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Discrete Analogs of the Comparison Theorem and Two-Sided Estimates of Solution of Parabolic Equations

Piotr Matus[1,2,*], *Ryszard Kozera*[2,3], *Agnieszka Paradzinska*[2] *and Denis Schadinskii*[1]

[1] Institute of Mathematics, NAS of Belarus, 11 Surganov St., 220072 Minsk, Belarus
[2] Institute of Mathematics and Computer Science, The John Paul II Catholic University of Lublin, Al. Racławickie 14, 20-950 Lublin, Poland
[3] Faculty of Applied Informatics and Mathematics, Warsaw University of Life Sciences - SGGW , Nowoursynowska 159, 02-776 Warsaw, Poland

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Abstract: In this paper, we prove the difference analogs of the comparison theorems for solutions of the Cauchy problem for a nonlinear ordinary differential equation (ODE). These theorems are used to analyse blow-up solution of finite-difference schemes (FDS) approximating the Neumann problem for a parabolic equation with a nonlinear source of power form. We also propose the method for obtaining the two-sided estimates of solution. This method is based on implicit and explicit FDS.

Keywords: Comparison theorems, blow-up, two-sided estimates, existence solution, guaranteed accuracy

1 Introduction

The comparison theorems are often used to study the properties of solution of partial differential equations [1, 2, 3, 4]. Differential inequalities were the first time considered by S. A. Chaplygin in the first half of the XXth century [2]. We know just some discrete analogs of these inequalities. For example, there is discrete analog of the Bihari lemma for explicit FDS [3]. Some generalizations of this Lemma for the implicit FDS one can find in [5, 4, 6, 7].

In works [8, 9, 10, 11] interval methods for obtaining two-sided estimates of solution of the initial-value problems for ordinary differential equation and partial differential equation were considered. There is a growing interest in methods to get two-sided estimates [12, 13]. These methods allow to determine the interval which contains exact solution and are consistent with the order of accuracy of numerical method.

In this paper, we prove the difference analogs of the comparison theorems for solutions of the Cauchy problem for a nonlinear ordinary differential equation. These theorems are based on the properties of implicit and explicit FDS [14, 15]. In problems of the parabolic type whose solution develops a singularity [5], such theorems are very important in studying of solution behavior, stability in the context of the application for the reconstruction of the maria of the Moon [16, 17] and blow-up time [18]. Here, the blow-up is phenomenon when solution tends to infinity in finite time. We find a blow-up condition for solution of the FDS approximating these problems. We present numerical results for nonlinear parabolic equation with Neumann boundary condition, whose solution blows up in finite time. Note, the blow-up time of implicit FDS converges to blow-up time of differential problem when the mesh size tends to zero.

Last section is devoted to the method for obtaining two-sided estimates of solution of parabolic linear equation. The disadvantage of this method is the requirement of a constant sign of the input data derivatives. However, using FDS with variable weights [19], we may be able to generalize the proposed method to the case of functions of alternating can signs.

* Corresponding author e-mail: matus@im.bas-net.by

2 The Cauchy problem for ODE

2.1 Statement of the problem and FDS

We assume the existence of a classical solution $u(t) \in C^1(0,T] \cap C[0,T]$ of the problem

$$\frac{du}{dt} = f(t,u), \quad 0 < t \leq T, \quad u(0) = u_0.$$

Let $\alpha(t)$, $\beta(t)$ be lower and upper solutions [20] satisfying the following inequalities

$$\frac{d\alpha}{dt} \leq f(t,\alpha(t)), \quad 0 < t \leq T, \quad \alpha(0) = u_0, \tag{1}$$

$$\frac{d\beta}{dt} \geq f(t,\beta(t)), \quad 0 < t \leq T, \quad \beta(0) = u_0. \tag{2}$$

Then

$$\alpha(t) \leq u(t) \leq \beta(t), \tag{3}$$

for all t belonging to the whole interval of existence [2, 20]. Let us introduce the range of the exact solutions $D_u = \{u(t) : u_1 \leq u(t) \leq u_2, 0 \leq t \leq T\}$ and its neighbourhood, $D_\varepsilon(u) = \{\tilde{u} : |\tilde{u} - u| < \varepsilon\}$, which can be sufficiently small.

Now let us consider the case when the function $f(t,v)$ satisfies the condition given by the inequalities

$$f_2(t)g_2(v) \leq f(t,v) \leq f_1(t)g_1(v),$$

where $f_k(t), g_k(v), k = 1,2$, are continuous and monotonically growing functions for all $t \in [0,T], v \in D_\varepsilon(u)$. According to Corollary 4.2 in Chapter III [20], in this case function $u(t)$ is a lower and upper solution of the following differential problem

$$\frac{dv_1}{dt} = f_1(t)g_1(v_1), \quad \frac{dv_2}{dt} = f_2(t)g_2(v_2), \quad v_k(0) = u_0, \quad k = 1,2, \tag{4}$$

i.e.

