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The Complete and Concrete Solution Procedures for Integrated Vendor-Buyer Cooperative Inventory Models with Trade Credit Financing in Supply Chain Management

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Abstract: In today's manufacturing environment, most enterprises are facing fierce competition from the pressure of the increasing costs. Thus, naturally, the integrated inventory models are adopted to increase the advantage of reducing the total cost. This paper explores some vendor-buyer inventory models with trade credit financing under suppliers' credits linked to the order quantity. The main purpose of this paper is to adopt and apply the rigorous methods and tools of mathematical analysis to develop the complete and concrete solution procedures for integrated vendor-buyer cooperative inventory models. The theoretical results presented in this paper are shown to improve some of the existing results in earlier articles.

Keywords: Inventory models; Trade credit; Cooperative solution; Supply chain management; Integrated vendor-buyer cooperative inventory models; Mathematical and analytical tools; Convex functions; Complete and concrete solution procedures.

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1 Introduction

The integrated inventory models usually have the advantage of reducing the total cost. In the modern global competitive market, the supplier and the retailer should be treated as strategic partners in the supply chain with a long-term cooperative relationship. Goyal [6] was probably the first researcher to develop the seller-customer inventory model. On the other hand, in the real world, suppliers usually provide a delay period in payment to encourage retailers to buy more order quantity. During the trade credit period, the retailers can obtain the interest from the nonpayment and sales income, while suppliers lose the interest income during the same time. Goyal [7] developed the economic order quantity model under conditions of permissible delay in payments. Chen and Kang [1] first incorporated the aforementioned works of Goyal ([6] and [7]) in order to establish integrated vendor-buyer cooperative inventory models with permissible delay in payments. In addition, Khouja and Mehrez [8] first explored supplier credit policies where the credit terms are linked to the order quantity. Teng *et al.* [12] combine the investigations by Khouja and Mehrez [8] and Chen and Kang [1] to consider vendor-buyer inventory models with trade credit financing under suppliers' credits linked to the order

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quantity. Basically, the integrated inventory models discussed in Teng *et al.* [12] are sufficiently interesting. However, the solution procedures to locate the optimal solutions in Teng *et al.* [12] are full of shortcomings such that their solution procedures can and should be further improved. The main purpose of this paper is not only to remove those shortcomings, but also to provide the complete and concrete solution procedures for the models considered by Teng *et al.* [12]. Finally, numerical examples are given to illustrate the theoretical results presented in this paper. The literature review in connection with this paper, the reader can refer to many of the closely-realated earlier works cited by Teng *et al.* [12]. Additionally, several other related works on the subject of this paper include (for example) the recent papers [2] to [5], [9] and [10].

2 Notations and Assumptions

The following notation and assumptions are adopted throughout this article.

- A the buyer's ordering cost per order
- S the vendor's set up cost per production run
- F the vendor's fixed process cost for handing each order
- h_v the vendor's holding cost per unit per year excluding interest charges
- h_b the buyer's holding cost per unit per year excluding interest charges
- c the buyer's procurement cost per unit
- p the buyer's selling price per unit with $p > c$
- D the buyer's annual demand rate
- R the vendor's annual production rate with $R > D$
- T the buyer's replenishment cycle time in years (a decision variable)
- M the permissible delay period in years offered by the vendor
- Q the buyer's order quantity
- Q_d the minimum order quantity at which the delay in payments is permitted
- I_d the buyer's annual investment return rate per \$
- I_c the buyer's annual interest rate to be paid per \$ in stock to the bank
- I_v the vendor's annual investment return rate on the opportunity loss due to the delay payment
- n the vendor's integral number of lots per production run (a decision variable)
- $TC(n,T)$ the annual total inventory cost for both the vendor and the buyer

The following assumptions are used in order to develop the model which we consider here.

1. The lead time is zero. Demand rate is constant over time. Shortages are not allowed.
2. If the order quantity is less than Q_d, then the payment must be made immediately after receiving the items. If the order quantity is greater than or equal to Q_d, then the delay in payments up to M is permitted. If the permissible delay M is granted, then the buyer invests the revenue and earns an annual return rate I_d by time M. When payment is due at time M, the buyer pays off all items sold to the vendor, and starts paying interest charged I_c on all unsold items as collaterals.
3. The vendor produces an integer multiple amount of the buyer's order quantity at one setup and ships to the buyer over multiple deliveries of n.
4. $p > c$ and $h_b \geqq h_v$.
5. Time horizon is infinite.

In this paper, we assume that neither $pI_d \geqq cI_c$ nor $I_c \geqq I_d$ in order to generalize the work by Teng *et al.* [12]. We also just explore the integrated inventory model only, in which case both the vendor and the buyer form a single firm.

3 The Mathematical Model to be Investigated

Based upon the above notations and assumptions (see Section 2), Teng *et al.* [12] divide discussions of the integrated annual total inventory costs $TC(n,T)$ for both the vendor and the buyer into two possible cases: (1) $T_d \leqq M$, and (2) $T_d > M$ in which both n and T are two decision variables.

Case 1. $T_d \leqq M$. In this case, the $TC(n,T)$ can be expressed as follows:

$$TC(n,T) = \begin{cases} TC_1(n,T) & (n \geqq 1;\ 0 < T < T_d) \quad (1a) \\ TC_2(n,T) & (n \geqq 1;\ T_d \leqq T \leqq M) \quad (1b) \\ TC_3(n,T) & (n \geqq 1;\ M \leqq T, \quad (1c) \end{cases}$$

where

$$TC_1(n,T) = \frac{A+F+\frac{S}{n}}{T} + \frac{D(h_b+cI_c)T}{2} + \frac{Dh_vT}{2}\left[(n-1)\left(1-\frac{D}{R}\right)+\frac{D}{R}\right], \quad (2)$$

$$TC_2(n,T) = \frac{A+F+\frac{S}{n}}{T} + \frac{D}{2}\left\{h_b + pI_d + h_v\left[(n-1)\left(1-\frac{D}{R}\right)+\frac{D}{R}\right]\right\}T + (cI_v - pI_d)MD \quad (3)$$

and

$$TC_3(n,T) = \frac{A+F+\frac{S}{n}}{T} + \frac{D}{2}\left\{h_b + h_v\left[(n-1)\left(1-\frac{D}{R}\right)+\frac{D}{R}\right]\right\}T + \frac{DcI_c(T-M)^2}{2T} - \frac{DpI_dM^2}{2T} + DcI_vM. \quad (4)$$

Equations (2) to (4) reveal the fact that

$$TC_1(n,T_d) - TC_2(n,T_d) = \frac{D(h_b+cI_c)T_d}{2} + (pI_d - cI_v)MD \quad (5)$$

and

$$TC_2(n,M) = TC_3(n,M). \tag{6}$$

In general, $TC_1(n,T_d) \neq TC_2(n,T_d)$. Hence, for any given n, $TC(n,T)$ is continuous except when $T = T_d$.

Our problem is to determine the optimal vendor's integral number of lots per production run n^* and the optimal buyer's replenishment cycle time T^* which minimizes $TC(n,T)$. So, the equations (1*a*) to (1*c*) imply that

$$TC(n^*,T^*) = \min\left\{TC_1(n^*_{11},T^*_{11}), TC_2(n^*_{21},T^*_{21}), TC_3(n^*_{31},T^*_{31})\right\}, \tag{7}$$

where

$$TC_1(n^*_{11},T^*_{11}) = \min\{TC_1(n,T) : n \geqq 1 \text{ and } 0 < T < T_d\}, \tag{8}$$

$$TC_2(n^*_{21},T^*_{21}) = \min\{TC_2(n,T) : n \geqq 1 \text{ and } T_d \leqq T \leqq M\}, \tag{9}$$

$$TC_3(n^*_{31},T^*_{31}) = \min\{TC_3(n,T) : n \geqq 1 \text{ and } M \leqq T\}, \tag{10}$$

where (n^*_{11},T^*_{11}), (n^*_{21},T^*_{21}) and (n^*_{31},T^*_{31}) are assumed to exist.

Case 2. $T_d > M$.

$$TC(n,T) = \begin{cases} TC_1(n,T) & (n \geqq 1;\ 0 < T < T_d) \quad (11a) \\ TC_3(n,T) & (n \geqq 1;\ T_d \leqq T). \quad (11b) \end{cases}$$

Equations (2) to (4) reveal the fact that

$$TC_1(n,T_d) - TC_3(n,T_d) = \frac{D(pI_d - cI_c)M^2}{2T_d}.$$

In general,

$$TC_1(n,T_d) \neq TC_3(n,T_d).$$

Hence, for any given n, $TC(n,T)$ is continuous except when $T = T_d$.

