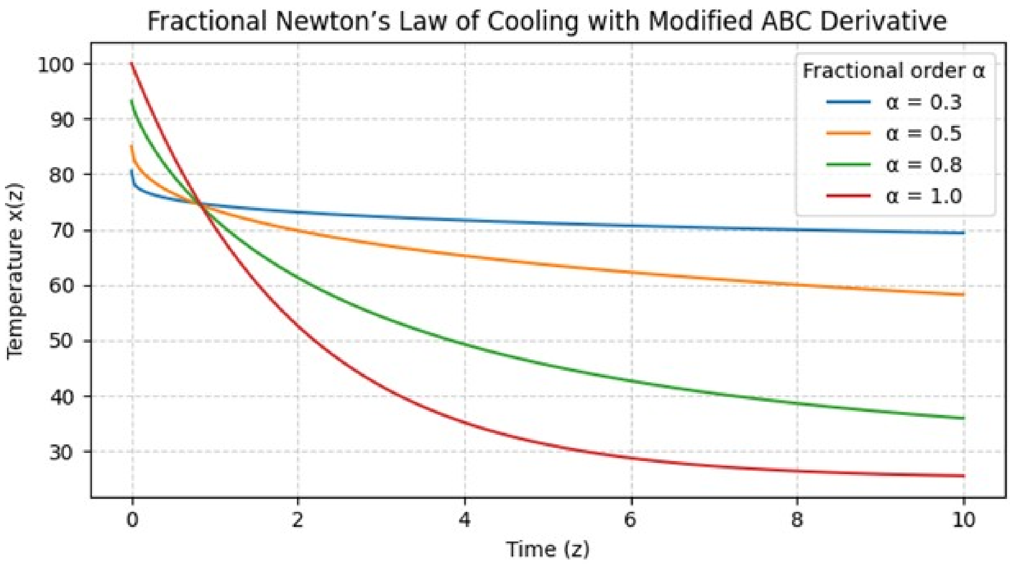

Figure 3 mABC type Newton's cooling model for different values of α

Problem 4. We investigate the fractional form of Newton's cooling model (1), expressed by the CPC derivative

$$^{CPC}{}_0D_t^{\alpha}x(z)=-k(x-x_m), \tag{50}$$

which has the following solution:

$$x(z)=x_mE^1_{1-\alpha,-\alpha,1}\left(\frac{-R_1(\alpha)}{k}z^{1-\alpha},\frac{-R_0(\alpha)}{k}z^{-\alpha}\right)+X_0E^1_{1,\alpha,1}\left(\frac{-R_1(\alpha)}{R_0(\alpha)}z,\frac{-k}{R_0(\alpha)}z^{\alpha}\right). \tag{51}$$

Proof. Applying GIT (5) on both sides of (50), we get

$$[R_1(\alpha)\psi(\hbar)^{\alpha-1}+R_0(\alpha)\psi(\hbar)^{\alpha}]X(\hbar)-R_0(\alpha)\psi(\hbar)^{\alpha-1}\phi(\hbar)X_0=-kX(\hbar)+k\frac{\phi(\hbar)}{\psi(\hbar)}x_m$$

$$[R_1(\alpha)\psi(\hbar)^{\alpha-1}+R_0(\alpha)\psi(\hbar)^{\alpha}+k]X(\hbar)=k\frac{\phi(\hbar)}{\psi(\hbar)}x_m+R_0(\alpha)\psi(\hbar)^{\alpha-1}\phi(\hbar)X_0,$$

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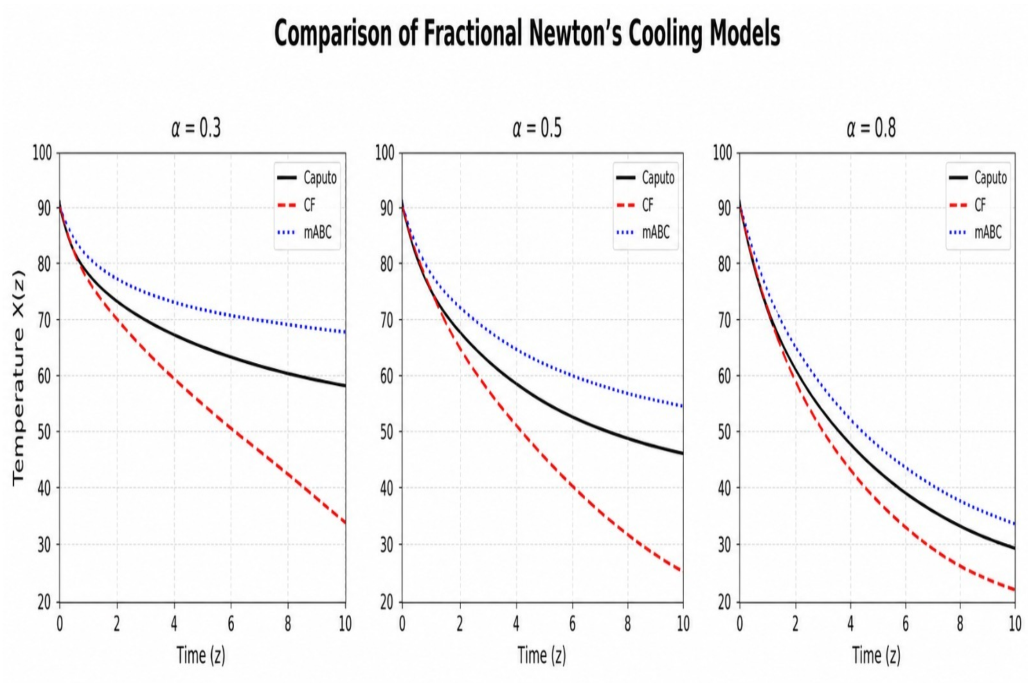

Figure 4 A comparison of Caputo, CF and mABC type Newton's law of cooling models for different values of $\boldsymbol{\alpha}$

on simplification

$$X(\hbar) = \frac{\phi(\hbar)}{\psi(\hbar)} \frac{kx_m}{k\left[1+\frac{R_1(\alpha)\psi(\hbar)^{\alpha-1}+R_0(\alpha)\psi(\hbar)^{\alpha}}{k}\right]} + \frac{\phi(\hbar)}{\psi(\hbar)} \frac{R_0(\alpha)X_0}{R_0(\alpha)\left[1+\frac{R_1(\alpha)\psi(\hbar)^{-1}+k\psi(\hbar)^{-\alpha}}{R_0(\alpha)}\right]}$$

$$= x_m \frac{\phi(\hbar)}{\psi(\hbar)} \sum_{\vartheta=0}^{\infty} \left[-\frac{R_1(\alpha)\psi(\hbar)^{\alpha-1}+R_0(\alpha)\psi(\hbar)^{\alpha}}{k}\right]^{\vartheta}$$

$$+ X_0 \frac{\phi(\hbar)}{\psi(\hbar)} \sum_{\varkappa=0}^{\infty} \left[-\frac{R_1(\alpha)\psi(\hbar)^{-1}+k\psi(\hbar)^{-\alpha}}{R_0(\alpha)}\right]^{\varkappa}$$

$$= x_m \frac{\phi(\hbar)}{\psi(\hbar)} \sum_{\vartheta=0}^{\infty} \left(\frac{-1}{k}\right)^{\vartheta} (R_1(\alpha)\psi(\hbar)^{\alpha-1}+R_0(\alpha)\psi(\hbar)^{\alpha})^{\vartheta}$$

$$+ X_0 \frac{\phi(\hbar)}{\psi(\hbar)} \sum_{\varkappa=0}^{\infty} \left(\frac{-1}{R_0(\alpha)}\right)^{\varkappa} (R_1(\alpha)\psi(\hbar)^{-1}+k\psi(\hbar)^{-\alpha})^{\varkappa}$$

$$= x_m \frac{\phi(\hbar)}{\psi(\hbar)} \sum_{\vartheta=0}^{\infty} \left(\frac{-1}{k}\right)^{\vartheta} \sum_{a=0}^{\vartheta} \binom{\vartheta}{a} (R_1(\alpha)\psi(\hbar)^{\alpha-1})^{\vartheta-a} (R_0(\alpha)\psi(\hbar)^{\alpha})^{a}$$

$$+ X_0 \frac{\phi(\hbar)}{\psi(\hbar)} \sum_{\varkappa=0}^{\infty} \left(\frac{-1}{R_0(\alpha)}\right)^{\varkappa} \sum_{b=0}^{\varkappa} \binom{\varkappa}{b} (R_1(\alpha)\psi(\hbar)^{-1})^{\varkappa-b} (k\psi(\hbar)^{-\alpha})^{b}$$

$$= x_m \sum_{\vartheta=0}^{\infty} \left(\frac{-1}{k}\right)^{\vartheta} \sum_{a=0}^{\vartheta} \binom{\vartheta}{a} (R_1(\alpha))^{\vartheta-a} (R_0(\alpha))^{a} \phi(\hbar)\psi(\hbar)^{\vartheta\alpha+a-\vartheta-1}$$

$$+ X_0 \sum_{\varkappa=0}^{\infty} \left(\frac{-1}{R_0(\alpha)}\right)^{\varkappa} \sum_{b=0}^{\varkappa} \binom{\varkappa}{b} (R_1(\alpha))^{\varkappa-b} (k)^{b} \phi(\hbar)\psi(\hbar)^{-\varkappa+b-b\alpha-1}.$$

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Now, by using the inverse of GIT, we get

$$x(z) = x_m \sum_{\vartheta=0}^{\infty} \left(\frac{-1}{k}\right)^{\vartheta} \sum_{a=0}^{\vartheta} \frac{\vartheta!}{(\vartheta-a)!a!}(R_1(\alpha))^{\vartheta-a}(R_0(\alpha))^a \phi(\hbar)\psi(\hbar)^{\vartheta\alpha+a-\vartheta-1}\frac{z^{\vartheta-a-\vartheta\alpha}}{\Gamma(\vartheta-a-\vartheta\alpha+1)}$$
$$+X_0 \sum_{\varkappa=0}^{\infty} \left(\frac{-1}{R_0(\alpha)}\right)^{\varkappa} \sum_{b=0}^{\varkappa} \frac{\varkappa!}{(\varkappa-b)!b!}(R_1(\alpha))^{\varkappa-b}(k)^b \phi(\hbar)\psi(\hbar)^{-\varkappa+b-b\alpha-1}\frac{z^{\varkappa-b+b\alpha}}{\Gamma(\varkappa-b+b\alpha+1)},$$

by assigning the parameters $\vartheta - a = p$ and $\varkappa - b = q$, respectively, the resulting expression can be written as follows:

$$\begin{aligned}
x(z) &= x_m \sum_{a=0}^{\infty}\sum_{p=0}^{\infty} \left(\frac{-1}{k}\right)^{a+p} \frac{(a+p)!}{p!a!}(R_1(\alpha))^p(R_0(\alpha))^a \frac{z^{-a\alpha+p(1-\alpha)}}{\Gamma(p-a\alpha-p\alpha+1)} \\
&\quad + X_0 \sum_{b=0}^{\infty}\sum_{q=0}^{\infty} \left(\frac{-1}{R_0(\alpha)}\right)^{b+q} \frac{(b+q)!}{q!b!}(R_1(\alpha))^q(k)^b \frac{z^{q+b\alpha}}{\Gamma(b\alpha+q+1)} \\
&= x_m \sum_{a=0}^{\infty}\sum_{p=0}^{\infty} \frac{(a+p)!}{p!a!}\left(-\frac{R_1(\alpha)}{k}z^{1-\alpha}\right)^p \left(-\frac{R_0(\alpha)}{k}z^{-\alpha}\right)^a \frac{1}{\Gamma(-a\alpha+p(1-\alpha)+1)} \\
&\quad + X_0 \sum_{b=0}^{\infty}\sum_{q=0}^{\infty} \frac{(b+q)!}{q!b!}\left(-\frac{R_1(\alpha)}{R_0(\alpha)}z\right)^q \left(-\frac{k}{R_0(\alpha)}z^{\alpha}\right)^b \frac{1}{\Gamma(b\alpha+q+1)}.
\end{aligned}$$

The use of bivariate Mittag–Leffler function yields the expression given in (51).