$$v_1(t) \leq u(t) \leq v_2(t). \tag{5}$$

Let us introduce a nonuniform grid $\bar{\omega}_\tau = \omega_\tau \cup \{0\}$, $\omega_\tau = \{t_{n+1} = t_n + \tau_n, \tau_n > 0, n = 0,1,\dots,N_0 - 1, t_0 = 0, t_{N_0} = T\}$ on the interval $[0,T]$. We approximate problems(1)-(4) using the following schemes

$$\frac{v_k^{n+1} - v_k^n}{\tau_n} = \varphi_k(t_{n+1})q_k(v_k^{n+1}), \quad v_k^0 = u_0, \quad k = 1,2, \tag{6}$$

$$\frac{\alpha_\tau^{n+1} - \alpha_\tau^n}{\tau_n} \leq f_1(t_n)g_1(\alpha_\tau^n), \quad \alpha_\tau^0 = u_0, \tag{7}$$

and

$$\frac{\beta_\tau^{n+1} - \beta_\tau^n}{\tau_n} \geq f_2(t_{n+1})g_2(\beta_\tau^{n+1}), \quad \beta_\tau^0 = u_0, \tag{8}$$

where

$$\varphi_k(t_{n+1}) = \frac{1}{\tau_n}\int_{t_n}^{t_{n+1}} f_k(t)dt,$$

$$q_k(v_k^{n+1}) = \left[\frac{1}{v_k^{n+1} - v_k^n}\int_{v_k^n}^{v_k^{n+1}} \frac{dv}{g_k(v)}\right]^{-1}$$

are chosen from the condition of exact FDS [21], and $\alpha_\tau^n, \beta_\tau^n$ are lower and upper approximate solutions of the Cauchy problem, satisfying differential inequalities (1)-(2). It is worth mentioning that the corresponding discrete analogs of the lower estimation $\alpha \leq v_1$ and the upper estimation $\beta \geq v_2$ can be obtained with use of the explicit and implicit approximation, respectively. Note that the nonuniform grids $\bar{\omega}_\tau$ in each of problems (6)-(8) can be different and the nodes t_n do not coincide.

Lemma 1. For any $y^k, y^n \in D_\varepsilon(u)(y^n > y^k)$ and $t_k, t_n \in \omega_\tau(t_n > t_k)$ the following estimates

$$f_1(t_k)g_1(y^k) \leq \frac{1}{t_n - t_k}\int_{t_k}^{t_n} f_1(t)dt \left[\frac{1}{y^n - y^k}\int_{y^k}^{y^n} \frac{du}{g_1(u)}\right]^{-1}$$

and

$$f_2(t_n)g_2(y^n) \geq \frac{1}{t_n - t_k}\int_{t_k}^{t_n} f_2(t)dt \left[\frac{1}{y^n - y^k}\int_{y^k}^{y^n} \frac{du}{g_2(u)}\right]^{-1}$$

hold.

The proof follows directly from the mean value theorem.

Lemma 2. For the solution of problem (6) the following equality takes place

$$F_k(v_k(t_n)) = \Phi_k(t_n), \quad k = 1,2, \quad t_n \in \omega_\tau,$$

where

$$F_k(v_k^n) = \int_{u_0}^{v_k^n} \frac{dv}{g_k(v)}, \quad \Phi_k(t_n) = \int_0^{t_n} f_k(t)dt.$$

Proof. Formula (6) with $n = m$ can be rewritten as

$$\int_{v_k^m}^{v_k^{m+1}} \frac{dv}{g_k(v)} = \int_{t_m}^{t_{m+1}} f_k(t)dt.$$

Summation of the last equalities by indices $m = 0,1,\dots,n-1$, yields the required relations. The lemma is proved.

Theorem 1. Let the classic solution of the Cauchy problem (4) exists with $k=1$ and inequalities (7) hold with $\alpha_\tau^n \in D_\varepsilon(u), n=0,1,...,N_0$. Then

$$\alpha_\tau^m \leq v_1^m, \quad m=0,1,...,N_0. \tag{9}$$

Proof. For $m=0$ we have $\alpha_\tau^0 = v_1^0$. Assume inequality (9) be true for all $m=1,2,...,n$. We show that it holds for $m=n+1$ as well. In fact, using (7), Lemma 1 and the assumption of mathematical induction we get the estimation

$$\alpha_\tau^{n+1} \leq \alpha_\tau^n + \tau_n f_1(t_n) g_1(\alpha_\tau^n) \leq v_1^n + \tau_n f_1(t_n) g_1(v_1^n) \leq$$
$$\leq v_1^n + \tau_n \varphi_1(t_{n+1}) q_1(v_1^{n+1}) = v_1^{n+1}.$$

Theorem is proved.

Remark 1. In [3,22] one can find the proof of Theorem 1 for $f_1(t)=1$ in more complicated way.

Theorem 2. Let the classic solution of the Cauchy problem (4) exist with $k=2$ and the grid function $\beta_\tau^n \in D_\varepsilon(u), n=0,1,...,N_0$ satisfies inequality (8). Then

$$\beta_\tau^m \geq v_2^m, \quad m=0,1,...,N_0. \tag{10}$$

Proof. For $m=0$ we have $\beta_\tau^0 = v_2^0$. Let condition (10) be satisfied for all $m=1,2,...,n$. We prove that it is true for $m=n+1$. Then, using Lemmas 1 and 2 and (8) we obtain the following inequalities

$$\int_{\beta_\tau^n}^{\beta_\tau^{n+1}} \frac{dv}{g_2(v)} \geq \int_{t_n}^{t_{n+1}} f_2(t)dt,$$

$$\int_{u_0}^{\beta_\tau^{n+1}} \frac{dv}{g_2(v)} \geq \int_0^{t_{n+1}} f_2(t)dt = \Phi_2(t_{n+1}),$$

$$\int_{u_0}^{v_2^{n+1}} \frac{dv}{g_2(v)} + \int_{v_2^{n+1}}^{\beta_\tau^{n+1}} \frac{dv}{g_2(v)} \geq \int_0^{t_{n+1}} f_2(t)dt,$$

$$\int_{v_2^{n+1}}^{\beta_\tau^{n+1}} \frac{dv}{g_2(v)} \geq 0, \quad \beta_\tau^{n+1} \geq v_2^{n+1}.$$

Theorem 2 is therefore proved.