Our problem is to determine the optimal solution (n^*,T^*) which minimizes $TC(n,T)$. So, clearly, the equations (11a) and (11b) imply that

$$TC(n^*,T^*) = \min\{TC_1(n^*_{12},T^*_{12}), TC_3(n^*_{32},T^*_{32})\}, \tag{12}$$

where

$$TC_1(n^*_{12},T^*_{12}) = \min\{TC_1(n,T) : n \geqq 1 \text{ and } 0 < T < T_d\} \tag{13}$$

and

$$TC_3(n^*_{32},T^*_{32}) = \min\{TC_3(n,T) : n \geqq 1 \text{ and } T_d \leqq T\}, \tag{14}$$

where (n^*_{12}, T^*_{12}) and (n^*_{32}, T^*_{32}) are assumed to exist.

4 The Convexity of $TC_i(n,T)$ $(i = 1,2,3)$

Equations (2) to (4) yield the first-order and the second-order partial derivatives with respect to T as follows:

$$\frac{\partial TC_1(n,T)}{\partial T} = -\frac{A+F+\frac{S}{n}}{T^2} + \frac{D}{2}\left\{h_b + cI_c + h_v\left[(n-1)\left(1-\frac{D}{R}\right)+\frac{D}{R}\right]\right\}, \tag{15}$$

$$\frac{\partial^2 TC_1(n,T)}{\partial T^2} = \frac{2\left(A+F+\frac{S}{n}\right)}{T^3} > 0, \tag{16}$$

$$\frac{\partial TC_2(n,T)}{\partial T} = -\frac{A+F+\frac{S}{n}}{T^2} + \frac{D}{2}\left\{h_b + pI_d + h_v\left[(n-1)\left(1-\frac{D}{R}\right)+\frac{D}{R}\right]\right\}, \tag{17}$$

$$\frac{\partial^2 TC_2(n,T)}{\partial T^2} = \frac{2\left(A+F+\frac{S}{n}\right)}{T^3} > 0, \tag{18}$$

$$\frac{\partial TC_3(n,T)}{\partial T} = -\frac{\left[2\left(A+F+\frac{S}{n}\right) + DM^2(cI_c - pI_d)\right]}{2T^2} + \frac{D}{2}\left\{h_b + cI_c + h_v\left[(n-1)\left(1-\frac{D}{R}\right)+\frac{D}{R}\right]\right\} \tag{19}$$

and

$$\frac{\partial^2 TC_3(n,T)}{\partial T^2} = \frac{2\left(A+F+\frac{S}{n}\right) + DM^2(cI_c - pI_d)}{T^3}. \tag{20}$$

For simplicity, Teng *et al.* [12] assumed that

$$2\left(A+F+\frac{S}{n}\right) + DM^2(cI_c - pI_d) \geqq 0 \text{ for all } n \geqq 1. \tag{21}$$

Equation (21) implies that

$$2(A+F) + DM^2(cI_c - pI_d) \geqq 0.$$

However, in order to make the equation (43) in Teng *et al.* [12] valid, we further assume that

$$2(A+F) + DM^2(cI_c - pI_d) > 0.$$

Hence, $TC_i(n,T)$ is convex on $T > 0$ for any given n and $i = 1,2,3$.

Upon solving

$$\frac{\partial TC_i(n,T)}{\partial T} = 0 \qquad (i = 1,2,3), \tag{22}$$

we obtain

$$T^*_1(n) = \sqrt{\frac{2\left(A+F+\frac{S}{n}\right)}{D\left\{h_b + cI_c + h_v\left[(n-1)\left(1-\frac{D}{R}\right)+\frac{D}{R}\right]\right\}}}, \tag{23}$$

$$T_2^*(n)=\sqrt{\frac{2\left(A+F+\frac{S}{n}\right)}{D\left\{h_b+pI_d+h_v\left[(n-1)\left(1-\frac{D}{R}\right)+\frac{D}{R}\right]\right\}}} \tag{24}$$

and

$$T_3^*(n)=\sqrt{\frac{2\left(A+F+\frac{S}{n}\right)+DM^2(cI_c-pI_d)}{D\left\{h_b+cI_c+h_v\left[(n-1)\left(1-\frac{D}{R}\right)+\frac{D}{R}\right]\right\}}} \tag{25}$$

as the respective roots of the equation (22). Furthermore, the convexity of the function $TC_i(n,T)$ for $T>0$ implies that (see, for details, [11])

$$\frac{\partial TC_i(n,T)}{\partial T}\begin{cases}<0 & \text{if } 0<T<T_i^*(n), \quad (26a)\\ =0 & \text{if } T=T_i^*(n), \quad (26b)\\ >0 & \text{if } T>T_i^*(n). \quad (26c)\end{cases}$$

Equations (26a) to (26c) reveal the fact that $TC_i(n,T)$ is decreasing on $(0,T_i^*(n)]$ and increasing on $[T_i^*(n),\infty)$ for any given n and $i=1,2,3$. Equations (15), (17) and (19) yield

$$\frac{\partial TC_1(n,T_d)}{\partial T}=\frac{\Delta_1(n,T_d)}{2T_d^2}, \tag{27}$$

$$\frac{\partial TC_2(n,T_d)}{\partial T}=\frac{\Delta_2(n,T_d)}{2T_d^2}, \tag{28}$$

$$\frac{\partial TC_3(n,T_d)}{\partial T}=\frac{\Delta_3(n,T_d)}{2T_d^2}, \tag{29}$$

$$\frac{\partial TC_2(n,M)}{\partial T}=\frac{\partial TC_3(n,M)}{\partial T}=\frac{\Delta_{23}(n,M)}{2M^2}, \tag{30}$$

$$\Delta_1(n,T_d)=-2\left(A+F+\frac{S}{n}\right)+DT_d^2\left\{h_b+cI_c+h_v\left[(n-1)\left(1-\frac{D}{R}\right)+\frac{D}{R}\right]\right\}, \tag{31}$$

$$\Delta_2(n,T_d)=-2\left(A+F+\frac{S}{n}\right)+DT_d^2\left\{h_b+pI_d+h_v\left[(n-1)\left(1-\frac{D}{R}\right)+\frac{D}{R}\right]\right\}, \tag{32}$$

$$\Delta_3(n,T_d)=-2\left(A+F+\frac{S}{n}\right)+DcI_c(T_d^2-M^2)+DM^2pI_d+DT_d^2\left\{h_b+h_v\left[(n-1)\left(1-\frac{D}{R}\right)+\frac{D}{R}\right]\right\} \tag{33}$$

and

$$\Delta_{23}(n,M)=-2\left(A+F+\frac{S}{n}\right)+DM^2\left\{h_b+pI_d+h_v\left[(n-1)\left(1-\frac{D}{R}\right)+\frac{D}{R}\right]\right\}. \tag{34}$$

Equations (32) to (34) imply that

$$\Delta_{23}(n,M)\geqq\Delta_2(n,T_d) \text{ if } T_d\leqq M, \tag{35}$$

$$\Delta_3(n,T_d)\geqq\Delta_2(n,T_d) \text{ if } T_d>M, \tag{36}$$

$$\Delta_{23}(n+1,M)>\Delta_{23}(n,M) \text{ if } n\geqq 1 \tag{37}$$

and

$$\Delta_i(n+1,T_d)>\Delta_i(n,T_d) \text{ if } n\geqq 1 \text{ and } i=1,2,3. \tag{38}$$

Based on the above arguments, we have the following results.

Lemma 1. *For any given n, the following two cases would occur.*

Case 1. $T_d\leqq M$.

(A) *If $\Delta_1(n,T_d)>0$, then $TC_1(n,T)$ is decreasing on $(0,T_1^*(n)]$ and increasing on $[T_1^*(n),T_d)$.*
(B) *If $\Delta_1(n,T_d)\leqq 0$, then $TC_1(n,T)$ is decreasing on $(0,T_d)$.*
(C) *If $\Delta_2(n,T_d)>0$, then $TC_2(n,T)$ is increasing on $[T_d,M]$ and $TC_3(n,T)$ is increasing on $[M,\infty)$.*
(D) *If $\Delta_2(n,T_d)\leqq 0<\Delta_{23}(n,M)$, then $TC_2(n,T)$ is decreasing on $(0,T_2^*(n)]$ and increasing on $[T_2^*(n),M)$.*
(E) *If $\Delta_{23}(n,M)\leqq 0$, then $TC_3(n,T)$ is decreasing on $[M,T_3^*(n)]$ and increasing on $[T_3^*(n),\infty)$.*
(F) *If $\Delta_{23}(n,M)>0$, then $TC_3(n,T)$ is increasing on $[M,\infty)$.*

Case 2. $M<T_d$.