5 Fractional Logistic Equation

This section analyze the logistic model by utilizing several types of fractional derivatives and present their corresponding solutions.

Problem 5. We investigate the fractional form of logistic's model (2), expressed by the Caputo derivative

$$ {}_0^C D_z^{\alpha} y(z) = -k(1-y(z)), \tag{52}$$

which has the following solution:

$$y(z) = 1 - E_{\alpha}(kz^{\alpha}) + Y_0 E_{\alpha}(kz^{\alpha}). \tag{53}$$

Proof. Applying GIT (5) on (5), we get

$$\begin{aligned}
\psi(\hbar)^{\alpha}\left[Y(\hbar) - \frac{\phi(\hbar)}{\psi(\hbar)}Y_0\right] &= -k\frac{\phi(\hbar)}{\psi(\hbar)} + kY(\hbar) \\
[\psi(\hbar)^{\alpha} - k]Y(\hbar) &= \frac{\phi(\hbar)}{\psi(\hbar)^{1-\alpha}} - k\frac{\phi(\hbar)}{\psi(\hbar)}.
\end{aligned}$$

This implies that

$$\begin{aligned}
Y(\hbar) &= \frac{\frac{\phi(\hbar)}{\psi(\hbar)^{1-\alpha}} - k\frac{\phi(\hbar)}{\psi(\hbar)}}{\psi(\hbar)^{\alpha} - k} \\
&= \frac{\phi(\hbar)}{\psi(\hbar)^{1-\alpha}} \frac{Y_0}{\psi(\hbar)(1-k\psi(\hbar)^{-\alpha})} - \frac{\phi(\hbar)}{\psi(\hbar)^{\alpha+1}} \frac{k}{(1-k\psi(\hbar)^{-\alpha})} \\
&= \frac{\phi(\hbar)}{\psi(\hbar)} \frac{Y_0}{1-\frac{k}{\psi(\hbar)^{\alpha}}} + \frac{\phi(\hbar)}{\psi(\hbar)} \frac{\frac{-k}{\psi(\hbar)^{\alpha}}}{1-\frac{-k}{\psi(\hbar)^{\alpha}}}.
\end{aligned}$$

On applying the inverse of GIT, we arrive at (53).

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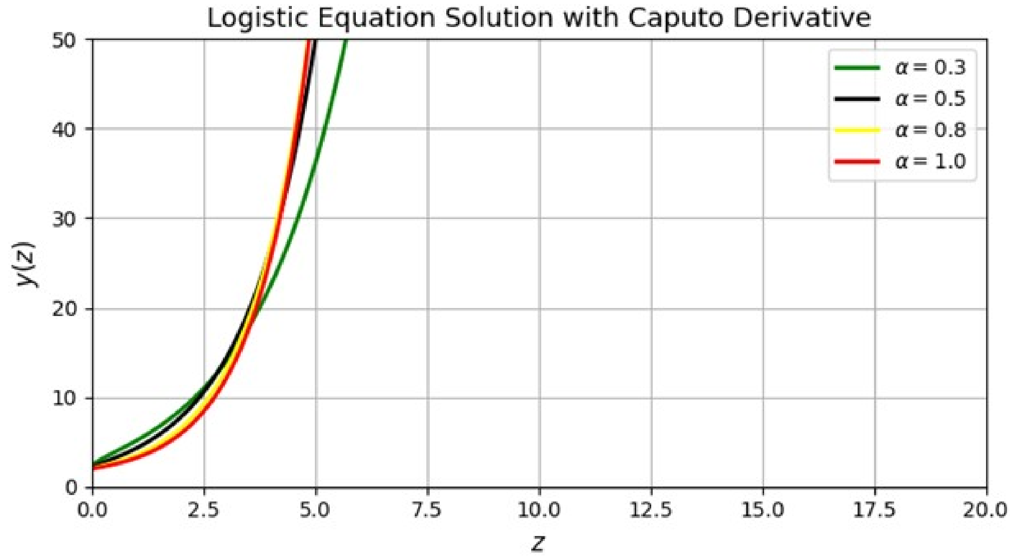

Figure 5 Logistic equation with Caputo derivative for different values of α

Problem 6. We investigate the fractional form of logistic's model (2), expressed by the CF derivative

$$ {}_0^{CF}D_z^{\alpha} y(z) = -k(1-y(z)), \tag{54}$$

which has the following solution:

$$\begin{aligned} y(z) = 1 - exp\left(\frac{k\alpha}{\mathscr{Z}(\alpha)-k(1-\alpha)}z\right) + \frac{\mathscr{Z}(\alpha)Y_0}{\mathscr{Z}(\alpha)-k(1-\alpha)} exp\left(\frac{k\alpha}{\mathscr{Z}(\alpha)-k(1-\alpha)}z\right) \\ - \frac{k(1-\alpha)}{\mathscr{Z}(\alpha)-k(1-\alpha)} exp\left(\frac{k\alpha}{\mathscr{Z}(\alpha)-k(1-\alpha)}z\right). \end{aligned} \tag{55}$$

Proof. Applying GIT (5) on (54), we get

$$\begin{aligned} &\frac{\mathscr{Z}(\alpha)}{1-\alpha+\alpha\frac{1}{\psi(\hbar)}}Y(\hbar) - \frac{\phi(\hbar)}{\psi(\hbar)}\frac{\mathscr{Z}(\alpha)}{1-\alpha+\alpha\frac{1}{\psi(\hbar)}}Y_0 = -k\frac{\phi(\hbar)}{\psi(\hbar)} + kY(\hbar) \\ &\left[\frac{\mathscr{Z}(\alpha)}{1-\alpha+\alpha\frac{1}{\psi(\hbar)}} - k\right]Y(\hbar) = -k\frac{\phi(\hbar)}{\psi(\hbar)} + \frac{\phi(\hbar)}{\psi(\hbar)}\frac{\mathscr{Z}(\alpha)}{1-\alpha+\alpha\frac{1}{\psi(\hbar)}} \\ &\left[\frac{\mathscr{Z}(\alpha)-k(1-\alpha)-k\alpha\frac{1}{\psi(\hbar)}}{1-\alpha+\alpha\frac{1}{\psi(\hbar)}}\right]Y(\hbar) = -k\frac{\phi(\hbar)}{\psi(\hbar)} + \frac{\phi(\hbar)}{\psi(\hbar)}\frac{\mathscr{Z}(\alpha)}{1-\alpha+\alpha\frac{1}{\psi(\hbar)}}. \end{aligned}$$

On simplification, this yields:

$$\begin{aligned} Y(\hbar) &= \frac{-k(1-\alpha)\frac{\phi(\hbar)}{\psi(\hbar)} - \frac{-k\alpha}{\psi(\hbar)}\frac{\phi(\hbar)}{\psi(\hbar)} + \frac{\phi(\hbar)}{\psi(\hbar)}\mathscr{Z}(\alpha)Y_0}{\mathscr{Z}(\alpha)-k(1-\alpha)-k\alpha\frac{1}{\psi(\hbar)}} \\ &= \frac{\phi(\hbar)}{\psi(\hbar)}\frac{-k(1-\alpha)}{\mathscr{Z}(\alpha)-k(1-\alpha)}\frac{1}{1-\frac{k\alpha}{\mathscr{Z}(\alpha)-k(1-\alpha)}\frac{1}{\psi(\hbar)}} + \frac{\phi(\hbar)}{\psi(\hbar)}\frac{-\frac{k\alpha}{\mathscr{Z}(\alpha)-k(1-\alpha)}\frac{1}{\psi(\hbar)}}{1-\frac{k\alpha}{\mathscr{Z}(\alpha)-k(1-\alpha)}\frac{1}{\psi(\hbar)}} \\ &\quad + \frac{\phi(\hbar)}{\psi(\hbar)}\frac{\mathscr{Z}(\alpha)}{\mathscr{Z}(\alpha)-k(1-\alpha)}\frac{1}{1-\frac{k\alpha}{\mathscr{Z}(\alpha)-k(1-\alpha)}\frac{1}{\psi(\hbar)}}Y_0. \end{aligned}$$

Now, by using the inverse of GIT, we arrive at (55).