Corollary 1. Let for the FDS

$$y_{\alpha t} = f_1(t_n) g_1(y_\alpha^n), \quad y_{\beta t} = f_2(t_{n+1}) g_2(y_\beta^{n+1}),$$
$$y_\alpha^0 = y_\beta^0 = u_0,$$

solutions $y_\alpha, y_\beta \in D_\varepsilon(u)$ exist and unique and inequalities (7), (8) be satisfied. Then for all $t_n \in \bar{\omega}_\tau$ the following relations

$$\alpha_\tau^n \leq y_\alpha^n \leq v_1^n, \quad \beta_\tau^n \geq y_\beta^n \geq v_2^n, \quad m=0,1,...,N_0, \tag{11}$$

hold.

Example 1. Let $f(t,u)=t^r u^p$, $r>0$, $p>1$, $u_0>0$. Then in (11) we have

$$v_1^n = v_2 = u(t_n) = \frac{u_0}{\left(1 - \frac{p-1}{r+1} u_0^{p-1} t_n^{r+1}\right)^{1/(p-1)}}.$$

Consider important corollaries of the discrete analogues of Theorem 1 and 2. First from inequalities (9), (11) it follows the boundedness of the difference solution on the arbitrary segment $[0,t_1]$

$$t_1 < T_{blow-up}, \quad T_{blow-up} = \left(\frac{r+1}{(p-1)u_0^{p-1}}\right)^{\frac{1}{r+1}}.$$

The occurance of blow-up phenomena in the case of (9) is only possible. From the second estimate

$$y_\beta^n \geq \frac{u_0}{\left(1 - \frac{p-1}{r+1} u_0^{p-1} t_n^{r+1}\right)^{\frac{1}{p-1}}},$$

it follows that global solution of the FDS exists [25].

2.2 Solvability of the implicit FDS for ODE

In this Section, we prove the existence and the uniqueness of the solution of implicit FDS

$$\frac{\beta_\tau^{n+1} - \beta_\tau^n}{\tau_n} = f_1(t_{n+1}) f_2(\beta_\tau^{n+1}), \quad \beta_\tau^0 = u_0, \tag{12}$$

approximating the solution of the Cauchy problem

$$\frac{du}{dt} = f_1(t) f_2(u), \quad 0 \leq t \leq T, \quad u(0)=u_0, \tag{13}$$

$$f_2(u) \in C^1(D_\varepsilon(u)).$$

Here, we use known results for proving existence and uniqueness conditions. We present this proof only for obtaining bound of τ_n under which solution of implicit scheme exists and is unique.

According to (12) let us consider the nonlinear equation of the form

$$\frac{x - \beta_\tau^n}{\tau_n} = f_1(t_{n+1}) f_2(x). \tag{14}$$

Here the question about the existence of root $x = \beta_\tau^{n+1}$ in the set R arises. For the clarity of presentation let us rewrite the equation (14) as

$$x = \Phi(x), \tag{15}$$

where

$$\Phi(x) = \beta_\tau^n + \tau_n f_1(t_{n+1}) f_2(x).$$

Let us introduce the segment

$$U_r(a) = \{x : |x-a| \le r\} \subset D_\varepsilon(u). \tag{16}$$

Note that $\Phi(x) \in C^1(D_\varepsilon(u))$. Below we use the result from [23].

Lemma 3. If the following condition is satisfied

$$|\Phi^{'}(x)| \le q < 1, \quad |\Phi(a) - a| \le (1-q)r, \tag{17}$$

then equation (15) has a unique solution x_* in the interval $U_r(a)$ and the simple iteration method

$$x_{k+1} = \Phi(x_k), \quad k = 0, 1, \ldots \tag{18}$$

converges to x_* for an arbitrary initial point $x_0 \in U_r(a)$. Moreover, the following estimation

$$|x_k - x_*| \le q^k |x_0 - x_*|, \quad k = 0, 1, 2 \ldots.$$

holds.

We will show that the conditions of the lemma are fulfilled when

$$\Theta(\tau_n) = f_1(t_n + \tau_n)\tau_n < \frac{1}{\frac{|f_2(\beta_\tau^n)|}{r} + \max\limits_{x\in U_r(\beta_\tau^n)} |f_2^{'}(x)|}. \tag{19}$$

From (12) we have $\beta_\tau^0 = u_0 \in R$. Note that $\Phi^{'}(x) = \tau_n f_1(t_{n+1}) f_2^{'}(x)$. Then inequalities (17) are fulfilled if

$$\begin{aligned} &\tau_n |f_1(t_{n+1})| \le \frac{q}{\max\limits_{x\in U_r(a)} |f_2^{'}(x)|}, \quad q < 1, \\ &|\beta_\tau^n + \tau_n f_1(t_{n+1}) f_2(a) - a| < r(1-q). \end{aligned} \tag{20}$$

Hence, according to Lemma 3 the unique root of (17) exists in $U_r(a)$. One can show that $a = \beta_\tau^n$ and $q = \max\limits_{x\in U_r(\beta_\tau^n)} |\Phi^{'}(x)|$. Then (20) can be written as

$$\begin{aligned} &\tau_n |f_1(t_{n+1})| \le \frac{1}{\max\limits_{x\in U_r(\beta_\tau^n)} |f_2^{'}(x)|}, \\ &\tau_n |f_1(t_{n+1})| \le \frac{1}{\frac{|f_2(\beta_\tau^n)|}{r} + \max\limits_{x\in U_r(\beta_\tau^n)} |f_2^{'}(x)|}. \end{aligned} \tag{21}$$

Inequality (19) follows from (21).