(A) *If $\Delta_1(n,T_d)>0$, then $TC_1(n,T)$ is decreasing on $(0,T_1^*(n)]$ and increasing on $[T_1^*(n),T_d)$.*
(B) *If $\Delta_1(n,T_d)\leqq 0$, then $TC_1(n,T)$ is decreasing on $(0,T_d)$.*
(C) *If $\Delta_3(n,T_d)>0$, then $TC_3(n,T)$ is increasing on $[T_d,\infty)$.*
(D) *If $\Delta_3(n,T_d)\leqq 0$, then $TC_3(n,T)$ is decreasing on $[T_d,T_3^*(n)]$ and increasing on $[T_3^*(n),\infty)$.*

5 Theorems for the Optimal Solution (n^*,T^*) of $T(n,T)$

Let

$$\overline{TC}_1(n)=TC_1(n,T_1^*(n)), \tag{39}$$

$$\overline{TC}_2(n)=TC_2(n,T_2^*(n)) \tag{40}$$

and

$$\overline{TC}_3(n)=TC_3(n,T_3^*(n)). \tag{41}$$

Then the equations (31), (37) and (42) in Teng *et al.* [12] reveal that

$$\overline{TC}_1(n)=\sqrt{2D\left(A+F+\tfrac{S}{n}\right)\left\{h_b+cI_c+h_v\left[(n-1)\left(1-\tfrac{D}{R}\right)+\tfrac{D}{R}\right]\right\}}, \quad (42)$$

$$\overline{TC}_2(n)=\sqrt{2D\left(A+F+\frac{S}{n}\right)\left\{h_b+pI_d+h_v\left[(n-1)\left(1-\frac{D}{R}\right)+\frac{D}{R}\right]\right\}}+(cI_v-pI_d)MD \quad (43)$$

and

$$\overline{TC}_3(n)=\sqrt{D\left[2\left(A+F+\frac{S}{n}\right)+DM^2(cI_c-pI_d)\right]}\cdot\sqrt{\left\{h_b+cI_c+h_v\left[(n-1)\left(1-\frac{D}{R}\right)+\frac{D}{R}\right]\right\}}+c(I_v-I_c)MD. \quad (44)$$

Taking the first-order derivatives of $\overline{TC}_i(n)$ $(i=1,2,3)$ with respect to n, we have

$$\frac{d\overline{TC}_1(n)}{dn}=\frac{D\left\{(A+F)h_v\left(1-\frac{D}{R}\right)-\frac{S}{n^2}\left[h_b+cI_c+h_v\left(\frac{2D}{R}-1\right)\right]\right\}}{\sqrt{2D\left(A+F+\frac{S}{n}\right)\left\{h_b+cI_c+h_v\left[(n-1)\left(1-\frac{D}{R}\right)+\frac{D}{R}\right]\right\}}}, \quad (45)$$

$$\frac{d\overline{TC}_2(n)}{dn}=\frac{D\left\{(A+F)h_v\left(1-\frac{D}{R}\right)-\frac{S}{n^2}\left[h_b+pI_d+h_v\left(\frac{2D}{R}-1\right)\right]\right\}}{\sqrt{2D\left(A+F+\frac{S}{n}\right)\left\{h_b+pI_d+h_v\left[(n-1)\left(1-\frac{D}{R}\right)+\frac{D}{R}\right]\right\}}}, \quad (46)$$

$$\frac{d\overline{TC}_3(n)}{dn}=\frac{D\left\{h_v\left(1-\frac{D}{R}\right)\left[2(A+F)+DM^2(cI_c-pI_d)\right]-\frac{2S}{n^2}\left[h_b+cI_c+h_v\left(\frac{2D}{R}-1\right)\right]\right\}}{\sqrt{D\left[2\left(A+F+\frac{S}{n}\right)+DM^2(cI_c-pI_d)\right]\left\{h_b+cI_c+h_v\left[(n-1)\left(1-\frac{D}{R}\right)+\frac{D}{R}\right]\right\}}}. \quad (47)$$

We now let

$$n_1=\sqrt{\frac{S\left[h_b+cI_c+h_v\left(\frac{2D}{R}-1\right)\right]}{(A+F)h_v\left(1-\frac{D}{R}\right)}}, \quad (48)$$

$$n_2=\sqrt{\frac{S\left[h_b+pI_d+h_v\left(\frac{2D}{R}-1\right)\right]}{(A+F)h_v\left(1-\frac{D}{R}\right)}} \quad (49)$$

and

$$n_3=\sqrt{\frac{2S\left[h_b+cI_c+h_v\left(\frac{2D}{R}-1\right)\right]}{h_v\left(1-\frac{D}{R}\right)\left[2(A+F)+DM^2(cI_c-pI_d)\right]}}. \quad (50)$$

Combining the equations (45) to (50), we obtain

$$\frac{d\overline{TC}_i(n)}{dn}\begin{cases}<0 & \text{if } 0<n<n_i, & (51a)\\ =0 & \text{if } n=n_i, & (51b)\\ >0 & \text{if } n>n_i, & (51c)\end{cases}$$

for $i=1,2,3$. Equations (51a) to (51c) imply that $\overline{TC}_i(n)$ is decreasing on $(0,n_i]$ and increasing on $[n_i,\infty)$ for $i=1,2,3$. Let

$$n_{i0}^*=\lfloor n_i\rfloor:=\text{the greatest integer}\leqq n_i \quad (52)$$

and

$$\overline{TC}_i(n_i^*)=\min\left\{\overline{TC}_i(n):n\geqq 1\right\}. \quad (53)$$

Then we have

$$\overline{TC}_i(n_i^*)=\min\left\{\overline{TC}_i(n_{i0}^*),\overline{TC}_i(n_{i0}^*+1)\right\} \quad (54)$$

for $i=1,2,3$. Based upon the above arguments, we have the following results.

Theorem 1. *Suppose that* $T_d\leqq M$.
1. *If* $\Delta_1(n_{10}^*,T_d)>0$, $\Delta_2(n_{20}^*,T_d)>0$ *and* $\Delta_{23}(n_{30}^*,M)>0$, *then*

$$TC(n^*,T^*)=\min\left\{\begin{array}{l}TC_1(n_{10}^*,T_1^*(n_{10}^*)),TC_1(n_{10}^*+1,T_1^*(n_{10}^*+1)),\\ TC_2(n_{20}^*,T_d),TC_2(n_{20}^*+1,T_d),\\ TC_3(n_{30}^*,M),TC_3(n_{30}^*+1,M)\end{array}\right\}. \quad (55)$$

2. *Suppose that*
If $\Delta_1(n_{10}^*,T_d)\leqq 0<\Delta_1(n_{10}^*+1,T_d)$, $\Delta_2(n_{20}^*,T_d)>0$ *and* $\Delta_{23}(n_{30}^*,M)>0$.

$$TC_1(n_{10}^*,T_d)\geqq\min\left\{\begin{array}{l}TC_1(n_{10}^*+1,T_1^*(n_{10}^*+1)),\\ TC_2(n_{20}^*,T_d),TC_2(n_{20}^*+1,T_d),\\ TC_3(n_{30}^*,M),TC_3(n_{30}^*+1,M)\end{array}\right\},$$

then

$$TC(n^*,T^*)=\min\left\{\begin{array}{l}TC_1(n_{10}^*+1,T_1^*(n_{10}^*+1)),\\ TC_2(n_{20}^*,T_d),TC_2(n_{20}^*+1,T_d),\\ TC_3(n_{30}^*,M),TC_3(n_{30}^*+1,M)\end{array}\right\}. \quad (56)$$

Otherwise, (n^*,T^*) *does not exist.*
3. *Suppose that*

$$\Delta_1(n_{10}^*+1,T_d)\leqq 0,\quad \Delta_2(n_{20}^*,T_d)>0\quad and\quad \Delta_{23}(n_{30}^*,M)>0.$$

If

$$\min\left\{TC_1(n_{10}^*,T_d),TC_1(n_{10}^*+1,T_d)\right\}\geqq\min\left\{\begin{array}{l}TC_2(n_{20}^*,T_d),TC_2(n_{20}^*+1,T_d),\\ TC_3(n_{30}^*,M),TC_3(n_{30}^*+1,M)\end{array}\right\},$$

then

$$TC(n^*,T^*)=\min\left\{\begin{array}{l}TC_2(n_{20}^*,T_d),TC_2(n_{20}^*+1,T_d),\\ TC_3(n_{30}^*,T_d),TC_3(n_{30}^*+1,M)\end{array}\right\}. \quad (57)$$