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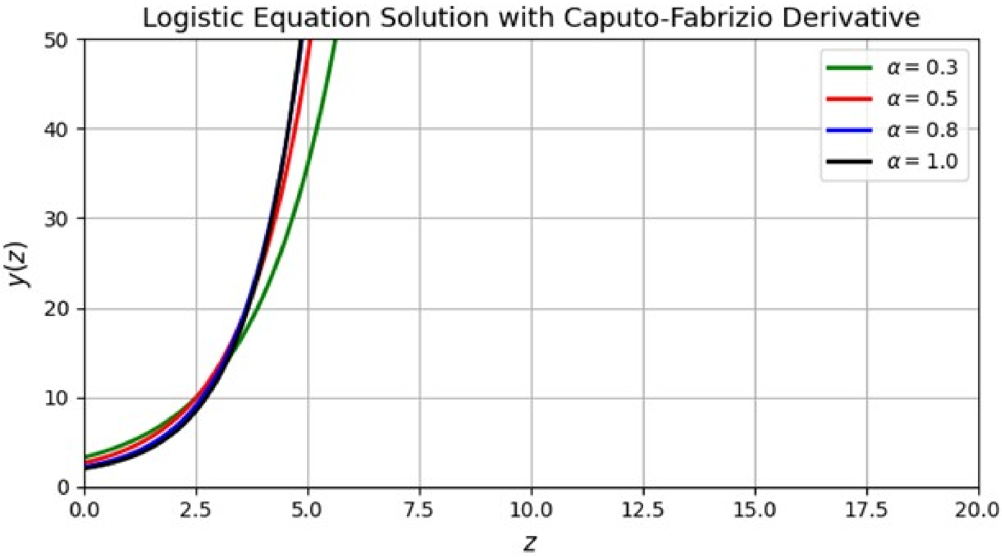

Figure 6 Logistic equation with CF derivative for different values of $\boldsymbol{\alpha}$

Problem 7. We investigate the fractional form of logistic's model (2), expressed by modified ABC derivative

$$ {}^{mABC}{}_0D_z^{\alpha} y(z) = -k(1-y(z)), \tag{56} $$

which has the following solution:

$$ \begin{aligned} y(z) = 1 - E_\alpha\left(\frac{k\alpha}{\mathscr{A}\mathscr{B}(\alpha)-k(1-\alpha)}z^\alpha\right) + \frac{\mathscr{A}\mathscr{B}(\alpha)Y_0}{\mathscr{A}\mathscr{B}(\alpha)-k(1-\alpha)}E_\alpha\left(\frac{k\alpha}{\mathscr{A}\mathscr{B}(\alpha)-k(1-\alpha)}z^\alpha\right) \\ - \frac{k(1-\alpha)}{\mathscr{A}\mathscr{B}(\alpha)-k(1-\alpha)}E_\alpha\left(\frac{k\alpha}{\mathscr{A}\mathscr{B}(\alpha)-k(1-\alpha)}z^\alpha\right). \end{aligned} \tag{57} $$

Proof. Applying GIT (5) on (7), we get

$$ \begin{aligned} &\frac{\mathscr{A}\mathscr{B}(\alpha)}{1-\alpha+\alpha\psi(\hbar)^{-\alpha}}\left[Y(\hbar) - \frac{\phi(\hbar)}{\psi(\hbar)}Y_0\right] = -k\frac{\phi(\hbar)}{\psi(\hbar)} + kY(\hbar) \\ &\left[\frac{\mathscr{A}\mathscr{B}(\alpha)}{1-\alpha+\alpha\psi(\hbar)^{-\alpha}} - k\right]Y(\hbar) = \frac{\mathscr{A}\mathscr{B}(\alpha)Y_0}{1-\alpha+\alpha\psi(\hbar)^{-\alpha}}\frac{\phi(\hbar)}{\psi(\hbar)} - k\frac{\phi(\hbar)}{\psi(\hbar)} \\ &\left[\frac{\mathscr{A}\mathscr{B}(\alpha)-k(1-\alpha)-k\alpha\psi(\hbar)^{-\alpha}}{1-\alpha+\alpha\psi(\hbar)^{-\alpha}}\right]Y(\hbar) = \frac{\mathscr{A}\mathscr{B}(\alpha)Y_0}{1-\alpha+\alpha\psi(\hbar)^{-\alpha}}\frac{\phi(\hbar)}{\psi(\hbar)} - k\frac{\phi(\hbar)}{\psi(\hbar)}. \end{aligned} $$

On simplification, this yields:

$$ \begin{aligned} Y(\hbar) &= \frac{\phi(\hbar)}{\psi(\hbar)}\frac{\mathscr{A}\mathscr{B}(\alpha)Y_0}{\mathscr{A}\mathscr{B}(\alpha)-k(1-\alpha)-k\alpha\psi(\hbar)^{-\alpha}} - \frac{\phi(\hbar)}{\psi(\hbar)}\frac{k(1-\alpha)}{\mathscr{A}\mathscr{B}(\alpha)-k(1-\alpha)-k\alpha\psi(\hbar)^{-\alpha}} \\ &\quad - \frac{\phi(\hbar)}{\psi(\hbar)}\frac{k\alpha\psi(\hbar)^{-\alpha}}{\mathscr{A}\mathscr{B}(\alpha)-k(1-\alpha)-k\alpha\psi(\hbar)^{-\alpha}} \\ &= \frac{\phi(\hbar)}{\psi(\hbar)}\frac{\mathscr{A}\mathscr{B}(\alpha)Y_0}{\mathscr{A}\mathscr{B}(\alpha)-k(1-\alpha)}\frac{1}{1-\frac{k\alpha\psi(\hbar)^{-\alpha}}{\mathscr{A}\mathscr{B}(\alpha)-k(1-\alpha)}} + \frac{\phi(\hbar)}{\psi(\hbar)}\frac{-\frac{k\alpha\psi(\hbar)^{-\alpha}}{\mathscr{A}\mathscr{B}(\alpha)-k(1-\alpha)}}{1-\frac{k\alpha\psi(\hbar)^{-\alpha}}{\mathscr{A}\mathscr{B}(\alpha)-k(1-\alpha)}} \\ &\quad - \frac{\phi(\hbar)}{\psi(\hbar)}\frac{k(1-\alpha)}{\mathscr{A}\mathscr{B}(\alpha)-k(1-\alpha)}\frac{1}{1-\frac{k\alpha\psi(\hbar)^{-\alpha}}{\mathscr{A}\mathscr{B}(\alpha)-k(1-\alpha)}}. \end{aligned} $$

Now, by using the inverse of GIT, we arrive at (57).

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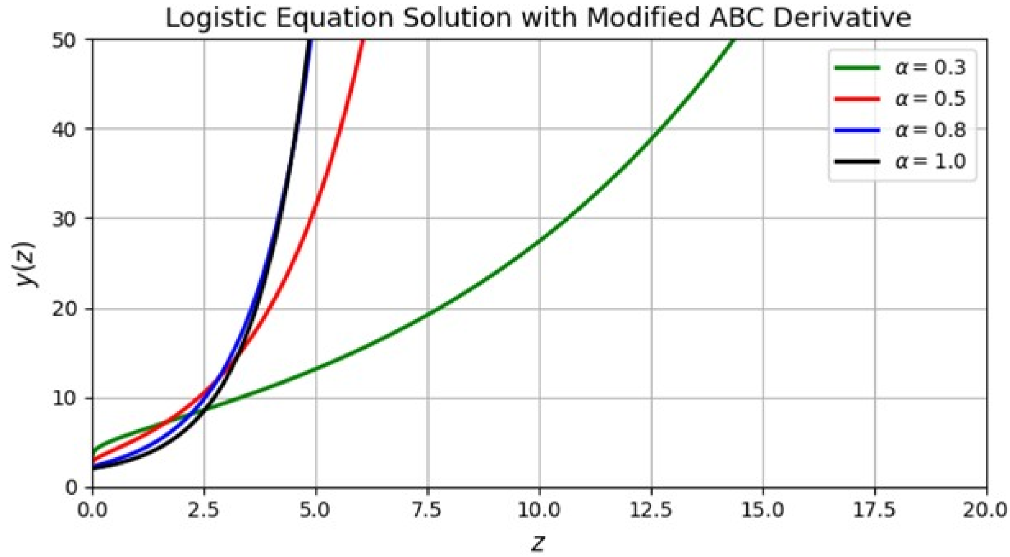

Figure 7 Logistic equation with mABC derivative for different values of α

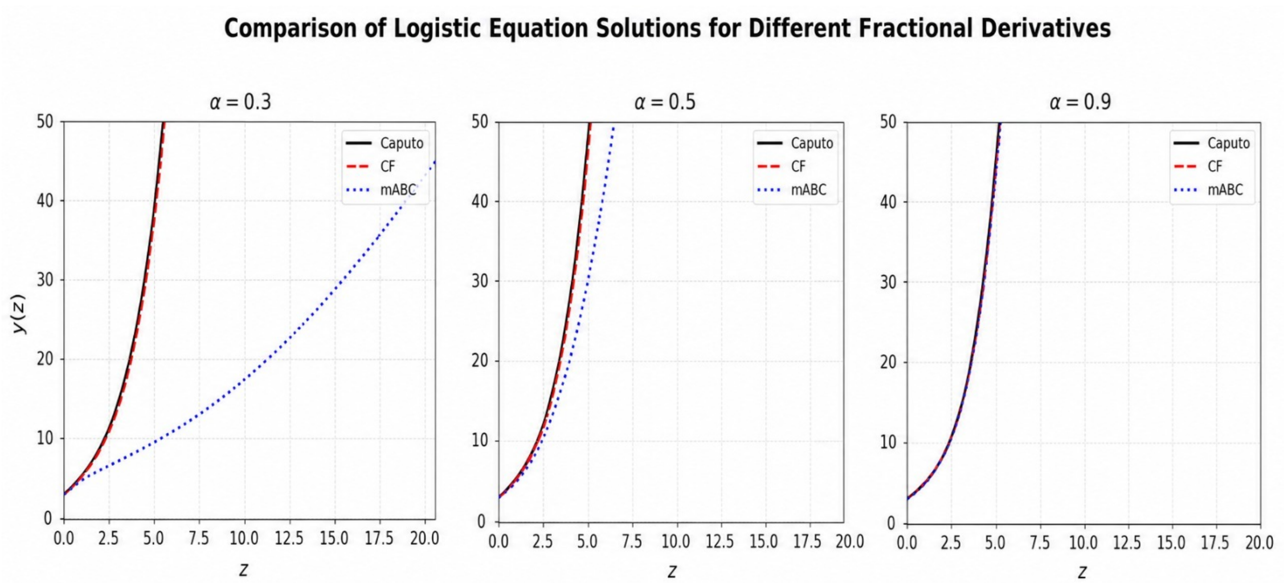

Figure 8 A comparison of logistic equation with Caputo, CF and mABC derivatives for different values of α

Problem 8. We investigate the fractional form of logistic's model (2), expressed by the CPC derivative

$$^{CPC}{}_{0}D_{z}^{\alpha}y(z) = -k(1-y(z)), \tag{58}$$

which has the following solution:

$$y(z) = Y_0 E^1_{1,\alpha,1}\left(-\frac{R_1(\alpha)}{R_0(\alpha)}z, \frac{k}{R_0(\alpha)}z^{\alpha}\right) + E^1_{1-\alpha,-\alpha,1}\left(\frac{R_1(\alpha)}{k}, \frac{R_0(\alpha)}{k}z^{-\alpha}\right). \tag{59}$$

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Proof. Applying GIT (5) on (58), we get

$$[R_1(\alpha)\psi(\hbar)^{\alpha-1}+R_0(\alpha)\psi(\hbar)^{\alpha}]Y(\hbar)-R_0(\alpha)\phi(\hbar)\psi(\hbar)^{\alpha-1}Y_0=-k\frac{\phi(\hbar)}{\psi(\hbar)}+kY(\hbar)$$

$$[R_1(\alpha)\psi(\hbar)^{\alpha-1}+R_0(\alpha)\psi(\hbar)^{\alpha}-k]Y(\hbar)=R_0(\alpha)\phi(\hbar)\psi(\hbar)^{\alpha-1}Y_0-k\frac{\phi(\hbar)}{\psi(\hbar)}.$$