2.3 Numerical experiment

Below we give the results of the numerical experiments. The explicit and implicit FDS are used for obtaining two-sided estimates of exact solution. To get more accurate results the schemes with weights $\sigma = 0.5 \pm \varepsilon$ are used.

Let $f_2(y) = y^2$ $f_1(t) \equiv 1$, $\alpha_0 = \beta_0 = 1$. Explicit and implicit FDS have a forms

$$\frac{\alpha^{n+1} - \alpha^n}{\tau_n} = (\alpha^n)^2, \quad n = 0, \ldots, N_0, \quad \alpha^0 = 1$$

and

$$\frac{\beta^{n+1} - \beta^n}{\tau_n^{'}} = (\beta^{n+1})^2, \quad n = 0, \ldots, N_0^{'}, \quad \beta^0 = 1,$$

respectively.

Using (19) we get the constraint for the step $\tau_n^{'} < \frac{1}{5\beta^n}$ with $r = \beta^n$.

The solution of the differential equation (13) with $F(t,u) = u^2$ and $u(0) = 1$ is given by the formula

$$u(t) = \frac{1}{1-t}.$$

Then, according to Theorems 2, 3 the estimates

$$\alpha^n \le u(t_n), \quad t_n = t_{n-1} + \tau_n, \quad n = 1, 2, \ldots, N_0,$$

and

$$\beta^n \ge u(t_n^{'}), \quad t_n^{'} = t_{n-1}^{'} + \tau_n^{'}, \quad n = 1, 2, \ldots, N_0^{'}$$

hold.

The results of numerical computations are given in Fig.1.

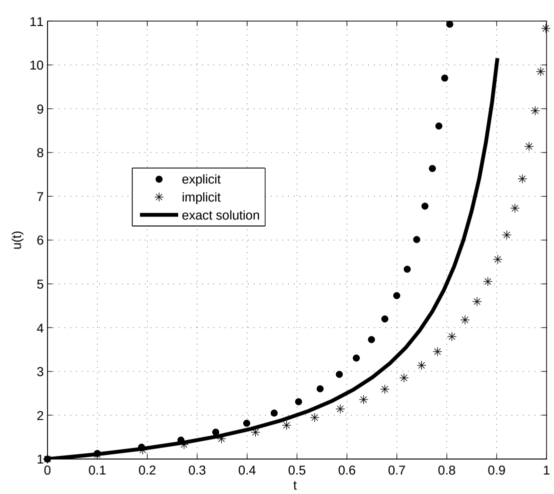

Fig. 1: Approximate solution given by explicit and implicit FDS.

In our experiment the steps were set as $\tau_n = (10\alpha^n)^{-1}$ and $\tau_n^{'} = (10\beta^n)^{-1}$.

3 The Neumann problem for a parabolic equation with a nonlinear source of power form

In this section the discrete analogues of Chaplygin and Hartman comparison theorems [2,20] are used for the analysis of boundary problems for PDE. The corresponding statement of IBVP (Initial Boundary Value Problem) in the one-dimensional case has the form [18]

$$\frac{\partial u}{\partial t}=\frac{\partial}{\partial x}\left(k(u)\frac{\partial u}{\partial x}\right)+f(u), \qquad (x,t)\in Q_T, \tag{22}$$

where

$$\begin{aligned} &u(x,0)=u_0(x)\geq 0,\\ &k(u)\frac{\partial u}{\partial x}\Big|_{x=0}=k(u)\frac{\partial u}{\partial x}\Big|_{x=l}=0,\\ &\bar{Q}_T=\{(x,t):0\leq x\leq l, 0\leq t\leq T\}. \end{aligned} \tag{23}$$

The coefficient $k(u)$ is assumed to satisfy the conditions of uniform parabolicity [24]: $0<k_1\leq k(u)\leq k_2, u\in \bar{D}_u, (x,t)\in \bar{Q}_T$. Further, we use the Jensen inequality for the convex functions in the continuous and discrete case, respectively.

Let the function $f(u)$ be convex on the arbitrary interval χ and $q_0,q_1,...,q_N>0$ and $q_0+q_1+...+q_N=1$. Then for arbitrary $u_0,u_1,...,u_N\in\chi$ and for an arbitrary function $u(t)$ integrable on χ the inequalities

$$f\left(\int_\chi u(x)dx\right)\leq \frac{1}{mes(\chi)}\int_\chi f(mes(\chi)u(x))dx, \tag{24}$$

and

$$f\left(\sum_{i=0}^N q_iu_i\right)\leq \sum_{i=0}^N q_if(u_i) \tag{25}$$

are satisfied.