Otherwise, (n^*,T^*) *does not exist.*

4. *If*

$$\Delta_1(n_{10}^*,T_d)>0,\quad \Delta_2(n_{20}^*,T_d)\leqq 0<\Delta_2(n_{20}^*+1,T_d),$$

$$\Delta_{23}(n_{20}^*,M)>0 \quad \text{and} \quad \Delta_{23}(n_{30}^*,M)>0,$$

then

$$TC(n^*,T^*)=\min\left\{\begin{array}{l} TC_1(n_{10}^*,T_1^*(n_{10}^*)),TC_1(n_{10}^*+1,T_1^*(n_{10}^*+1)),\\ TC_2(n_{20}^*,T_d),TC_2(n_{20}^*+1,T_d),\\ TC_3(n_{30}^*,M),TC_3(n_{30}^*+1,M)\end{array}\right\}. \tag{58}$$

5. *Suppose that*

$$\Delta_1(n_{10}^*,T_d)\leqq 0<\Delta_1(n_{10}^*+1,T_d),$$

$$\Delta_2(n_{20}^*,T_d)\leqq 0<\Delta_2(n_{20}^*+1,T_d),$$

$$\Delta_{23}(n_{20}^*,M)>0 \qquad \text{and} \qquad \Delta_{23}(n_{30}^*,M)>0.$$

If

$$TC(n_{10}^*,T_d)\geqq \min\left\{\begin{array}{l} TC_1(n_{10}^*+1,T_1^*(n_{10}^*+1)),\\ TC_2(n_{20}^*,T_2^*(n_{20}^*)),TC_2(n_{20}^*+1,T_d),\\ TC_3(n_{30}^*,M),TC_3(n_{30}^*+1,M)\end{array}\right\},$$

then

$$TC(n^*,T^*)=\min\left\{\begin{array}{l} TC_1(n_{10}^*+1,T_1^*(n_{10}^*+1)),\\ TC_2(n_{20}^*,T_2^*(n_{20}^*)),TC_2(n_{20}^*+1,T_d),\\ TC_3(n_{30}^*,M),TC_3(n_{30}^*+1,M)\end{array}\right\}. \tag{59}$$

Otherwise, (n^*,T^*) *does not exist.*

6. *Suppose that*

$$\Delta_1(n_{10}^*+1,T_d)\leqq 0,\qquad \Delta_2(n_{20}^*,T_d)\leqq 0<\Delta_2(n_{20}^*+1,T_d),$$

$$\Delta_{23}(n_{20}^*,M)>0 \qquad \text{and} \qquad \Delta_{23}(n_{30}^*,M)>0.$$

If

$$\min\{TC_1(n_{10}^*,T_d),TC_1(n_{10}^*+1,T_d)\} \geqq \min\left\{\begin{array}{l} TC_2(n_{20}^*,T_2^*(n_{20}^*)),\\ TC_2(n_{20}^*+1,T_d),\\ TC_3(n_{30}^*,M),TC_3(n_{30}^*+1,M)\end{array}\right\},$$

then

$$TC(n^*,T^*)=\min\left\{\begin{array}{l} TC_2(n_{20}^*,T_2^*(n_{20}^*)),TC_2(n_{20}^*+1,T_d),\\ TC_3(n_{30}^*,M),TC_3(n_{30}^*+1,M)\end{array}\right\}. \tag{60}$$

Otherwise, (n^*,T^*) *does not exist.*

7. *If*

$$\Delta_1(n_{10}^*,T_d)>0,$$

$$\Delta_2(n_{20}^*,T_d)\leqq \Delta_{23}(n_{20}^*,M)\leqq 0<\Delta_2(n_{20}^*+1,T_d)$$

and

$$\Delta_{23}(n_{30}^*,M)>0,$$

then

$$TC(n^*,T^*) =\min\left\{\begin{array}{l} TC_1(n_{10}^*,T_1^*(n_{10}^*)),TC_1(n_{10}^*+1,T_1^*(n_{10}^*+1)),\\ TC_2(n_{20}^*,M),TC_2(n_{20}^*+1,T_d),\\ TC_3(n_{30}^*,M),TC_3(n_{30}^*+1,M)\end{array}\right\}. \tag{61}$$

8. Suppose that

$$\Delta_1(n_{10}^*,T_d)\leqq 0<\Delta_1(n_{10}^*+1,T_d),$$

$$\Delta_2(n_{20}^*,T_d)\leqq \Delta_{23}(n_{20}^*,M)\leqq 0<\Delta_2(n_{20}^*+1,T_d)$$

and

$$\Delta_{23}(n_{30}^*,M)>0.$$

If

$$TC_1(n_{10}^*,T_d) \geqq \min\left\{\begin{array}{l} TC_1(n_{10}^*+1,T_1^*(n_{10}^*+1)),TC_2(n_{20}^*,M),\\ TC_2(n_{20}^*+1,T_d),TC_3(n_{30}^*,M),TC_3(n_{30}^*+1,M)\end{array}\right\}, \tag{62}$$

then

$$TC(n^*,T^*) =\min\left\{\begin{array}{l} TC_1(n_{10}^*+1,T_1^*(n_{10}^*+1)),TC_2(n_{20}^*,M),\\ TC_2(n_{20}^*+1,T_d),TC_3(n_{30}^*,M),TC_3(n_{30}^*+1,M)\end{array}\right\}. \tag{63}$$

Otherwise, (n^*,T^*) doest not exist.

9. Suppose that

$$\Delta_1(n_{10}^*+1,T_d)\leqq 0,$$

$$\Delta_2(n_{20}^*,T_d)\leqq \Delta_{23}(n_{20}^*,M)\leqq 0<\Delta_2(n_{20}^*+1,T_d)$$

and

$$\Delta_{23}(n_{30}^*,M)>0.$$

If

$$\min\{TC_1(n_{10}^*,T_d),TC_1(n_{10}^*+1,T_d)\} \geqq \min\left\{\begin{array}{l} TC_2(n_{20}^*,M),TC_2(n_{20}^*+1,T_d),\\ TC_3(n_{30}^*,M),TC_3(n_{30}^*+1,M)\end{array}\right\},$$

then

$$TC(n^*,T^*)=\min\left\{\begin{array}{l} TC_2(n_{20}^*,M),TC_2(n_{20}^*+1,T_d),\\ TC_3(n_{30}^*,M),TC_3(n_{30}^*+1,M)\end{array}\right\}. \tag{64}$$

Otherwise, (n^*,T^*) does not exist.

10. If

$$\Delta_1(n_{10}^*,T_d)>0,$$

$$\Delta_2(n_{20}^*+1,T_d)\leqq 0<\Delta_{23}(n_{20}^*,M) \quad \text{and} \quad \Delta_{23}(n_{30}^*,M)>0,$$

then

$$TC(n^*,T^*) = \min\left\{\begin{array}{l} TC_1(n_{10}^*,T_1^*(n_{10}^*)), TC_1(n_{10}^*+1,T_1^*(n_{10}^*+1)), \\ TC_2(n_{20}^*,T_2^*(n_{20}^*)), TC_2(n_{20}^*+1,T_2^*(n_{20}^*+1)), \\ TC_3(n_{30}^*,M), TC_3(n_{30}^*+1,M) \end{array}\right\}. \quad (65)$$

11. Suppose that $\Delta_1(n_{10}^*,T_d) \leqq 0 < \Delta_1(n_{10}^*+1,T_d)$, $\Delta_2(n_{20}^*+1,T_d) \leqq 0 < \Delta_{23}(n_{20}^*,M)$ and $\Delta_{23}(n_{30}^*,M) > 0$. If

$$TC_1(n_{10}^*,T_d) \geqq \min\left\{\begin{array}{l} TC_1(n_{10}^*+1,T_1^*(n_{10}^*+1)), TC_2(n_{20}^*,T_2^*(n_{20}^*)), \\ TC_2(n_{20}^*+1,T_2^*(n_{20}^*+1)), TC_3(n_{30}^*,M), TC_3(n_{30}^*+1,M) \end{array}\right\}, \quad (66)$$

then

$$TC(n^*,T^*) = \min\left\{\begin{array}{l} TC_1(n_{10}^*+1,T_1^*(n_{10}^*+1)), TC_2(n_{20}^*,T_2^*(n_{20}^*)), \\ TC_2(n_{20}^*+1,T_2^*(n_{20}^*+1)), TC_3(n_{30}^*,M), TC_3(n_{30}^*+1,M) \end{array}\right\}. \quad (67)$$

Otherwise, (n^*,T^*) does not exist.