On simplification

$$\begin{aligned}
Y(\hbar)&=\frac{\phi(\hbar)}{\psi(\hbar)}\frac{R_0(\alpha)Y_0}{R_1(\alpha)\psi(\hbar)^{-1}+R_0(\alpha)-k\psi(\hbar)^{-\alpha}}+\frac{\phi(\hbar)}{\psi(\hbar)}\frac{k}{k-R_1(\alpha)\psi(\hbar)^{\alpha-1}-R_0(\alpha)\psi(\hbar)^{\alpha}}\\
&=\frac{\phi(\hbar)}{\psi(\hbar)}\frac{R_0(\alpha)Y_0}{R_0(\alpha)\left[1+\frac{R_1(\alpha)\psi(\hbar)^{-1}-k\psi(\hbar)^{-\alpha}}{R_0(\alpha)}\right]}+\frac{\phi(\hbar)}{\psi(\hbar)}\frac{k}{k\left[1-\frac{R_1(\alpha)\psi(\hbar)^{\alpha-1}+R_0(\alpha)\psi(\hbar)^{\alpha}}{k}\right]}\\
&=\frac{\phi(\hbar)}{\psi(\hbar)}Y_0\sum_{\vartheta=0}^{\infty}\left[-\frac{R_1(\alpha)\psi(\hbar)^{-1}-k\psi(\hbar)^{-\alpha}}{R_0(\alpha)}\right]^{\vartheta}+\frac{\phi(\hbar)}{\psi(\hbar)}\sum_{\varkappa=0}^{\infty}\left[\frac{R_1(\alpha)\psi(\hbar)^{\alpha-1}+R_0(\alpha)\psi(\hbar)^{\alpha}}{k}\right]^{\varkappa}\\
&=\frac{\phi(\hbar)}{\psi(\hbar)}Y_0\sum_{\vartheta=0}^{\infty}\left(\frac{-1}{R_0(\alpha)}\right)^{\vartheta}\left(R_1(\alpha)\psi(\hbar)^{-1}-k\psi(\hbar)^{-\alpha}\right)^{\vartheta}\\
&\qquad+\frac{\phi(\hbar)}{\psi(\hbar)}\sum_{\varkappa=0}^{\infty}\left(\frac{1}{k}\right)^{\varkappa}\left(R_1(\alpha)\psi(\hbar)^{\alpha-1}+R_0(\alpha)\psi(\hbar)^{\alpha}\right)^{\varkappa}\\
&=\frac{\phi(\hbar)}{\psi(\hbar)}Y_0\sum_{\vartheta=0}^{\infty}\left(\frac{-1}{R_0(\alpha)}\right)^{\vartheta}\sum_{a=0}^{\vartheta}\binom{\vartheta}{a}\left(R_1(\alpha)\psi(\hbar)^{-1}\right)^{\vartheta-a}\left(-k\psi(\hbar)^{-\alpha}\right)^{a}\\
&\qquad+\frac{\phi(\hbar)}{\psi(\hbar)}\sum_{\varkappa=0}^{\infty}\left(\frac{1}{k}\right)^{\varkappa}\sum_{b=0}^{\varkappa}\binom{\varkappa}{b}\left(R_1(\alpha)\psi(\hbar)^{\alpha-1}\right)^{\varkappa-b}\left(R_0(\alpha)\psi(\hbar)^{\alpha}\right)^{b}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
Y(\hbar)=Y_0&\sum_{\vartheta=0}^{\infty}\left(\frac{-1}{R_0(\alpha)}\right)^{\vartheta}\sum_{a=0}^{\vartheta}\binom{\vartheta}{a}(R_1(\alpha))^{\vartheta-a}(-k)^{a}\phi(\hbar)(\psi(\hbar))^{-\vartheta+a-a\alpha-1}\\
&+\sum_{\varkappa=0}^{\infty}\left(\frac{1}{k}\right)^{\varkappa}\sum_{b=0}^{\varkappa}\binom{\varkappa}{b}(R_1(\alpha))^{\varkappa-b}(R_0(\alpha))^{b}\phi(\hbar)(\psi(\hbar))^{\varkappa\alpha-j+b-1}.
\end{aligned}$$

By using the inverse of GIT, we get

$$\begin{aligned}
Y(\hbar)=Y_0&\sum_{\vartheta=0}^{\infty}\left(\frac{-1}{R_0(\alpha)}\right)^{\vartheta}\sum_{a=0}^{\vartheta}\frac{\vartheta!}{(\vartheta-a)!a!}(R_1(\alpha))^{\vartheta-a}(-k)^{a}\frac{z^{\vartheta-a+a\alpha}}{\Gamma(\vartheta-a+a\alpha+1)}\\
&+\sum_{\varkappa=0}^{\infty}\left(\frac{1}{k}\right)^{\varkappa}\sum_{b=0}^{\varkappa}\frac{\varkappa!}{(\varkappa-b)!b!}(R_1(\alpha))^{\varkappa-b}(R_0(\alpha))^{b}\frac{z^{\varkappa-\varkappa\alpha-b}}{\Gamma(\varkappa-\varkappa\alpha-b+1)},
\end{aligned}$$

by assigning the parameters $\vartheta-a=p$ and $\varkappa-b=q$, respectively, the resulting expression can be written as follows:

$$\begin{aligned}
Y(\hbar)&=Y_0\sum_{a=0}^{\infty}\sum_{p=0}^{\infty}\left(\frac{-1}{R_0(\alpha)}\right)^{a+p}\frac{(a+p)!}{p!a!}(R_1(\alpha))^{p}(-k)^{a}\frac{z^{p+a\alpha}}{\Gamma(p+a\alpha+1)}\\
&\qquad+\sum_{b=0}^{\infty}\sum_{q=0}^{\infty}\left(\frac{1}{k}\right)^{b+q}\frac{(b+q)!}{q!b!}(R_1(\alpha))^{q}(R_0(\alpha))^{b}\frac{z^{q(1-\alpha)-b\alpha}}{\Gamma(q(1-\alpha)-b\alpha+1)}\\
&=Y_0\sum_{a=0}^{\infty}\sum_{p=0}^{\infty}\frac{(a+p)!}{p!a!}\left(\frac{-R_1(\alpha)}{R_0(\alpha)}z\right)^{p}\left(\frac{k}{R_0(\alpha)}z^{\alpha}\right)^{a}\frac{1}{\Gamma(p+a\alpha+1)}\\
&\qquad+\sum_{b=0}^{\infty}\sum_{q=0}^{\infty}\frac{(b+q)!}{q!b!}\left(\frac{R_1(\alpha)}{k}z^{1-\alpha}\right)^{q}\left(\frac{R_0(\alpha)}{k}z^{-\alpha}\right)^{b}\frac{1}{\Gamma(q(1-\alpha)-b\alpha+1)}.
\end{aligned}$$

The use of bivariate Mittag–Leffler function yields the expression given in (59).

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6 Fractional Blood Alcohol Model

This section analyze the Blood Alcohol model by utilizing several types of fractional derivatives and present their corresponding solutions.

Problem 9. We investigate the fractional form of Blood Alcohol Model (3), expressed by the Caputo derivative

$$ {}_0^C D_z^\alpha f(z) = -\ell_1 f(z), \tag{60} $$

which has the following solution:

$$ f(z) = f_0 E_\alpha(-\ell_1 z^\alpha). \tag{61} $$

Proof. Applying the GIT on both sides of (60), we get

$$ \begin{aligned} \psi(s)^\alpha \left[F(\hbar) \frac{\phi(\hbar)}{\psi(\hbar)} f_0 \right] &= -\ell_1 F(\hbar) \\ [\psi(\hbar)^\alpha + \ell_1] F(\hbar) &= \frac{\phi(\hbar)}{\psi(\hbar)^{1-\alpha}} f_0 \\ F(\hbar) &= \frac{\phi(\hbar)}{\psi(\hbar)} \frac{f_0 \psi(\hbar)^\alpha}{\psi(\hbar)^\alpha + \ell_1} \\ &= \frac{\phi(\hbar)}{\psi(\hbar)} \frac{f_0}{1 + \ell_1 \psi(\hbar)^{-\alpha}}. \end{aligned} \tag{62} $$

By using the inverse of GIT, we arrive at (61).

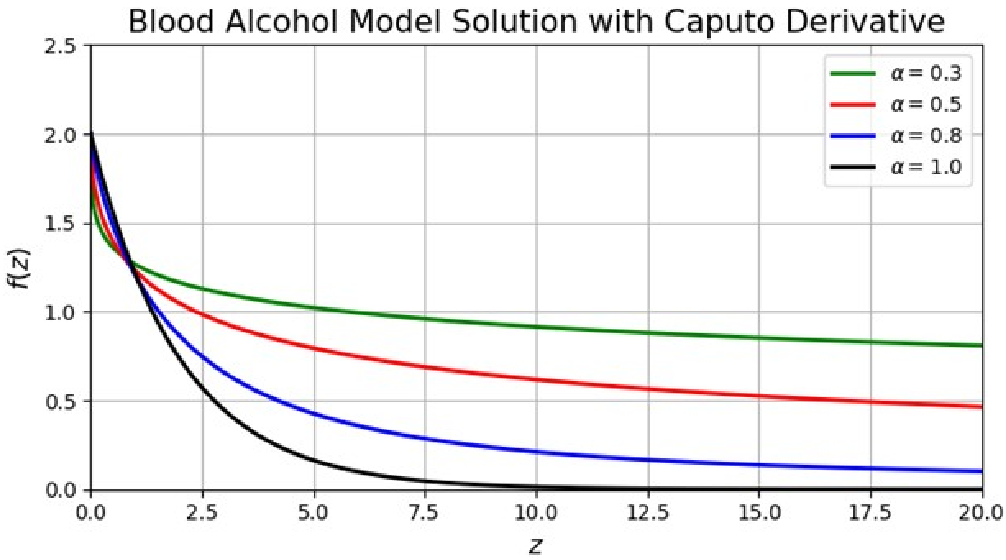

Figure 9 Blood alcohol model with Caputo derivative for different values of α

Problem 10. We investigate the fractional form of Blood Alcohol Model (4), expressed by the Caputo derivative

$$ {}_0^C D_z^\alpha g(z) = \ell_1 f(z) - \ell_2 g(z), \tag{63} $$

has the following solution:

$$ g(z) = \frac{f_0 \ell_1}{\ell_1 + \ell_2} E^1_{-\alpha,\alpha,1} \left(\frac{-1}{\ell_1 + \ell_2} z^{-\alpha}, \frac{-\ell_1 \ell_2}{\ell_1 + \ell_2} z^\alpha \right). \tag{64} $$

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Proof. Applying the GIT on both sides of (63), we get

$$\psi(\hbar)^{\alpha}\left[G(\hbar)-\frac{\phi(\hbar)}{\psi(\hbar)}g(0)\right]=\ell_1F(\hbar)-\ell_2G(\hbar),$$

in view of (62), we have

$$\begin{aligned}[\psi(s)^{\alpha}+\ell_2]G(\hbar)&=\ell_1\frac{\phi(\hbar)}{\psi(\hbar)}\frac{f_0}{1+\ell_1\psi(\hbar)^{-\alpha}}\\ G(\hbar)&=\frac{\phi(\hbar)}{\psi(\hbar)}\frac{\ell_1f_0}{1+\ell_1\psi(\hbar)^{-\alpha}}\frac{1}{\psi(s)^{\alpha}+\ell_2}.\end{aligned}$$