According to [18], we integrate equation (22) with respect to $x\in[0,l]$. Then using (24),we get the following estimate

$$\frac{dv}{dt}\geq f(v), \qquad v(0)=v_0, \qquad v(t)=\frac{1}{l}\int_0^l u(x,t)dx.$$

For the continuous function $f(x)$ the estimate

$$\int_0^l f(x)dx\leq \max_{x\in[0,l]}|f(x)|l=l||f||_\infty$$

is valid. Hence, on the basis of Theorem 2 we have the following relation

$$||u(t)||_{L_\infty}\geq v(t)\geq w(t), \tag{26}$$

where $w(t)$ is the solution of Cauchy problem

$$\frac{dw}{dt}=f(w), \qquad w(0)=v_0.$$

Let us introduce the grid $\omega_\tau=\{t_{n+1}=t_n+\tau_n,\ \tau_n>0,\ n=0,1,...,N_0-1, t_0=0, t_{N_0}=T\}$ $\omega_h=\{x_i=ih, h=l/N, n=0,1,...,N\}$. Now consider the FDS

$$y_t=(ay_{\bar{x}})_x^{(\sigma)}+f(\hat{y}), \qquad (x,t)\in\omega, \qquad \omega=\omega_h\times\omega_\tau, \tag{27}$$

where

$$y(x,0)=u_0(x), \qquad a_i=0.5(k(y_{i-1})+k(y_i)), \tag{28}$$

$$\begin{aligned} &(a_1y_{x,0})^{(\sigma)}-\frac{h}{2}(y_{t,0}+f(\hat{y}_0))=\\ &=(a_Ny_{\bar{x},N})^{(\sigma)}+\frac{h}{2}(y_{t,N}+f(\hat{y}_N))=0, \end{aligned} \tag{29}$$

which approximates problem of (22)-(23) with an order $O(h^2+\tau)$.

Multiplying the difference equation (27) by h and summing the result over the internal nodes ω_h, we obtain

$$\sum_{i=1}^{N-1}hy_t=(a_Ny_{\bar{x},N})^{(\sigma)}-(a_1y_{x,0})^{(\sigma)}+\sum_{i=1}^{N-1}hf(\hat{y}_i).$$

Using the homogeneous boundary conditions (29) we rewrite the last expression as

$$\begin{aligned} &\frac{1}{l}\left(\frac{h}{2}y_{t,0}+\sum_{i=1}^{N-1}hy_t+\frac{h}{2}y_{t,N}\right)=\\ &=\frac{1}{l}\left(\frac{h}{2}f(\hat{y}_0)+\sum_{i=1}^{N-1}hf(\hat{y}_i)+\frac{h}{2}f(\hat{y}_N)\right). \end{aligned}$$

Since the sum of the coefficients of the function f is equal to 1 we can apply the discrete analogue Jensen's inequality (25). The following estimate

$$\frac{1}{l}\left(\frac{h}{2}y_0+h\sum_{i=1}^{N-1}y_i+\frac{h}{2}y_N\right)_t\geq f\left(\frac{h}{2}\hat{y}_0+h\sum_{i=1}^{N-1}\hat{y}_i+\frac{h}{2}\hat{y}_N\right),$$

is valid. For $v_h=\frac{1}{l}\left(\frac{h}{2}y_0+\sum\limits_{i=1}^{N-1}hy_i+\frac{h}{2}y_N\right)$, the last estimate can be rewritten in the form of

$$v_{ht}\geq f(\hat{v}_h). \tag{30}$$

Using the Theorem 1 we arrive at the estimate

$$v_h^n\geq w(t_n), \tag{31}$$

where $w(t)$ is the solution of the following Cauchy problem

$$\frac{dw}{dt}=f(w), \qquad w(0)=v_h^0. \tag{32}$$

Consider the following initial-boundary value problem

$$\frac{\partial u}{\partial t}=\frac{\partial}{\partial x}\left(u\frac{\partial u}{\partial x}\right)+u^2,\quad -\frac{L_s}{2}<x<\frac{L_s}{2},\quad 0<t<T_0,$$
$$u(x,0)=u_0(x),$$
$$\left(k(u)\frac{\partial u}{\partial x}\right)\left(-\frac{L_s}{2},t\right)=\left(k(u)\frac{\partial u}{\partial x}\right)\left(\frac{L_s}{2},t\right)=0,$$

which has solution [18]

$$u(x,t)=(T_0-t)^{-1}u_0(x),\quad T_0=T_{blow-up}=1,$$

$$u_0(x)=\begin{cases}\frac{4}{3}\cos^2\frac{\pi x}{L_s}, & |x|<L_s/2;\\ 0, & |x|\geq L_s/2.\end{cases}$$

where $L_s=2\sqrt{2}\pi$.

When T is finite and the solution u develops a singularity in a finite time, namely

$$\lim_{t\to T}||u(t)||_\infty=\infty,$$

then we say that solution u blows up in a finite time and time T is called the blow-up time of the solution u.

Similarly, if

$$||y^n||_{\infty,h}=\max_{x\in\bar{\omega}_h}|y^n|\geq\omega(t_n),\quad \lim_{t\to T_h}\omega(t)=\infty,$$

then we say that y_i^n blows-up for finite time $T_h\leq T$.

The theoretical study of blow-up of differential and numerical solutions for quasilinear parabolic equations has been the subject of investigation of many authors [25, 24, 26, 27, 28].