12. Suppose that

$$\Delta_1(n_{10}^*+1,T_d) \leqq 0, \ \ \Delta_2(n_{20}^*+1,T_d) \leqq 0 < \Delta_{23}(n_{20}^*,M)$$

and

$$\Delta_{23}(n_{30}^*,M) > 0.$$

If

$$\min\{TC_1(n_{10}^*,T_d), TC_1(n_{10}^*+1,T_d)\} \geqq \min\left\{\begin{array}{l} TC_2(n_{20}^*,T_2^*(n_{20}^*)), \\ TC_2(n_{20}^*+1,T_2^*(n_{20}^*+1)), \\ TC_3(n_{30}^*,M), TC_3(n_{30}^*+1,M) \end{array}\right\},$$

then

$$TC(n^*,T^*) = \min\left\{\begin{array}{l} TC_2(n_{20}^*,T_2^*(n_{20}^*)), \\ TC_2(n_{20}^*+1,T_2^*(n_{20}^*+1)), \\ TC_3(n_{30}^*,M), TC_3(n_{30}^*+1,M) \end{array}\right\}. \quad (68)$$

Otherwise, (n^*,T^*) does not exist.

13. If

$$\Delta_1(n_{10}^*,T_d) > 0, \ \ \Delta_2(n_{20}^*+1,T_d) \leqq 0 < \Delta_{23}(n_{20}^*+1,M)$$

and

$$\Delta_{23}(n_{30}^*,M) > 0,$$

then

$$TC(n^*,T^*) = \min\left\{\begin{array}{l} TC_1(n_{10}^*,T_1^*(n_{10}^*)), TC_1(n_{10}^*+1,T_1^*(n_{10}^*+1)), \\ TC_2(n_{20}^*,M), TC_2(n_{20}^*+1,T_2^*(n_{20}^*+1)), \\ TC_3(n_{30}^*,M), TC_3(n_{30}^*+1,M) \end{array}\right\}. \quad (69)$$

14. Suppose that

$$\Delta_1(n_{10}^*,T_d) \leqq 0 < \Delta_1(n_{10}^*+1,T_d), \qquad \Delta_2(n_{20}^*+1,T_d) \leqq 0,$$

$$\Delta_{23}(n_{20}^*,M) \leqq 0 < \Delta_{23}(n_{20}^*+1,M) \qquad \textit{and} \qquad \Delta_{23}(n_{30}^*,M) > 0.$$

If

$$TC_1(n_{10}^*,T_d) \geqq \min\left\{\begin{array}{l} TC_1(n_{10}^*+1,T_1^*(n_{10}^*+1)), TC_2(n_{20}^*,M), \\ TC_2(n_{20}^*+1,T_2^*(n_{20}^*+1)), TC_3(n_{30}^*,M), TC_3(n_{30}^*+1,M) \end{array}\right\},$$

then

$$TC(n^*,T^*) = \min\left\{\begin{array}{l} TC_1(n_{10}^*+1,T_1^*(n_{10}^*+1)), TC_2(n_{20}^*,M), \\ TC_2(n_{20}^*+1,T_2^*(n_{20}^*+1)), TC_3(n_{30}^*,M), TC_3(n_{30}^*+1,M) \end{array}\right\}. \quad (70)$$

Otherwise, (n^*,T^*) does not exist.

15. Suppose that

$$\Delta_1(n_{10}^*+1,T_d) \leqq 0, \qquad \Delta_2(n_{20}^*+1,T_d) \leqq 0,$$

$$\Delta_{23}(n_{20}^*,M) \leqq 0 < \Delta_{23}(n_{20}^*+1,M)$$

and

$$\Delta_{23}(n_{30}^*,M) > 0.$$

If

$$\min\{TC_1(n_{10}^*,T_d), TC_1(n_{10}^*+1,T_d)\} \geqq \min\left\{\begin{array}{l} TC_2(n_{20}^*,M,), \\ TC_2(n_{20}^*+1,T_2^*(n_{20}^*+1)), \\ TC_3(n_{30}^*,M), TC_3(n_{30}^*+1,M) \end{array}\right\},$$

then

$$TC(n^*,T^*) = \min\left\{\begin{array}{l} TC_2(n_{20}^*,M), TC_2(n_{20}^*+1,T_2^*(n_{20}^*+1)), \\ TC_3(n_{30}^*,M), TC_3(n_{30}^*+1,M) \end{array}\right\}. \quad (71)$$

16. If

$$\Delta_1(n_{10}^*,T_d) > 0, \quad \Delta_{23}(n_{20}^*+1,M) \leqq 0$$

and

$$\Delta_{23}(n_{30}^*,M) > 0,$$

then

$$TC(n^*,T^*) = \min\left\{\begin{array}{l} TC_1(n_{10}^*,T_1^*(n_{10}^*)), TC_1(n_{10}^*+1,T_1^*(n_{10}^*+1)), TC_2(n_{20}^*,M), \\ TC_2(n_{20}^*+1,M), TC_3(n_{30}^*,M), TC_3(n_{30}^*+1,M) \end{array}\right\}. \quad (72)$$

17. Suppose that

$$\Delta_1(n_{10}^*,T_d) \leqq 0 < \Delta_1(n_{10}^*+1,T_d), \ \ \Delta_{23}(n_{20}^*+1,M) \leqq 0$$

and

$$\Delta_{23}(n_{30}^*,M) > 0.$$

If

$$TC_1(n_{10}^*,T_d) \geqq \min\left\{\begin{array}{l} TC_1(n_{10}^*+1,T_1^*(n_{10}^*+1)), TC_2(n_{20}^*,M), \\ TC_2(n_{20}^*+1,M), TC_3(n_{30}^*,M), TC_3(n_{30}^*+1,M) \end{array}\right\},$$

then

$$TC(n^*,T^*) = \min\left\{\begin{array}{l} TC_1(n_{10}^*+1,T_1^*(n_{10}^*+1)), TC_2(n_{20}^*,M), \\ TC_2(n_{20}^*+1,M), TC_3(n_{30}^*,M), TC_3(n_{30}^*+1,M) \end{array}\right\}. \quad (73)$$

Otherwise, (n^*,T^*) does not exist.

18. Suppose that

$$\Delta_1(n_{10}^*+1,T_d) \leqq 0,\ \ \Delta_{23}(n_{20}^*+1,M) \leqq 0 \quad and \quad \Delta_{23}(n_{30}^*,M) > 0.$$

If

$$\min\{TC_1(n_{10}^*,T_d),TC_1(n_{10}^*+1,T_d)\} \geqq \min\left\{\begin{array}{l} TC_2(n_{20}^*,M),TC_2(n_{20}^*+1,M), \\ TC_3(n_{30}^*,M),TC_3(n_{30}^*+1,M)\end{array}\right\},$$

then

$$TC(n^*,T^*) = \min\left\{\begin{array}{l} TC_2(n_{20}^*,M),TC_2(n_{20}^*+1,M), \\ TC_3(n_{30}^*,M),TC_3(n_{30}^*+1,M)\end{array}\right\}. \tag{74}$$

Otherwise, (n^*,T^*) does not exist.

19. If

$$\Delta_1(n_{10}^*,T_d) > 0,\ \ \Delta_2(n_{20}^*,T_d) > 0$$

and

$$\Delta_{23}(n_{30}^*,M) \leqq 0 < \Delta_{23}(n_{30}^*+1,M),$$

then

$$TC(n^*,T^*) = \min\left\{\begin{array}{l} TC_1(n_{10}^*,T_1^*(n_{10}^*)),TC_1(n_{10}^*+1,T_1^*(n_{10}^*+1)),TC_2(n_{20}^*,T_d), \\ TC_2(n_{20}^*+1,T_d),TC_3(n_{30}^*,T_3^*(n_{30}^*)),TC_3(n_{30}^*+1,M)\end{array}\right\}. \tag{75}$$

20. Suppose that

$$\Delta_1(n_{10}^*,T_d) \leqq 0 < \Delta_1(n_{10}^*+1,T_d), \qquad \Delta_2(n_{20}^*,T_d) > 0$$

and

$$\Delta_{23}(n_{30}^*,M) \leqq 0 < \Delta_{23}(n_{30}^*+1,M).$$

If

$$TC_1(n_{10}^*,T_d) \geqq \min\left\{\begin{array}{l} TC_1(n_{10}^*+1,T_1^*(n_{10}^*+1)),TC_2(n_{20}^*,T_d), \\ TC_2(n_{20}^*+1,T_d),TC_3(n_{30}^*,T_3^*(n_{30}^*)),TC_3(n_{30}^*+1,M)\end{array}\right\},$$

then

$$TC(n^*,T^*) = \min\left\{\begin{array}{l} TC_1(n_{10}^*+1,T_1^*(n_{10}^*+1)),TC_2(n_{20}^*,T_d), \\ TC_2(n_{20}^*+1,T_d),TC_3(n_{30}^*,T_3^*(n_{30}^*)),TC_3(n_{30}^*+1,M)\end{array}\right\}. \tag{76}$$

Otherwise, (n^*,T^*) does not exist.