On simplification

$$\begin{aligned}G(\hbar)&=\frac{\phi(\hbar)}{\psi(\hbar)}\frac{\ell_1f_0}{\ell_1+\ell_2+\ell_1\ell_2\psi(\hbar)^{-\alpha}+\psi(\hbar)^{\alpha}}\\ &=\frac{\phi(\hbar)}{\psi(\hbar)}\frac{\ell_1f_0}{(\ell_1+\ell_2)\left[1+\frac{\ell_1\ell_2\psi(\hbar)^{-\alpha}+\psi(\hbar)^{\alpha}}{\ell_1+\ell_2}\right]},\end{aligned}$$

for $\frac{\ell_1\ell_2\psi(\hbar)^{-\alpha}+\psi(\hbar)^{\alpha}}{\ell_1+\ell_2}<1$, we can write

$$\begin{aligned}G(\hbar)&=\frac{\phi(\hbar)}{\psi(\hbar)}\frac{\ell_1f_0}{(\ell_1+\ell_2)}\sum_{\vartheta=0}^{\infty}\left[-\frac{\ell_1\ell_2\psi(\hbar)^{-\alpha}+\psi(\hbar)^{\alpha}}{\ell_1+\ell_2}\right]^{\vartheta}\\ &=\frac{\phi(\hbar)}{\psi(\hbar)}\frac{\ell_1f_0}{(\ell_1+\ell_2)}\sum_{\vartheta=0}^{\infty}\left(\frac{-1}{\ell_1+\ell_2}\right)^{\vartheta}\left(\ell_1\ell_2\psi(\hbar)^{-\alpha}+\psi(\hbar)^{\alpha}\right)^{\vartheta}.\end{aligned}$$

Therefore,

$$\begin{aligned}G(\hbar)&=\frac{\phi(\hbar)}{\psi(\hbar)}\frac{\ell_1f_0}{(\ell_1+\ell_2)}\sum_{\vartheta=0}^{\infty}\left(\frac{-1}{\ell_1+\ell_2}\right)^{\vartheta}\sum_{a=0}^{\vartheta}\binom{\vartheta}{a}(\psi(\hbar)^{\alpha})^{\vartheta-a}\left(\ell_1\ell_2\psi(\hbar)^{-\alpha}\right)^{a}\\ &=\frac{\ell_1f_0}{(\ell_1+\ell_2)}\sum_{\vartheta=0}^{\infty}\left(\frac{-1}{\ell_1+\ell_2}\right)^{\vartheta}\sum_{a=0}^{\vartheta}\binom{\vartheta}{a}(\ell_1\ell_2)^{a}\phi(\hbar)\psi(\hbar)^{\alpha(\vartheta-a)-a\alpha-1}.\end{aligned}$$

Using the inverse of GIT, we get

$$G(\hbar)=\frac{\ell_1f_0}{(\ell_1+\ell_2)}\sum_{\vartheta=0}^{\infty}\left(\frac{-1}{\ell_1+\ell_2}\right)^{\vartheta}\sum_{a=0}^{\vartheta}\frac{\vartheta!}{(\vartheta-a)!a!}(\ell_1\ell_2)^{a}\frac{z^{-\alpha(\vartheta-a)+a\alpha}}{\Gamma(-\alpha(\vartheta-a)+a\alpha+1)}.$$

By assigning the parameters $\vartheta-a$ with p the resulting expression can be written as follows:

$$\begin{aligned}G(\hbar)&=\frac{\ell_1f_0}{(\ell_1+\ell_2)}\sum_{a=0}^{\infty}\sum_{p=0}^{\infty}\left(\frac{-1}{\ell_1+\ell_2}\right)^{a+p}\frac{(a+p)!}{p!a!}(\ell_1\ell_2)^{a}\frac{z^{-\alpha p+a\alpha}}{\Gamma(-\alpha p+a\alpha+1)}\\ &=\frac{\ell_1f_0}{(\ell_1+\ell_2)}\sum_{a=0}^{\infty}\sum_{p=0}^{\infty}\frac{(a+p)!}{p!a!}\left(\frac{-\ell_1\ell_2}{\ell_1+\ell_2}z^{\alpha}\right)^{a}\left(\frac{-1}{\ell_1+\ell_2}z^{-\alpha}\right)^{p}\frac{1}{\Gamma(-\alpha p+a\alpha+1)}.\end{aligned}$$

The use of bivariate Mittag–Leffler function yields the expression given in (64).

Problem 11. We investigate the fractional form of Blood Alcohol Model (3), expressed by the CF derivative

$${}_{0}^{CF}D_{z}^{\alpha}f(z)=-\ell_1f(z), \tag{65}$$

which has the following solution:

$$f(z)=\frac{\mathscr{L}(\alpha)f_0}{\mathscr{L}(\alpha)+\ell_1(1-\alpha)}\cdot exp\left(\frac{-\ell_1\alpha}{\mathscr{L}(\alpha)+\ell_1(1-\alpha)}z\right). \tag{66}$$

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Proof. Applying the GIT on both sides of (65), we get

$$\frac{\mathscr{Z}(\alpha)}{1-\alpha+\alpha\frac{1}{\psi(\hbar)}}\left[F(\hbar)-\frac{\phi(\hbar)}{\psi(\hbar)}f_0\right]=-\ell_1 F(\hbar)$$

$$\left[\frac{\mathscr{Z}(\alpha)}{1-\alpha+\alpha\frac{1}{\psi(\hbar)}}+\ell_1\right]F(\hbar)=\frac{\phi(\hbar)}{\psi(\hbar)}\frac{\mathscr{Z}(\alpha)f_0}{1-\alpha+\alpha\frac{1}{\psi(\hbar)}}$$

$$\left[\mathscr{Z}(\alpha)+\ell_1(1-\alpha)+\ell_1\alpha\frac{1}{\psi(\hbar)}\right]F(\hbar)=\frac{\phi(\hbar)}{\psi(\hbar)}\mathscr{Z}(\alpha)f_0$$

$$\begin{aligned}F(\hbar)&=\frac{\phi(\hbar)}{\psi(\hbar)}\frac{\mathscr{Z}(\alpha)f_0}{\mathscr{Z}(\alpha)+\ell_1(1-\alpha)+\ell_1\alpha\frac{1}{\psi(\hbar)}}\\&=\frac{\phi(\hbar)}{\psi(\hbar)}\frac{\mathscr{Z}(\alpha)f_0}{\mathscr{Z}(\alpha)+\ell_1(1-\alpha)}\frac{1}{1+\frac{\ell_1\alpha}{\mathscr{Z}(\alpha)+\ell_1(1-\alpha)}\frac{1}{\psi(\hbar)}}.\end{aligned}\tag{67}$$

Using the inverse of GIT, we arrive at (66).

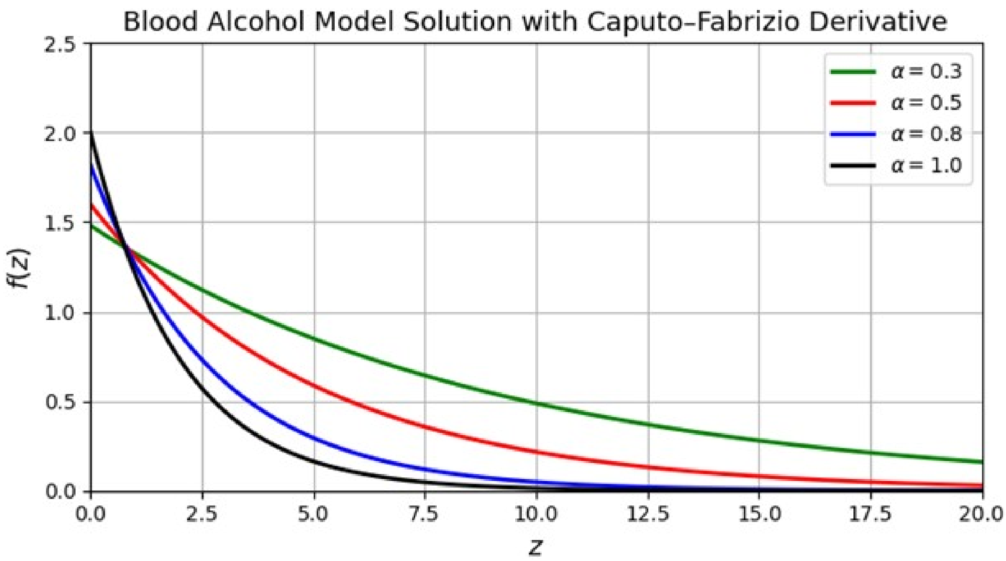

Figure 10 Blood alcohol model with CF derivative for different values of α

Problem 12. We investigate the fractional form of Blood Alcohol Model (4), expressed by the CF derivative

$${}^{CF}_{\ 0}D^{\alpha}_{z}g(z)=\ell_1 f(z)-\ell_2 g(z),\tag{68}$$

which has the following solution:

$$g(z)=XE^{1}_{1,2,1}\left(\frac{-B\alpha}{A}z,\frac{-\ell_1\ell_2\alpha^2}{A}z^2\right)+YE^{1}_{-1,1,1}\left(\frac{-A}{B\alpha}z^{-1},\frac{-\ell_1\ell_2\alpha}{B}z\right).\tag{69}$$

Proof. Applying the GIT on both sides of (68), we get

$$\frac{\mathscr{Z}(\alpha)}{1-\alpha+\alpha\frac{1}{\psi(\hbar)}}\left[G(\hbar)-\frac{\phi(\hbar)}{\psi(\hbar)}g(0)\right]=\ell_1 F(\hbar)-\ell_2 G(\hbar)$$

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in view of (67), we have

$$\left[\frac{\mathscr{Z}(\alpha)}{1-\alpha+\alpha\frac{1}{\psi(\hbar)}}+\ell_2\right]G(\hbar)=\frac{\phi(\hbar)}{\psi(\hbar)}\frac{\mathscr{Z}(\alpha)f_0\ell_1}{\mathscr{Z}(\alpha)+\ell_1(1-\alpha)+\ell_1\alpha\frac{1}{\psi(\hbar)}}$$
$$\left[\frac{\mathscr{Z}(\alpha)+\ell_2(1-\alpha)+\ell_2\alpha\frac{1}{\psi(\hbar)}}{1-\alpha+\alpha\frac{1}{\psi(\hbar)}}\right]G(\hbar)=\frac{\phi(\hbar)}{\psi(\hbar)}\frac{\mathscr{Z}(\alpha)f_0\ell_1}{\mathscr{Z}(\alpha)+\ell_1(1-\alpha)+\ell_1\alpha\frac{1}{\psi(\hbar)}}.$$