According to (31), (32), solution of the FDS (27)-(29) satisfies

$$\max_{(x,t)\in\bar{\omega}_\tau}|y(x,t)|\geq v_h^n\geq\omega(t_n)=\frac{v_h^0}{1-v_h^0 t},\quad v_h^0=\frac{2}{3}. \tag{33}$$

Inequality (33) implies that solution blows up in finite time.

In our numerical experiment, we use the conservative FDS (27)-(29) with $\sigma=0.5$. The time steps are set as $\tau_n=0.03/||y^n||_{\infty,h}$. Computing is stopped when $y^N>10^{150}$.

Numerical result for scheme with the explicit approximation of source with weight $\sigma=0,5$ is also presented. It is widely known that blow-up time of implicit FDS is less than the blow-up time of differential problem $T_0=1$ and vice versa for explicit FDS.

Numerical results are presented in Table 1.

Table 1: Numerical results.

Approximation of sources	h	N	τ_N	y_N	t_N
Explicit	0.27	46211	$1.01\cdot10^{-152}$	$1.00\cdot10^{150}$	1.017597
Implicit	0.27	45516	$1.00\cdot10^{-152}$	$1.01\cdot10^{150}$	0.976533

Here, we can see that solution of implicit FDS very rapidly tends to infinity and blow-up time is less than T_1 in accordance with (33). For the implicit FDS the blow-up time is less than T_1 and this corresponds to the above estimate (31). Since $v_h^0=v_0$, blow-up time of differential problem T_0 is less than T_1 that follows from (26).

4 Maximum principle for continuous and discrete cases

Here, we will need the maximum principle [29, 30, 31].

Let $T>0$, $S_T=[0,T]\times\partial\Omega$, $G_T=S_T\cup\Omega$, $Q_T=(0,T)\times\Omega$. We consider the equation

$$L(u)=f, \tag{34}$$

where

$$L(u)=\sum_{i,j=1}^{n}a_{ij}\frac{\partial^2 u}{\partial x_i\partial x_j}+\sum_{i=1}^{n}b_i\frac{\partial u}{\partial x_i}+cu-\frac{\partial u}{\partial t},$$

and $a_{i,j},b_i,c$ are real and finite functions independent of t and x.

We assume hereafter $a_{ij}=a_{ji}$ and

$$\sum_{i,j=1}^{n}a_{ij}(t,x)\xi_i\xi_j>0\quad\forall(t,x)\in\bar{Q}_T/G_T,$$

are valid for all nonzero $\xi\in R^n$.

Theorem 3. [29] Suppose that function u is continuous in $\bar{Q}_T$, derivatives of u in operator L are continuous in $\bar{Q}_T/G_T$ and

$$L(u(t,x))\leq 0,\quad (t,x)\in\bar{Q}_T/G_T,$$
$$u(t,x)\geq m_2\geq 0,\quad (t,x)\in G_T$$

are valid.

Let the coefficient c in operator L is bounded above by constant $m_1(c(t,x)<m_1),\forall(t,x)\in\bar{Q}_T)$. Then

$$u(t,x)\geq m_2,\quad (t,x)\in\bar{Q}_T.$$

Theorem 4. Suppose that function $u(t,x)$ is continuous in $\bar{Q}_T$, satisfies equation (34) in $\bar{Q}_T/G_T$ and $|u(t,x)|_{G_T}\leq q$. Let the function f is bounded and the coefficient c is not positive

$$|f(t,x)|\leq p,\quad c(t,x)\leq 0,\quad\forall(x,t)\in\bar{Q}_T.$$

Then inside $\bar{Q}_T$ the following inequality

$$|u(t,x)|\leq pt+q$$

is valid.

Let Ω_{grid} is grid, $S(x)$ is stencil (any subset of Ω_{grid}), $S^{'}(x) = S(x)/x$. Define grid operator L_{grid} :

$$L_{grid}y(x) = A(x)y(x) - \sum_{\xi \in S^{'}(x)} B(x,\xi)y(\xi),$$

and denote

$$D(x) = A(x) - \sum_{\xi \in S^{'}(x)} B(x,\xi).$$

We shall say that at point $x \in \Omega_{grid}$ the conditions of positiveness of coefficients holds, if

$$A(x) > 0, \quad B(x,\xi) > 0 \quad \forall \xi \in S^{'}(x), \quad D(x) \geq 0. \quad (35)$$

Theorem 5. [32] Let grid Ω_{grid} and its subset ω are connected, and $\bar{\omega} \subseteq \Omega_{grid}$. Suppose the condition of positiveness of coefficients (35) holds for for ω. If function $y(x)$, defined on Ω_{grid} , is not constant in $\bar{\omega}$ and $L_{grid}y(x) \leq 0$ for all $x \in \omega$ (or $L_{grid}y(x) \geq 0$ for all $x \in \omega$), then $y(x)$ does not take the largest positive (or smallest negative) value in ω for all its values on $\bar{\omega}$.