21. Suppose that

$$\Delta_1(n_{10}^*+1,T_d) \leqq 0, \qquad \Delta_2(n_{20}^*,T_d) > 0$$

and

$$\Delta_{23}(n_{30}^*,M) \leqq 0 < \Delta_{23}(n_{30}^*+1,M).$$

If

$$\min\{TC_1(n_{10}^*,T_d),TC_1(n_{10}^*+1,T_d)\} \geqq \min\left\{\begin{array}{l} TC_2(n_{20}^*,T_d),TC_2(n_{20}^*+1,T_d), \\ TC_3(n_{30}^*,T_3^*(n_{30}^*)),TC_3(n_{30}^*+1,M)\end{array}\right\},$$

then

$$TC(n^*,T^*) = \min\left\{\begin{array}{l} TC_2(n_{20}^*,T_d),TC_2(n_{20}^*+1,T_d), \\ TC_3(n_{30}^*,T_3^*(n_{30}^*)),TC_3(n_{30}^*+1,M)\end{array}\right\}. \tag{77}$$

Otherwise, (n^*,T^*) does not exist.

22. If

$$\Delta_1(n_{10}^*,T_d) > 0, \qquad \Delta_2(n_{20}^*,T_d) \leqq 0 < \Delta_2(n_{20}^*+1,T_d),$$
$$\Delta_{23}(n_{30}^*,M) > 0$$

and

$$\Delta_{23}(n_{30}^*,M) \leqq 0 < \Delta_{23}(n_{30}^*+1,M),$$

then

$$TC(n^*,T^*) = \min\left\{\begin{array}{l} TC_1(n_{10}^*,T_1^*(n_{10}^*)),TC_1(n_{10}^*+1,T_1^*(n_{10}^*+1)),TC_2(n_{20}^*,T_2^*(n_{20}^*)), \\ TC_2(n_{20}^*+1,T_d),TC_3(n_{30}^*,T_3^*(n_{30}^*)),TC_3(n_{30}^*+1,M)\end{array}\right\}. \tag{78}$$

23. Suppose that

$$\Delta_1(n_{10}^*,T_d) \leqq 0 < \Delta_1(n_{10}^*+1,T_d),$$
$$\Delta_2(n_{20}^*,T_d) \leqq 0 < \Delta_2(n_{20}^*+1,T_d),$$
$$\Delta_{23}(n_{20}^*,M) > 0$$

and

$$\Delta_{23}(n_{30}^*,M) \leqq 0 < \Delta_{23}(n_{30}^*+1,M).$$

If

$$TC_1(n_{10}^*,T_d) \geqq \min\left\{\begin{array}{l} TC_1(n_{10}^*+1,T_1^*(n_{10}^*+1)),TC_2(n_{20}^*,T_2^*(n_{20}^*)), \\ TC_2(n_{20}^*+1,T_d),TC_3(n_{30}^*,T_3^*(n_{30}^*)),TC_3(n_{30}^*+1,M)\end{array}\right\},$$

then

$$TC(n^*,T^*) = \min\left\{\begin{array}{l} TC_1(n_{10}^*+1,T_1^*(n_{10}^*+1)),TC_2(n_{20}^*,T_2^*(n_{20}^*)), \\ TC_2(n_{20}^*+1,T_d),TC_3(n_{30}^*,T_3^*(n_{30}^*)),TC_3(n_{30}^*+1,M)\end{array}\right\}. \tag{79}$$

Otherwise, (n^*,T^*) does not exist.

24. Suppose that

$$\Delta_1(n_{10}^*+1,T_d) \leqq 0, \qquad \Delta_2(n_{20}^*,T_d) \leqq 0 < \Delta_2(n_{20}^*+1,T_d),$$
$$\Delta_{23}(n_{20}^*,M) > 0$$

and

$$\Delta_{23}(n_{30}^*,M) \leqq 0 < \Delta_{23}(n_{30}^*+1,M).$$

If

$$\min\{TC_1(n_{10}^*,T_d),TC_1(n_{10}^*+1,T_d)\} \geqq \min\left\{\begin{array}{l} TC_2(n_{20}^*,T_2^*(n_{20}^*)),TC_2(n_{20}^*+1,T_d), \\ TC_3(n_{30}^*,T_3^*(n_{30}^*)),TC_3(n_{30}^*+1,M)\end{array}\right\},$$

then

$$TC(n^*,T^*) = \min\left\{\begin{array}{l} TC_2(n_{20}^*,T_2^*(n_{20}^*)),TC_2(n_{20}^*+1,T_d), \\ TC_3(n_{30}^*,T_3^*(n_{30}^*)),TC_3(n_{30}^*+1,M)\end{array}\right\}. \tag{80}$$

Otherwise, (n^*,T^*) does not exist.

25. If

$$\Delta_1(n_{10}^*,T_d) > 0,$$
$$\Delta_2(n_{20}^*,T_d) \leqq \Delta_{23}(n_{20}^*,M) \leqq 0 < \Delta_2(n_{20}^*+1,T_d)$$

and

$$\Delta_{23}(n_{30}^*, M) \leqq 0 < \Delta_{23}(n_{30}^* + 1, M),$$

then

$$TC(n^*, T^*) = \min \left\{ \begin{array}{l} TC_1(n_{10}^*, T_1^*(n_{10}^*)), TC_1(n_{10}^* + 1, T_1^*(n_{10}^* + 1)), TC_2(n_{20}^*, M), \\ TC_2(n_{20}^* + 1, T_d), TC_3(n_{30}^*, T_3^*(n_{30}^*)), TC_3(n_{30}^* + 1, M) \end{array} \right\}. \quad (81)$$

26. Suppose that

$$\Delta_1(n_{10}^*, T_d) \leqq 0 < \Delta_1(n_{10}^* + 1, T_d),$$

$$\Delta_2(n_{20}^*, T_d) \leqq \Delta_{23}(n_{20}^*, M) \leqq 0 < \Delta_2(n_{20}^* + 1, T_d)$$

and

$$\Delta_{23}(n_{30}^*, M) \leqq 0 < \Delta_{23}(n_{30}^* + 1, M).$$

If

$$TC_1(n_{10}^*, T_d) \geqq \min \left\{ \begin{array}{l} TC_1(n_{10}^* + 1, T_1^*(n_{10}^* + 1)), TC_2(n_{20}^*, M), \\ TC_2(n_{20}^* + 1, T_d), TC_3(n_{30}^*, T_3^*(n_{30}^*))), TC_3(n_{30}^* + 1, M) \end{array} \right\},$$

then

$$TC(n^*, T^*) = \min \left\{ \begin{array}{l} TC_1(n_{10}^* + 1, T_1^*(n_{10}^* + 1)), TC_2(n_{20}^*, M), \\ TC_2(n_{20}^* + 1, T_d), TC_3(n_{30}^*, T_3^*(n_{30}^*))), TC_3(n_{30}^* + 1, M) \end{array} \right\}. \quad (82)$$

Otherwise, (n^*, T^*) does not exist.

27. Suppose the

$$\Delta_1(n_{10}^* + 1, T_d) \leqq 0,$$

$$\Delta_2(n_{20}^*, T_d) \leqq \Delta_{23}(n_{20}^*, M) \leqq 0 < \Delta_2(n_{20}^* + 1, T_d)$$

and

$$\Delta_{23}(n_{30}^*, M) \leqq 0 < \Delta_{23}(n_{30}^* + 1, M).$$

If

$$\min\{TC_1(n_{10}^*, T_d), TC_1(n_{10}^* + 1, T_d)\} \geqq \min \left\{ \begin{array}{l} TC_2(n_{20}^*, M), TC_2(n_{20}^* + 1, T_d), \\ TC_3(n_{30}^*, T_3^*(n_{30}^*)), TC_3(n_{30}^* + 1, M) \end{array} \right\},$$

then

$$TC(n^*, T^*) = \min \left\{ \begin{array}{l} TC_2(n_{20}^*, M), TC_2(n_{20}^* + 1, T_d), \\ TC_3(n_{30}^*, T_3^*(n_{30}^*)), TC_3(n_{30}^* + 1, M) \end{array} \right\}. \quad (83)$$

Otherwise, (n^*, T^*) does not exist.