On simplification, we have

$$\begin{aligned}G(\hbar)&=\frac{\phi(\hbar)}{\psi(\hbar)}\frac{\mathscr{Z}(\alpha)f_0\ell_1(1-\alpha+\alpha\frac{1}{\psi(\hbar)})}{\left(\mathscr{Z}(\alpha)+\ell_1(1-\alpha)+\ell_1\alpha\frac{1}{\psi(\hbar)}\right)\left(\mathscr{Z}(\alpha)+\ell_2(1-\alpha)+\ell_2\alpha\frac{1}{\psi(\hbar)}\right)}\\&=\frac{\phi(\hbar)}{\psi(\hbar)}\frac{\mathscr{Z}(\alpha)f_0\ell_1(1-\alpha+\alpha\frac{1}{\psi(\hbar)})}{A+B\frac{\alpha}{\psi(\hbar)}+\ell_1\ell_2\alpha^2\frac{1}{\psi(\hbar)^2}},\end{aligned}$$

where $A=(\mathscr{Z}(\alpha)+\ell_1(1-\alpha))(\mathscr{Z}(\alpha)+\ell_2(1-\alpha))$
and $B=\ell_2(\mathscr{Z}(\alpha)+\ell_1(1-\alpha))+\ell_1(\mathscr{Z}(\alpha)+\ell_2(1-\alpha))$. Therefore,

$$G(\hbar)=\frac{\phi(\hbar)}{\psi(\hbar)}\frac{\mathscr{Z}(\alpha)f_0\ell_1(1-\alpha)}{A\left[1+\frac{B\frac{\alpha}{\psi(\hbar)}+\ell_1\ell_2\frac{\alpha^2}{\psi(\hbar)^2}}{A}\right]}+\frac{\phi(\hbar)}{\psi(\hbar)}\frac{\mathscr{Z}(\alpha)f_0\ell_1\alpha}{B\alpha\left[1+\frac{A\psi(\hbar)+\ell_1\ell_2\frac{\alpha^2}{\psi(\hbar)}}{B\alpha}\right]},$$

for $\frac{B\frac{\alpha}{\psi(\hbar)}+\ell_1\ell_2\frac{\alpha^2}{\psi(\hbar)^2}}{A}<1$ and $\frac{A\psi(\hbar)+\ell_1\ell_2\frac{\alpha^2}{\psi(\hbar)}}{B\alpha}<1$, we reach at

$$\begin{aligned}G(\hbar)=\frac{\phi(\hbar)}{\psi(\hbar)}\frac{\mathscr{Z}(\alpha)f_0\ell_1(1-\alpha)}{A}\sum_{\vartheta=0}^{\infty}\left[-\frac{B\frac{\alpha}{\psi(\hbar)}+\ell_1\ell_2\frac{\alpha^2}{\psi(\hbar)^2}}{A}\right]^{\vartheta}\\+\frac{\phi(\hbar)}{\psi(\hbar)}\frac{\mathscr{Z}(\alpha)f_0\ell_1}{B}\sum_{\varkappa=0}^{\infty}\left[-\frac{A\psi(\hbar)+\ell_1\ell_2\frac{\alpha^2}{\psi(\hbar)}}{B\alpha}\right]^{\varkappa}.\end{aligned}$$

Let $X=\frac{\mathscr{Z}(\alpha)f_0\ell_1(1-\alpha)}{A}$ and $Y=\frac{\mathscr{Z}(\alpha)f_0\ell_1}{B}$, respectively. Then

$$\begin{aligned}G(\hbar)&=X\frac{\phi(\hbar)}{\psi(\hbar)}\sum_{\vartheta=0}^{\infty}\left(\frac{-1}{A}\right)^{\vartheta}\sum_{a=0}^{\vartheta}\binom{\vartheta}{a}\left(\frac{B\alpha}{\psi(\hbar)}\right)^{\vartheta-a}\left(\frac{\ell_1\ell_2\alpha^2}{\psi(\hbar)^2}\right)^{a}\\&\quad+Y\frac{\phi(\hbar)}{\psi(\hbar)}\sum_{\varkappa=0}^{\infty}\left(\frac{-1}{B\alpha}\right)^{\varkappa}\sum_{b=0}^{\varkappa}\binom{\varkappa}{b}\left(\frac{A}{\psi(\hbar)^{-1}}\right)^{\varkappa-b}\left(\frac{\ell_1\ell_2\alpha^2}{\psi(\hbar)}\right)^{b}\\&=X\sum_{\vartheta=0}^{\infty}\left(\frac{-1}{A}\right)^{\vartheta}\sum_{a=0}^{\vartheta}\binom{\vartheta}{a}(B\alpha)^{\vartheta-a}\left(\ell_1\ell_2\alpha^2\right)^{a}\phi(\hbar)\psi(\hbar)^{-(\vartheta-a)-2a-1}\\&\quad+Y\sum_{\varkappa=0}^{\infty}\left(\frac{-1}{B\alpha}\right)^{\varkappa}\sum_{b=0}^{\varkappa}\binom{\varkappa}{b}(A)^{\varkappa-b}\left(\ell_1\ell_2\alpha^2\right)^{b}\phi(\hbar)\psi(\hbar)^{(\varkappa-b)-b-1}.\end{aligned}$$

Using the inverse of GIT, we have

$$\begin{aligned}G(\hbar)=X\sum_{\vartheta=0}^{\infty}\left(\frac{-1}{A}\right)^{\vartheta}\sum_{a=0}^{\vartheta}\frac{\vartheta!}{(\vartheta-a)!a!}(B\alpha)^{\vartheta-a}\left(\ell_1\ell_2\alpha^2\right)^{a}\frac{z^{\vartheta-a+2a}}{\Gamma(\vartheta-a+2a+1)}\\+Y\sum_{\varkappa=0}^{\infty}\left(\frac{-1}{B\alpha}\right)^{\varkappa}\sum_{b=0}^{\varkappa}\frac{\varkappa!}{(\varkappa-b)!b!}(A)^{\varkappa-b}\left(\ell_1\ell_2\alpha^2\right)^{b}\frac{z^{-(\varkappa-b)+b}}{\Gamma(b-(\varkappa-b)+1)}.\end{aligned}$$

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By assigning the parameters $\vartheta - a$ with p and $\varkappa - b$ with q, respectively, the resulting expression can be written as follows:

$$G(\hbar) = X\sum_{a=0}^{\infty}\sum_{p=0}^{\infty}\frac{(a+p)!}{p!a!}\left(\frac{-B\alpha}{A}z\right)^p\left(\frac{-\ell_1\ell_2\alpha^2}{A}z^2\right)^a\frac{1}{\Gamma(p+2a+1)}$$
$$+Y\sum_{b=0}^{\infty}\sum_{q=0}^{\infty}\frac{(b+q)!}{q!b!}\left(\frac{-A}{B\alpha}z^{-1}\right)^q\left(\frac{-\ell_1\ell_2\alpha}{B}z\right)^b\frac{1}{\Gamma(-q+b+1)}.$$

The use of bivariate Mittag–Leffler function yields the expression given in (69).

Problem 13. We investigate the fractional form of Blood Alcohol Model (3), expressed by the modified ABC derivative

$$^{mABC}{}_0D_z^\alpha f(z) = -\ell_1 f(z), \tag{70}$$

which has the following solution:

$$f(z) = \frac{\mathscr{A}\mathscr{B}(\alpha)f_0}{\mathscr{A}\mathscr{B}(\alpha)+\ell_1(1-\alpha)}\cdot E_\alpha\left(\frac{-\ell_1\alpha}{\mathscr{A}\mathscr{B}(\alpha)+\ell_1(1-\alpha)}z^\alpha\right). \tag{71}$$

Proof. Applying the GIT on both sides of (70), we get

$$\frac{\mathscr{A}\mathscr{B}(\alpha)}{1-\alpha+\alpha\frac{1}{\psi(\hbar)^\alpha}}\left[F(\hbar)-\frac{\phi(\hbar)}{\psi(\hbar)}f_0\right] = -\ell_1F(\hbar)$$
$$\left[\frac{\mathscr{A}\mathscr{B}(\alpha)}{1-\alpha+\alpha\frac{1}{\psi(\hbar)^\alpha}}+\ell_1\right]F(\hbar) = \frac{\phi(\hbar)}{\psi(\hbar)}\frac{\mathscr{A}\mathscr{B}(\alpha)f_0}{1-\alpha+\alpha\frac{1}{\psi(\hbar)^\alpha}}$$
$$\left[\mathscr{A}\mathscr{B}(\alpha)+\ell_1(1-\alpha)+\ell_1\alpha\frac{1}{\psi(\hbar)^\alpha}\right]F(\hbar) = \frac{\phi(\hbar)}{\psi(\hbar)}\mathscr{A}\mathscr{B}(\alpha)f_0.$$

On simplification, we get

$$\begin{aligned}F(\hbar) &= \frac{\phi(\hbar)}{\psi(\hbar)}\frac{\mathscr{A}\mathscr{B}(\alpha)f_0}{\mathscr{A}\mathscr{B}(\alpha)+\ell_1(1-\alpha)+\ell_1\alpha\frac{1}{\psi(\hbar)^\alpha}}\\ &= \frac{\phi(\hbar)}{\psi(\hbar)}\frac{\mathscr{A}\mathscr{B}(\alpha)f_0}{\mathscr{A}\mathscr{B}(\alpha)+\ell_1(1-\alpha)}\frac{1}{1+\frac{\ell_1\alpha}{\mathscr{A}\mathscr{B}(\alpha)+\ell_1(1-\alpha)}\psi(\hbar)^{-\alpha}}.\end{aligned} \tag{72}$$

Now, using the inverse of GIT, we arrive at (71).