5 Two-sided estimates of solutions for IBVP for parabolic equations

Consider the simplest problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad (x,t) \in Q_T \quad (36)$$

where

$$\begin{aligned} &u(x,0) = u_0(x), \quad x \in [0,l], \\ &u(0,t) = \mu_1(t), \quad u(l,t) = \mu_2(t) \geq 0, \quad t \in [0,T]. \end{aligned} \quad (37)$$

Define uniform grid $\bar{\omega} = \bar{\omega}_\tau \times \bar{\omega}_h$ with a constant step h in space and τ in time. The differential problem (36), (37) is approximated by explicit FDS

$$y_{1t} = y_{1x\bar{x}}, \quad (x,t) \in \omega, \quad (38)$$

where

$$\begin{aligned} &y_1(x,0) = u_0(x), \quad x \in \bar{\omega}_h, \\ &y_1(0,\hat{t}) = \mu_1(\hat{t}), \quad y_1(l,\hat{t}) = \mu_2(\hat{t}), \quad \hat{t} \in \omega_\tau. \end{aligned} \quad (39)$$

The error of the scheme takes the form

$$z_1 = y_1 - u. \quad (40)$$

Then the error problem is written as

$$z_{1t} = z_{1x\bar{x}} + \psi_1, \quad (x,t) \in \omega, \quad (41)$$

$$z_1(x,0) = 0, \quad x \in \omega_h, \quad z_1(0,\hat{t}) = z_1(l,\hat{t}) = 0. \quad (42)$$

Using the Taylor expansion and equation (36) we get truncation error ψ_1 for FDS (38), (39)

$$\begin{aligned} \psi_1 = -u_t + u_{x\bar{x}} = -\frac{\tau}{2}\frac{\partial^3 u}{\partial t \partial x^2}(x_i,t_{n,n+1}) \\ + \frac{h^2}{24}\left(\frac{\partial^3 u}{\partial t \partial x^2}(x_{i,i+1},t_n) + \frac{\partial^3 u}{\partial t \partial x^2}(x_{i-1,i},t_n)\right), \end{aligned} \quad (43)$$

$$t_{n,n+1} \in (t_n,t_{n+1}), \quad x_{i,i+1} \in (x_i,x_{i+1}), \quad x_{i-1,i} \in (x_{i-1},x_i).$$

To use the discrete maximum principle we need to know the sign of the truncation error. For these purposes, we shall use the results given in Section 4. For function

$$w(x,t) = \frac{\partial^3 u}{\partial^2 x \partial t}, \quad (44)$$

using equation (36) we obtain the following initial boundary value problem

$$\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial x^2}, \quad (x,t) \in Q_T, \quad (45)$$

where

$$\begin{aligned} &w(x,0) = u_0^{(4)}(x), \quad x \in [0,L], \\ &u(0,t) = \mu_1^{''}(t), \quad u(L,t) = \mu_2^{''}(t), \quad t \in [0,T]. \end{aligned} \quad (46)$$

We assume that the right-hand sides of equalities in (46) are positive or

$$w|_{S_T} \geq m_2 > 0. \quad (47)$$

Then by Theorem_3 we obtain the estimate for the function w in domain Q_T

$$w(x,t) \geq m_2 > 0. \quad (48)$$

Using the properties of input data of the problem, we learn behaviour of solution derivatives in domain Q_T.

Based on (43), (48), from inequality

$$\tau \geq \frac{h^2}{12}\frac{\frac{\partial^3 u}{\partial t \partial x^2}(x_{i,i+1},t_n) + \frac{\partial^3 u}{\partial t \partial x^2}(x_{i-1,i},t_n)}{\frac{\partial^3 u}{\partial t \partial x^2}(x_i,t_{n,n+1})}, \quad (49)$$

it follows that truncation error of explicit FDS is a non-negative function

$$\psi_{1i} \leq 0. \quad (50)$$

Using grid maximum principle (Theorem 5) we get that for all grid nodes $z_i^n \leq 0$ or

$$u_i^n \geq y_{1i}^n. \quad (51)$$

Now we consider the implicit FDS

$$y_{2t} = \hat{y}_{2x\bar{x}}, \quad (x,t) \in \omega, \quad (52)$$

Table 2: Numerical results for $T=1$ and $h=\frac{\pi}{8}, \tau=\frac{1}{25}$.

Nodes	Exact solution u	Approximated solution $\bar{y}$	Error estimate $\frac{1}{2}\lvert y_{1i}^n - y_{2i}^n\rvert$
x_0	0.2180	0.2180	0.0000
x_1	0.3615	0.3625	0.0015
x_2	0.4738	0.4753	0.0023
x_3	0.5452	0.5473	0.0031
x_4	0.5698	0.5718	0.0032
x_5	0.5452	0.5473	0.0031
x_6	0.4738	0.4753	0.0023
x_7	0.3615	0.3625	0.0015
x_8	0.2180	0.2180	0.0000

Table 3: Numerical results for $T=1$ and $h=\frac{\pi}{16}, \tau=\frac{1}{100}$.

Nodes	Exact solution u	Approximated solution $\bar{y}$	Error estimate $\frac{1}{2}\lvert y_{1i}^n - y_{2i}^n\rvert$
x_0	0.2180	0.2180	0.0000
x_2	0.3615	0.3617	0.0003
x_4	0.4738	0.4741	0.0006
x_6	0.5452	0.5457	0.0007
x_8	0.5698	0.5703	0.0008
x_{10}	0.5452	0.5457	0.0007
x_{12}	0.4738	0.4741	0.0006
x_{14}	0.3615	0.3617	0.0003
x_{16}	0.2180	0.2180	0.0000

where

$$\begin{aligned} &y_2(x,0)=u_0(x), \quad x\in\bar{\omega}_h, \\ &y_2(0,\hat{t})=\mu_1(\hat{t}), \quad y_2(L,\hat{t})=\mu_2(\hat{t}), \quad \hat{t}\in\omega_\tau. \end{aligned} \tag{53}$$