28. If

$$\Delta_1(n_{10}^*, T_d) > 0, \qquad \Delta_2(n_{20}^* + 1, T_d) \leqq 0 < \Delta_{23}(n_{20}^*, M)$$

and

$$\Delta_{23}(n_{30}^*, M) \leqq 0 < \Delta_{23}(n_{30}^* + 1, M),$$

then

$$TC(n^*, T^*) = \min \left\{ \begin{array}{l} TC_1(n_{10}^*, T_1^*(n_{10}^*)), TC_1(n_{10}^* + 1, T_1^*(n_{10}^* + 1)), TC_2(n_{20}^*, T_2^*(n_{20}^*)), \\ TC_2(n_{20}^* + 1, T_2^*(n_{20}^* + 1)), TC_3(n_{30}^*, T_3^*(n_{30}^*)), TC_3(n_{30}^* + 1, M) \end{array} \right\}. \quad (84)$$

29. Suppose that

$$\Delta_1(n_{10}^*, T_d) \leqq 0 < \Delta_1(n_{10}^* + 1, T_d),$$

$$\Delta_2(n_{20}^* + 1, T_d) \leqq 0 < \Delta_{23}(n_{20}^*, M)$$

and

$$\Delta_{23}(n_{30}^*, M) \leqq 0 < \Delta_{23}(n_{30}^* + 1, M).$$

If

$$TC_1(n_{10}^*, T_d) \geqq \min \left\{ \begin{array}{l} TC_1(n_{10}^* + 1, T_1^*(n_{10}^* + 1)), TC_2(n_{20}^*, T_2^*(n_{20}^*)), \\ TC_2(n_{20}^* + 1, T_2^*(n_{20}^* + 1)), TC_3(n_{30}^*, T_3^*(n_{30}^*)), \\ TC_3(n_{30}^* + 1, M) \end{array} \right\},$$

then

$$TC(n^*, T^*) = \min \left\{ \begin{array}{l} TC_1(n_{10}^* + 1, T_1^*(n_{10}^* + 1)), TC_2(n_{20}^*, T_2^*(n_{20}^*)), \\ TC_2(n_{20}^* + 1, T_2^*(n_{20}^* + 1)), TC_3(n_{30}^*, T_3^*(n_{30}^*)), \\ TC_3(n_{30}^* + 1, M) \end{array} \right\}. \quad (85)$$

Otherwise, (n^*, T^*) does not exist.

30. Suppose that

$$\Delta_1(n_{10}^* + 1, T_d) \leqq 0, \quad \Delta_2(n_{20}^* + 1, T_d) \leqq 0 < \Delta_{23}(n_{20}^*, M)$$

and

$$\Delta_{23}(n_{30}^*, M) \leqq 0 < \Delta_{23}(n_{30}^* + 1, M).$$

If

$$\min\{TC_1(n_{10}^*, T_d), TC_1(n_{10}^* + 1, T_d)\} \geqq \min \left\{ \begin{array}{l} TC_2(n_{20}^*, T_2^*(n_{20}^*)), \\ TC_2(n_{20}^* + 1, T_2^*(n_{20}^* + 1)), \\ TC_3(n_{30}^*, T_3^*(n_{30}^*)), TC_3(n_{30}^* + 1, M) \end{array} \right\},$$

then

$$TC(n^*, T^*) = \min \left\{ \begin{array}{l} TC_2(n_{20}^*, T_2^*(n_{20}^*)), TC_2(n_{20}^* + 1, T_2^*(n_{20}^* + 1)), \\ TC_3(n_{30}^*, T_3^*(n_{30}^*)), TC_3(n_{30}^* + 1, M) \end{array} \right\}. \quad (86)$$

Otherwise, (n^*, T^*) does not exist.

31. If

$$\Delta_1(n_{10}^*, T_d) > 0,$$

$$\Delta_2(n_{20}^* + 1, T_d) \leqq 0, \quad \Delta_{23}(n_{20}^*, M) \leqq 0 < \Delta_{23}(n_{20}^* + 1, M)$$

and

$$\Delta_{23}(n_{30}^*, M) \leqq 0 < \Delta_{23}(n_{30}^* + 1, M),$$

then

$$TC(n^*, T^*) = \min \left\{ \begin{array}{l} TC_1(n_{10}^*, T_1^*(n_{10}^*)), TC_1(n_{10}^* + 1, T_1^*(n_{10}^* + 1)), TC_2(n_{20}^*, M), \\ TC_2(n_{20}^* + 1, T_2^*(n_{20}^* + 1)), TC_3(n_{30}^*, T_3^*(n_{30}^*)), TC_3(n_{30}^* + 1, M) \end{array} \right\}. \quad (87)$$

32. Suppose that

$$\Delta_1(n_{10}^*, T_d) \leqq 0 < \Delta_1(n_{10}^* + 1, T_d), \qquad \Delta_2(n_{20}^* + 1, T_d) \leqq 0,$$

$$\Delta_{23}(n_{20}^*, M) \leqq 0 < \Delta_{23}(n_{20}^*+1, M)$$

and

$$\Delta_{23}(n_{30}^*, M) \leqq 0 < \Delta_{23}(n_{30}^*+1, M).$$

If

$$TC_1(n_{10}^*, T_d) \geqq \min\left\{ \begin{array}{l} TC_1(n_{10}^*+1, T_1^*(n_{10}^*+1)), TC_2(n_{20}^*, M), \\ TC_2(n_{20}^*+1, T_2^*(n_{20}^*+1)), TC_3(n_{30}^*, T_3^*(n_{30}^*)), \\ TC_3(n_{30}^*+1, M) \end{array} \right\},$$

then

$$TC(n^*, T^*) = \min\left\{ \begin{array}{l} TC_1(n_{10}^*+1, T_1^*(n_{10}^*+1)), TC_2(n_{20}^*, M), \\ TC_2(n_{20}^*+1, T_2^*(n_{20}^*+1)), TC_3(n_{30}^*, T_3^*(n_{30}^*)), \\ TC_3(n_{30}^*+1, M) \end{array} \right\}. \tag{88}$$

Otherwise, (n^*, T^*) does not exist.

33. Suppose that

$$\Delta_1(n_{10}^*+1, T_d) \leqq 0, \quad \Delta_2(n_{20}^*+1, T_d) \leqq 0,$$

$$\Delta_{23}(n_{20}^*, M) \leqq 0 < \Delta_{23}(n_{20}^*+1, M)$$

and

$$\Delta_{23}(n_{30}^*, M) \leqq 0 < \Delta_{23}(n_{30}^*+1, M).$$

If

$$\min\{TC_1(n_{10}^*, T_d), TC_1(n_{10}^*+1, T_d)\} \geqq \min\left\{ \begin{array}{l} TC_2(n_{20}^*, M), TC_2(n_{20}^*+1, T_2^*(n_{20}^*+1)), \\ TC_3(n_{30}^*, T_3^*(n_{30}^*)), TC_3(n_{30}^*+1, M) \end{array} \right\},$$

then

$$TC(n^*, T^*) = \min\left\{ \begin{array}{l} TC_2(n_{20}^*, M), TC_2(n_{20}^*+1, T_2^*(n_{20}^*+1)), \\ TC_3(n_{30}^*, T_3^*(n_{30}^*)), TC_3(n_{30}^*+1, M) \end{array} \right\}. \tag{89}$$

Otherwise, (n^*, T^*) does not exist.

34. If

$$\Delta_1(n_{10}^*, T_d) > 0, \ \Delta_{23}(n_{20}^*+1, M) \leqq 0$$

and

$$\Delta_{23}(n_{30}^*, M) \leqq 0 < \Delta_{23}(n_{30}^*+1, M),$$

then

$$TC(n^*, T^*) = \min\left\{ \begin{array}{l} TC_1(n_{10}^*, T_1^*(n_{10}^*)), TC_1(n_{10}^*+1, T_1^*(n_{10}^*+1)), TC_2(n_{20}^*, M), \\ TC_2(n_{20}^*+1, M), TC_3(n_{30}^*, T_3^*(n_{30}^*)), TC_3(n_{30}^*+1, M) \end{array} \right\}. \tag{90}$$

35. Suppose that

$$\Delta_1(n_{10}^*, T_d) \leqq 0 < \Delta_1(n_{10}^*+1, T_d), \qquad \Delta_{23}(n_{20}^*+1, M) \leqq 0$$

and

$$\Delta_{23}(n_{30}^*, M) \leqq 0 < \Delta_{23}(n_{30}^*+1, M).$$

If

$$TC_1(n_{10}^*, T_d) \geqq \min\left\{ \begin{array}{l} TC_1(n_{10}^*+1, T_1^*(n_{10}^*+1)), TC_2(n_{20}^*, M), \\ TC_2(n_{20}^*+1, M), TC_3(n_{30}^*, T_3^*(n_{30}^*)), \\ TC_3(n_{30}^*+1, M) \end{array} \right\},$$

then

$$TC(n^*, T^*) = \min\left\{ \begin{array}{l} TC_1(n_{10}^*+1, T_1^*(n_{10}^*+1)), TC_2(n_{20}^*, M), \\ TC_2(n_{20}^*+1, M), TC_3(n_{30}^*, T_3^*(n_{30}^*)), \\ TC_3(n_{30}^*+1, M) \end{array} \right\}. \tag{91}$$

Otherwise, (n^*, T^*) does not exist.