Problem 14. We investigate the fractional form of Blood Alcohol Model (4), expressed by the modified ABC derivative

$$^{mABC}{}_0D_z^\alpha g(z) = \ell_1 f(z) - \ell_2 g(z), \tag{73}$$

which has the following solution:

$$\begin{aligned}g(z) &= CE^1_{\alpha,2\alpha,1}\left(-\frac{P\ell_2\alpha+Q\ell_1\alpha}{PQ}z^\alpha, -\frac{\ell_1\ell_2\alpha^2}{PQ}z^{2\alpha}\right)\\ &\quad + DE^1_{-\alpha,\alpha,1}\left(-\frac{PQ}{P\ell_2\alpha+Q\ell_1\alpha}z^{-\alpha}, -\frac{\ell_1\ell_2\alpha^2}{P\ell_2\alpha+Q\ell_1\alpha}z^\alpha\right).\end{aligned} \tag{74}$$

Proof. Applying the GIT on both sides of (73), we get

$$\frac{\mathscr{A}\mathscr{B}(\alpha)}{1-\alpha+\alpha\frac{1}{\psi(\hbar)^\alpha}}\left[G(\hbar)-\frac{\phi(\hbar)}{\psi(\hbar)}g(0)\right] = \ell_1F(\hbar)-\ell_2G(\hbar),$$

in view of (72), we have

$$\left[\frac{\mathscr{A}\mathscr{B}(\alpha)}{1-\alpha+\alpha\frac{1}{\psi(\hbar)^\alpha}}+\ell_2\right]G(\hbar) = \frac{\phi(\hbar)}{\psi(\hbar)}\frac{\mathscr{A}\mathscr{B}(\alpha)f_0}{\mathscr{A}\mathscr{B}(\alpha)+\ell_1(1-\alpha)+\ell_1\alpha\frac{1}{\psi(\hbar)^\alpha}}$$
$$\left[\frac{\mathscr{A}\mathscr{B}(\alpha)+\ell_2(1-\alpha)+\ell_2\alpha\psi(\hbar)^{-\alpha}}{1-\alpha+\alpha\frac{1}{\psi(\hbar)^\alpha}}\right]G(\hbar) = \frac{\phi(\hbar)}{\psi(\hbar)}\frac{\mathscr{A}\mathscr{B}(\alpha)f_0}{\mathscr{A}\mathscr{B}(\alpha)+\ell_1(1-\alpha)+\ell_1\alpha\frac{1}{\psi(\hbar)^\alpha}}$$

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$$G(\hbar)=\frac{\phi(\hbar)}{\psi(\hbar)}\frac{\mathscr{A}\mathscr{B}(\alpha)f_0\ell_1\left(1-\alpha+\alpha\frac{1}{\psi(\hbar)^\alpha}\right)}{\left(\mathscr{A}\mathscr{B}(\alpha)+\ell_1(1-\alpha)+\ell_1\alpha\frac{1}{\psi(\hbar)^\alpha}\right)\left(\mathscr{A}\mathscr{B}(\alpha)+\ell_2(1-\alpha)+\ell_2\alpha\frac{1}{\psi(\hbar)^\alpha}\right)}$$

$$G(\hbar)=\frac{\phi(\hbar)}{\psi(\hbar)}\frac{\mathscr{A}\mathscr{B}(\alpha)f_0\ell_1(1-\alpha)}{PQ+(P\ell_2\alpha+Q\ell_1\alpha)\psi(\hbar)^{-\alpha}+\ell_1\ell_2\alpha^2\psi(\hbar)^{-2\alpha}}+\frac{\phi(\hbar)}{\psi(\hbar)}\frac{\mathscr{A}\mathscr{B}(\alpha)f_0\ell_1\alpha}{PQ\psi(\hbar)^\alpha+P\ell_2\alpha+Q\ell_1\alpha+\ell_1\ell_2\alpha^2\psi(\hbar)^{-\alpha}},$$

where the constants are given by
$P=\mathscr{A}\mathscr{B}(\alpha)+\ell_1(1-\alpha)$, $Q=\mathscr{A}\mathscr{B}(\alpha)+\ell_2(1-\alpha)$, respectively.

$$G(\hbar)=\frac{\phi(\hbar)}{\psi(\hbar)}\frac{\mathscr{A}\mathscr{B}(\alpha)f_0\ell_1(1-\alpha)}{PQ\left[1+\frac{(P\ell_2\alpha+Q\ell_1\alpha)\psi(\hbar)^{-\alpha}+\ell_1\ell_2\alpha^2\psi(\hbar)^{-2\alpha}}{PQ}\right]}+\frac{\phi(\hbar)}{\psi(\hbar)}\frac{\mathscr{A}\mathscr{B}(\alpha)f_0\ell_1\alpha}{(P\ell_2\alpha+Q\ell_1\alpha)\left[1+\frac{PQ\psi(\hbar)^\alpha+\ell_1\ell_2\alpha^2\psi(\hbar)^{-\alpha}}{P\ell_2\alpha+Q\ell_1\alpha}\right]},$$

for $\frac{(P\ell_2\alpha+Q\ell_1\alpha)\psi(\hbar)^{-\alpha}+\ell_1\ell_2\alpha^2\psi(\hbar)^{-2\alpha}}{PQ}<1$ and $\frac{PQ\psi(\hbar)^\alpha+\ell_1\ell_2\alpha^2\psi(\hbar)^{-\alpha}}{P\ell_2\alpha+Q\ell_1\alpha}<1$, we have

$$G(\hbar)=\frac{\phi(\hbar)}{\psi(\hbar)}\frac{\mathscr{A}\mathscr{B}(\alpha)f_0\ell_1(1-\alpha)}{PQ}\sum_{\vartheta=0}^{\infty}\left(\frac{-1}{PQ}\right)^\vartheta\left((P\ell_2\alpha+Q\ell_1\alpha)\psi(\hbar)^{-\alpha}+\ell_1\ell_2\alpha^2\psi(\hbar)^{-2\alpha}\right)^\vartheta+\frac{\phi(\hbar)}{\psi(\hbar)}\frac{\mathscr{A}\mathscr{B}(\alpha)f_0\ell_1}{(P\ell_2+Q\ell_1)}\sum_{\varkappa=0}^{\infty}\left(\frac{-1}{P\ell_2\alpha+Q\ell_1\alpha}\right)^\varkappa\left(PQ\psi(\hbar)^\alpha+\ell_1\ell_2\alpha^2\psi(\hbar)^{-\alpha}\right)^\varkappa.$$

Let $C=\frac{\mathscr{A}\mathscr{B}(\alpha)f_0\ell_1(1-\alpha)}{PQ}$ and $D=\frac{\mathscr{A}\mathscr{B}(\alpha)f_0\ell_1}{P\ell_2+Q\ell_1}$. Then

$$\begin{aligned}G(\hbar)&=\frac{\phi(\hbar)}{\psi(\hbar)}C\sum_{\vartheta=0}^{\infty}\left(\frac{-1}{PQ}\right)^\vartheta\sum_{a=0}^{\vartheta}\binom{\vartheta}{a}\left((P\ell_2\alpha+Q\ell_1\alpha)\psi(\hbar)^{-\alpha}\right)^{\vartheta-a}\left(\ell_1\ell_2\alpha^2\psi(\hbar)^{-2\alpha}\right)^a\\&+\frac{\phi(\hbar)}{\psi(\hbar)}D\sum_{\varkappa=0}^{\infty}\left(\frac{-1}{P\ell_2\alpha+Q\ell_1\alpha}\right)^\varkappa\sum_{b=0}^{\varkappa}\binom{\varkappa}{b}\left(PQ\psi(\hbar)^\alpha\right)^{\varkappa-b}\left(\ell_1\ell_2\alpha^2\psi(\hbar)^{-\alpha}\right)^b\\&=C\sum_{\vartheta=0}^{\infty}\left(\frac{-1}{PQ}\right)^\vartheta\sum_{a=0}^{\vartheta}\binom{\vartheta}{a}(P\ell_2\alpha+Q\ell_1\alpha)^{\vartheta-a}(\ell_1\ell_2\alpha^2)^a\phi(\hbar)\psi(\hbar)^{-\alpha(\vartheta-a)-2a\alpha-1}\\&+D\sum_{\varkappa=0}^{\infty}\left(\frac{-1}{P\ell_2\alpha+Q\ell_1\alpha}\right)^\varkappa\sum_{b=0}^{\varkappa}\binom{\varkappa}{b}(PQ)^{\varkappa-b}(\ell_1\ell_2\alpha^2)^b\phi(\hbar)\psi(\hbar)^{\alpha(\varkappa-b)-b\alpha-1}.\end{aligned}$$

Now, using the inverse of GIT, we get

$$G(\hbar)=C\sum_{\vartheta=0}^{\infty}\left(\frac{-1}{PQ}\right)^\vartheta\sum_{a=0}^{\vartheta}\frac{\vartheta!}{(\vartheta-a)!a!}(P\ell_2\alpha+Q\ell_1\alpha)^{\vartheta-a}(\ell_1\ell_2\alpha^2)^a\frac{z^{\alpha(\vartheta-a)+2a\alpha}}{\Gamma(\alpha(\vartheta-a)+2a\alpha+1)}+D\sum_{\varkappa=0}^{\infty}\left(\frac{-1}{P\ell_2\alpha+Q\ell_1\alpha}\right)^\varkappa\sum_{b=0}^{\varkappa}\frac{\varkappa!}{(\varkappa-b)!b!}(PQ)^{\varkappa-b}(\ell_1\ell_2\alpha^2)^b\frac{z^{-\alpha(\varkappa-b)+b\alpha}}{\Gamma(-\alpha(\varkappa-b)+b\alpha+1)}.$$

By assigning the parameters $\vartheta-a$ with p and $\varkappa-b$ with q, respectively, the resulting expression can be written as follows:

$$\begin{aligned}G(\hbar)&=C\sum_{a=0}^{\infty}\sum_{p=0}^{\infty}\left(\frac{-1}{PQ}\right)^{p+a}\frac{(a+p)!}{p!a!}(P\ell_2\alpha+Q\ell_1\alpha)^p(\ell_1\ell_2\alpha^2)^a\frac{z^{\alpha p+2a\alpha}}{\Gamma(\alpha p+2a\alpha+1)}\\&+D\sum_{b=0}^{\infty}\sum_{q=0}^{\infty}\left(\frac{-1}{P\ell_2\alpha+Q\ell_1\alpha}\right)^{q+b}\frac{(b+q)!}{(q!b!}(PQ)^q(\ell_1\ell_2\alpha^2)^b\frac{z^{-\alpha q+b\alpha}}{\Gamma(-\alpha q+b\alpha+1)}\\&=C\sum_{a=0}^{\infty}\sum_{p=0}^{\infty}\frac{(a+p)!}{p!a!}\left(-\frac{P\ell_2\alpha+Q\ell_1\alpha}{PQ}z^\alpha\right)^p\left(-\frac{\ell_1\ell_2\alpha^2}{PQ}z^{2\alpha}\right)^a\frac{1}{\Gamma(\alpha p+2a\alpha+1)}\\&+D\sum_{b=0}^{\infty}\sum_{q=0}^{\infty}\frac{(b+q)!}{(q!b!}\left(\frac{-PQ}{P\ell_2\alpha+Q\ell_1\alpha}z^{-\alpha}\right)^q\left(-\frac{\ell_1\ell_2\alpha^2}{P\ell_2\alpha+Q\ell_1\alpha}z^\alpha\right)^b\frac{1}{\Gamma(-\alpha q+b\alpha+1)}.\end{aligned}$$

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The use of bivariate Mittag–Leffler function yields the expression given in (74).