The problem for error $z_2=y_2-u$ has the following form

$$z_{2t}=\hat{z}_{2x\bar{x}}+\psi_2, \quad x\in\omega, \tag{54}$$

$$z_2(x,0)=0, \quad x\in\omega_h, \quad z_2(0,\hat{t})=z_2(L,\hat{t})=0. \tag{55}$$

Then truncation error ψ_2 can be written in the following form

$$\begin{aligned} \psi_2 = -u_t+\hat{u}_{x\bar{x}} = \frac{\tau}{2}\frac{\partial^3 u}{\partial t\partial x^2}(x_i,t_{n,n+1})+ \\ +\frac{h^2}{24}\left(\frac{\partial^3 u}{\partial t\partial x^2}(x_{i,i+1},t_{n+1})+\frac{\partial^3 u}{\partial t\partial x^2}(x_{i-1,i},t_{n+1})\right). \end{aligned}$$

Note that for any τ and h function ψ_2 is non-negative, i.e.

$$\psi_2\geq 0, \quad (x,t)\in\bar{\omega}. \tag{56}$$

Using the grid maximum principle we obtain that for all $i=\overline{0,N}, n=\overline{0,N_0}$,

$$u_i^n\leq y_{2i}^n. \tag{57}$$

From inequalities (51) and (57) the desired result follows

$$y_{1i}^n\leq u_i^n\leq y_{2i}^n. \tag{58}$$

Inequality (58) is valid if stability condition for explicit FDS and inequality (49) are fulfilled

$$\min_{i,n}\frac{h^2}{6}\frac{M_i^n}{m_i^n}\leq\tau\leq\frac{h^2}{2}, \tag{59}$$

where M_i^n and m_i^n is minimum and maximum of function w in rectangle $G_i^n=\{(x,t): x_{i-1}\leq x\leq x_{i+1}, t_n\leq t_{n+1}\}$, respectively.

If input data (45), (46) are negative, similar result can be obtained

$$w|_{S_T}\leq m_2<0.$$

In this case, under the condition

$$\tau\leq\min\left\{\frac{h^2}{2},\min_{i,n}\frac{h^2}{6}\frac{M_i^n}{m_i^n}\right\},$$

we get two-sided estimate

$$y_{2i}^n\leq u_i^n\leq y_{1i}^n, \quad i=\overline{0,N}, \quad n=\overline{0,N_0}.$$

We can also approximate the solution by the half-sum of the solutions of explicit and implicit FDS $\bar{y}_i^n=\frac{1}{2}(y_{1i}^n+y_{2i}^n), i=\overline{0,N}, n=\overline{0,N_0}$. The error estimate follows from the inequality

$$\max_{i,n}|\bar{y}_i^n-u_i^n|\leq\frac{1}{2}\max_{i,n}|y_{1i}^n-y_{2i}^n|.$$

5.1 Numerical experiment

In Table 2, we present numerical results for problem (36), (37), which has the exact solution

$$u(x,t) = e^{-\frac{9}{16}t} cos(-\frac{\pi}{2} + \frac{3}{4}x).$$

Reducing time step τ four times and space h step two times we established that the error estimate was reduced four time. These results can be found in Table 3.

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Piotr Matus holds Professor position at the Institute of Mathematics, National Academy of Sciences of Belarus and John Paul II Catholic University of Lublin. He is a member of the editorial board of 8 international journals. Professor Matus directed and advised 18 Ph.D. students in Belarus and Poland. He published more than 150 scientific papers in international journals and two monographs. An important milestone in Professor Matus' scientific activity was the creation of the journal "Computational methods in applied mathematics" in 2001. His research interests are finite difference schemes for initial boundary value problem for nonlinear equation of mathematical physics.

Ryszard Kozera is a Professor at the Faculty of Applied Informatics and Mathematics at Warsaw University of Life Sciences - SGGW. He is also the Adjunct Associate Professor at the School of Information Science and Software Engineering at The University of Western Australia in Perth. He received his M.Sc. degree in Pure Mathematics at Warsaw University, Poland (1985), a Ph.D. degree in Computer Science at Flinders

University of South Australia, Adelaide, Australia (1991) and his D.Sc. degree in Computer Science (a Habilitation) at The Silesian University of Technology, Gliwice, Poland (2006). Professor Kozera was awarded three times Alexander von Humboldt Fellowships between 1995-2004. He delivered about 60 seminars at various prestigious international research centers, published 90 refereed journal and conference papers, organized international conferences and participated in the collaborative international grants. He has 21 postgraduate supervisions including Ph.D. and M.Sc. students. Professor Kozera's research interests include computer vision, numerical analysis, optimization, interpolation and approximation, partial differential equations and artificial intelligence. More information can be found under Professor Kozera's homepage (see URL: http://www.wzim.sggw.pl/ryszard_kozera or URL: http://www.csse.uwa.edu.au/~ryszard).

Agnieszka Paradzinska was the Ph.D. student and Assistant at John Paul II Catholic University of Lublin. She has worked with Professor Peter Matus from The National Academy of Sciences of Belarus. Her main research interest is Mathematical Modelling. Her investigation and papers regarded High Accuracy Difference Scheme for Transfer Equation.

Schadinskii Denis graduated from the Belarusian State University; he earned his master's degree in computational mathematics. At present he is a Ph.D. student, Institute of mathematics of National Academy of Sciences of Belarus. Schadinskii's research interests are numerical methods for solving equations of mathematical physics with blow-up.