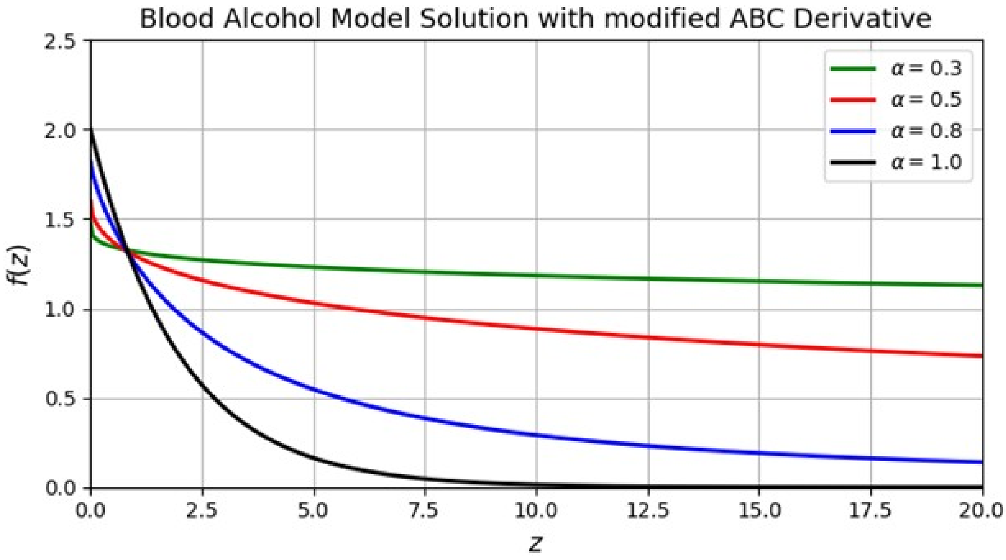

Figure 11 Blood alcohol model with mABC derivative for different values of α

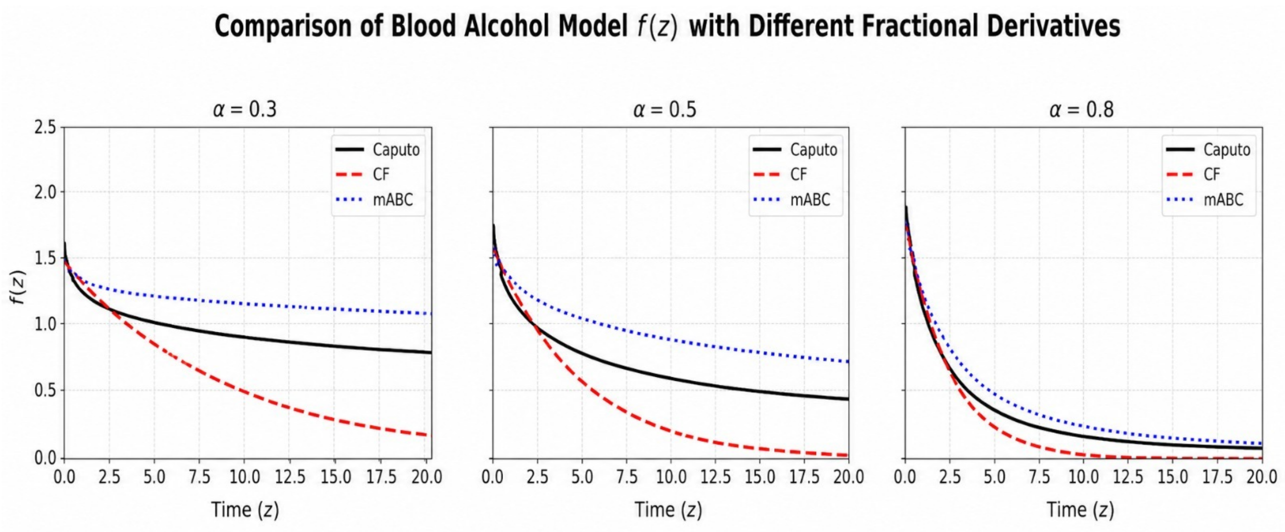

Figure 12 A comparison of blood alcohol model with Caputo, CF and mABC derivatives for different values of α

Problem 15. We investigate the fractional form of Blood Alcohol Model (3), expressed by the CPC derivative

$$ {}^{CPC}_{0}D^{\alpha}_{z} f(z) = -\ell_1 f(z) \tag{75} $$

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which has the following solution:

$$f(z) = f_0 E^1_{1,\alpha,1}\left(\frac{-R_1(\alpha)}{R_0(\alpha)}z, \frac{-\ell_1}{R_0(\alpha)}z^{\alpha}\right). \tag{76}$$

Proof. Applying the GIT on both sides of (75), we get

$$[R_1(\alpha)\psi(\hbar)^{\alpha-1} + R_0(\alpha)\psi(\hbar)^{\alpha}]F(\hbar) - R_0(\alpha)\phi(\hbar)\psi(\hbar)^{\alpha-1}f_0 = -\ell_1 F(\hbar)$$
$$[R_1(\alpha)\psi(\hbar)^{\alpha-1} + R_0(\alpha)\psi(\hbar)^{\alpha} + \ell_1]F(\hbar) = R_0(\alpha)\phi(\hbar)\psi(\hbar)^{\alpha-1}f_0.$$

On simplification, this yields:

$$\begin{aligned} F(\hbar) &= \frac{R_0(\alpha)\phi(\hbar)\psi(\hbar)^{\alpha-1}f_0}{R_1(\alpha)\psi(\hbar)^{\alpha-1} + R_0(\alpha)\psi(\hbar)^{\alpha} + \ell_1} \\ &= \frac{\phi(\hbar)}{\psi(\hbar)} \frac{R_0(\alpha)}{R_0(\alpha)\left[1 + \frac{R_1(\alpha)\psi(\hbar)^{-1} + \ell_1\psi(\hbar)^{-\alpha}}{R_0(\alpha)}\right]}, \end{aligned}$$

for $\frac{R_1(\alpha)\psi(\hbar)^{-1} + \ell_1\psi(\hbar)^{-\alpha}}{R_0(\alpha)} < 1$, we have

$$\begin{aligned} F(\hbar) &= \frac{\phi(\hbar)}{\psi(\hbar)} f_0 \sum_{\vartheta=0}^{\infty}\left[-\frac{R_1(\alpha)\psi(\hbar)^{-1} + \ell_1\psi(\hbar)^{-\alpha}}{R_0(\alpha)}\right]^{\vartheta} \\ &= \frac{\phi(\hbar)}{\psi(\hbar)} f_0 \sum_{\vartheta=0}^{\infty}\left(\frac{-1}{R_0(\alpha)}\right)^{\vartheta}\left(R_1(\alpha)\psi(\hbar)^{-1} + \ell_1\psi(\hbar)^{-\alpha}\right)^{\vartheta} \\ &= \frac{\phi(\hbar)}{\psi(\hbar)} f_0 \sum_{\vartheta=0}^{\infty}\left(\frac{-1}{R_0(\alpha)}\right)^{\vartheta}\sum_{a=0}^{\vartheta}\binom{\vartheta}{a}\left(R_1(\alpha)\psi(\hbar)^{-1}\right)^{\vartheta-a}\left(\ell_1\psi(\hbar)^{-\alpha}\right)^{a}. \end{aligned}$$

Therefore, this implies

$$F(\hbar) = f_0 \sum_{\vartheta=0}^{\infty}\left(\frac{-1}{R_0(\alpha)}\right)^{\vartheta}\sum_{a=0}^{\vartheta}\binom{\vartheta}{a}(R_1(\alpha))^{\vartheta-a}(\ell_1)^a\phi(\hbar)\psi(\hbar)^{-\vartheta-a-a\alpha-1}.$$

Using the inverse of GIT, we have

$$F(\hbar) = f_0 \sum_{\vartheta=0}^{\infty}\left(\frac{-1}{R_0(\alpha)}\right)^{\vartheta}\sum_{a=0}^{\vartheta}\frac{\vartheta!}{(\vartheta-a)!a!}(R_1(\alpha))^{\vartheta-a}(\ell_1)^a\frac{z^{\vartheta-a+a\alpha}}{\Gamma(\vartheta-a+a\alpha+1)}.$$

By assigning the parameters $\vartheta - a$ with p, the resulting expression can be written as follows:

$$\begin{aligned} F(\hbar) &= f_0 \sum_{p=0}^{\infty}\sum_{a=0}^{\infty}\left(\frac{-1}{R_0(\alpha)}\right)^{a+p}\frac{(a+p)!}{p!a!}(R_1(\alpha))^p(\ell_1)^a\frac{z^{p+a\alpha}}{\Gamma(p+a\alpha+1)} \\ &= f_0 \sum_{p=0}^{\infty}\sum_{a=0}^{\infty}\frac{(a+p)!}{p!a!}\left(\frac{-R_1(\alpha)}{R_0(\alpha)}z\right)^p\left(\frac{-\ell_1}{R_0(\alpha)}z^{\alpha}\right)^a\frac{1}{\Gamma(p+a\alpha+1)}. \end{aligned}$$

Therefore, bivariate Mittag–Leffler yields the expression given in (76).

7 Figure Analysis

The characteristic of the fractional Newtonion law of cooling is illustrated in Figures 1–4. They use Caputo, CF, and mABC derivatives for various α values. As α rises, the rate of cooling speeds up and gets closer to what it would be in a classical situation when $\alpha = 1$. The Caputo model's temperature drops the fastest in Fig. 1. The CF model's shift is smoother because its kernel is non-singular in Fig. 2. Figure 3 shows that the mABC model cools down the least, which means that the memory effect is greater. The comparison plot in Fig. 4 shows that the choice of fractional derivative has a

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big effect on the cooling dynamics. The mABC derivative better models long-term memory behaviour than the computed and CF models.

The fraction logistics equation using Caputo, CF and mABC derivative for different values of α is shown in Figures 5– 8. When α goes up, the growth rate of the answer goes up faster, getting closer to how logic works when $\alpha = 1$. The Caputo model (Fig. 5) has a sharper rise, while the CF model (Fig. 6) has a softer rise because of its exponential kernel, which shows that memory effects are not too strong. The mABC model (Fig. 7), which has a Mittag-Leffler kernel, grows the most gradually, which means that memory has a bigger effect and the system responds more slowly. The comparison plot in Fig. 8 shows that the fractional order and kernel type have a big impact on how the system changes over time. For processes that use long-term memory, the mABC derivative gives the most accurate picture.

The fractional blood alcohol model using Caputo, CF and mABC derivatives for different values of α are shown in Figures 9–12. When α goes up, the alcohol concentration f(z) goes down more quickly, getting closer to the classical case where $\alpha = 1$. Because its kernel is exponential, the CF model (Fig. 10) decays more slowly and smoothly than the Caputo model (Fig. 9). The mABC model (Fig. 11) shows the slowest decrease, which shows its strong memory effect and non-local activity. The comparison in Fig. 12 shows that the type of fractional derivative used has a big effect on the pattern of decay. The mABC derivative is a better representation of systems that have long-term memory and slow diffusion.

8 Conclusion

This study employs a generalized integral transform technique to examine fractional formulations of Newton's cooling process, the logistic growth equation, and the blood alcohol concentration system under Caputo, Caputo-Fabrizio, modified Atangana-Baleanu-Caputo, and constant proportional Caputo derivatives. The analytical as well as graphical outcomes produced, illustrate the impact of fractional parameters on system dynamics returning to classical examples as the order approaches one. In addressing various fractional differential equations the results validate the efficacy and versatility of the generalized transform. It also underscores the capacity for modelling real-world phenomena showing memory effects.

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