36. Suppose that

$$\Delta_1(n_{10}^*+1, T_d) \leqq 0, \qquad \Delta_{23}(n_{20}^*+1, M) \leqq 0$$

and

$$\Delta_{23}(n_{30}^*, M) \leqq 0 < \Delta_{23}(n_{30}^*+1, M).$$

If

$$\min\{TC_1(n_{10}^*, T_d), TC_1(n_{10}^*+1, T_d)\} \geqq \min\left\{ \begin{array}{l} TC_2(n_{20}^*, M), TC_2(n_{20}^*+1, M), \\ TC_3(n_{30}^*, T_3^*(n_{30}^*)), TC_3(n_{30}^*+1, M) \end{array} \right\},$$

then

$$TC(n^*, T^*) = \min\left\{ \begin{array}{l} TC_2(n_{20}^*, M), TC_2(n_{20}^*+1, M), \\ TC_3(n_{30}^*, T_3^*(n_{30}^*)), TC_3(n_{30}^*+1, M) \end{array} \right\}. \tag{92}$$

Otherwise, (n^*, T^*) does not exist.

37. If

$$\Delta_1(n_{10}^*, T_d) > 0, \quad \Delta_2(n_{20}^*, T_d) > 0$$

and

$$\Delta_{23}(n_{30}^*+1, M) \leqq 0,$$

then

$$TC(n^*, T^*) = \min\left\{ \begin{array}{l} TC_1(n_{10}^*, T_1^*(n_{10}^*)), TC_1(n_{10}^*+1, T_1^*(n_{10}^*+1)), TC_2(n_{20}^*, T_d), \\ TC_2(n_{20}^*+1, T_d), TC_3(n_{30}^*, T_3^*(n_{30}^*)), TC_3(n_{30}^*+1, T_3^*(n_{30}^*+1)) \end{array} \right\}. \tag{93}$$

38. Suppose that

$$\Delta_1(n_{10}^*, T_d) \leqq 0 < \Delta_1(n_{10}^*+1, T_d), \qquad \Delta_2(n_{20}^*, T_d) > 0$$

and

$$\Delta_{23}(n_{30}^*+1, M) \leqq 0.$$

If

$$TC_1(n_{10}^*, T_d) \geqq \min\left\{ \begin{array}{l} TC_1(n_{10}^*+1, T_1^*(n_{10}^*+1)), TC_2(n_{20}^*, T_d), \\ TC_2(n_{20}^*+1, T_d), TC_3(n_{30}^*, T_3^*(n_{30}^*)), \\ TC_3(n_{30}^*+1, T_3^*(n_{30}^*+1)) \end{array} \right\},$$

then

$$TC(n^*,T^*) = \min\left\{\begin{array}{l} TC_1(n_{10}^*+1,T_1^*(n_{10}^*+1)), TC_2(n_{20}^*,T_d), \\ TC_2(n_{20}^*+1,T_d), TC_3(n_{30}^*,T_3^*(n_{30}^*)), \\ TC_3(n_{30}^*+1,T_3^*(n_{30}^*+1)) \end{array}\right\}. \quad (94)$$

Otherwise, (n^*,T^*) does not exist.

39. Suppose that

$$\Delta_1(n_{10}^*+1,T_d) \leqq 0, \ \ \Delta_2(n_{20}^*,T_d) > 0$$

and

$$\Delta_{23}(n_{30}^*+1,M) \leqq 0.$$

If

$$\min\{TC_1(n_{10}^*,T_d), TC_1(n_{10}^*+1,T_d)\} \geqq \min\left\{\begin{array}{l} TC_2(n_{20}^*,T_d), TC_2(n_{20}^*+1,T_d), \\ TC_3(n_{30}^*,T_3^*(n_{30}^*)), \\ TC_3(n_{30}^*+1,T_3^*(n_{30}^*+1)) \end{array}\right\},$$

then

$$TC(n^*,T^*) = \min\left\{\begin{array}{l} TC_2(n_{20}^*,T_d), TC_2(n_{20}^*+1,T_d), \\ TC_3(n_{30}^*,T_3^*(n_{30}^*)), TC_3(n_{30}^*+1,T_3^*(n_{30}^*+1)) \end{array}\right\}. \quad (95)$$

Otherwise, (n^*,T^*) does not exist.

40. If

$$\Delta_1(n_{10}^*,T_d) > 0, \ \ \Delta_2(n_{20}^*,T_d) \leqq 0 < \Delta_2(n_{20}^*+1,T_d)$$

and

$$\Delta_{23}(n_{30}^*+1,M) \leqq 0,$$

then

$$TC(n^*,T^*) = \min\left\{\begin{array}{l} TC_1(n_{10}^*,T_1^*(n_{10}^*)), TC_1(n_{10}^*+1,T_1^*(n_{10}^*+1)), TC_2(n_{20}^*,M), \\ TC_2(n_{20}^*+1,T_d), TC_3(n_{30}^*,T_3^*(n_{30}^*)), TC_3(n_{30}^*+1,T_3^*(n_{30}^*+1)) \end{array}\right\}. \quad (96)$$

41. Suppose that

$$\Delta_1(n_{10}^*,T_d) \leqq 0 < \Delta_1(n_{10}^*+1,T_d),$$

$$\Delta_2(n_{20}^*,T_d) \leqq 0 < \Delta_2(n_{20}^*+1,T_d)$$

and

$$\Delta_{23}(n_{30}^*+1,M) \leqq 0.$$

If

$$TC_1(n_{10}^*,T_d) \geqq \min\left\{\begin{array}{l} TC_1(n_{10}^*+1,T_1^*(n_{10}^*+1)), TC_2(n_{20}^*,M), \\ TC_2(n_{20}^*+1,T_d), TC_3(n_{30}^*,T_3^*(n_{30}^*)), \\ TC_3(n_{30}^*+1,T_3^*(n_{30}^*+1)) \end{array}\right\},$$

then

$$TC(n^*,T^*) = \min\left\{\begin{array}{l} TC_1(n_{10}^*+1,T_1^*(n_{10}^*+1)), TC_2(n_{20}^*,M), \\ TC_2(n_{20}^*+1,T_d), TC_3(n_{30}^*,T_3^*(n_{30}^*)), \\ TC_3(n_{30}^*+1,T_3^*(n_{30}^*+1)) \end{array}\right\}. \quad (97)$$

Otherwise, (n^*,T^*) does not exist.

42. Suppose that

$$\Delta_1(n_{10}^*+1,T_d) \leqq 0, \qquad \Delta_2(n_{20}^*,T_d) \leqq 0 < \Delta_2(n_{20}^*+1,T_d)$$

and

$$\Delta_{23}(n_{30}^*+1,M) \leqq 0.$$

If

$$\min\{TC_1(n_{10}^*,T_d), TC_1(n_{10}^*+1,T_d)\} \geqq \min\left\{\begin{array}{l} TC_2(n_{20}^*,M), TC_2(n_{20}^*+1,T_d), \\ TC_3(n_{30}^*,T_3^*(n_{30}^*)), \\ TC_3(n_{30}^*+1,T_3^*(n_{30}^*+1)) \end{array}\right\},$$

then

$$T(n^*,T^*) = \min\left\{\begin{array}{l} TC_2(n_{20}^*,M), TC_2(n_{20}^*+1,T_d), \\ TC_3(n_{30}^*,T_3^*(n_{30}^*)), TC_3(n_{30}^*+1,T_3^*(n_{30}^*+1)) \end{array}\right\}. \quad (98)$$

Otherwise, (n^*,T^*) does not exist.

43. If

$$\Delta_1(n_{10}^*,T_d) > 0,$$

$$\Delta_2(n_{20}^*,T_d) \leqq \Delta_{23}(n_{20}^*,M) \leqq 0 < \Delta_2(n_{20}^*+1,T_d)$$

and

$$\Delta_{23}(n_{30}^*+1,M) \leqq 0,$$

then

$$TC(n^*,T^*) = \min\left\{\begin{array}{l} TC_1(n_{10}^*,T_1^*(n_{10}^*)), TC_1(n_{10}^*+1,T_1^*(n_{10}^*+1)), TC_2(n_{20}^*,M), \\ TC_2(n_{20}^*+1,T_d), TC_3(n_{30}^*,T_3^*(n_{30}^*)), TC_3(n_{30}^*+1,T_3^*(n_{30}^*+1)) \end{array}\right\}. \quad (99)$$

44. Suppose that

$$\Delta_2(n_{20}^*,T_d) \leqq \Delta_{23}(n_{20}^*,M) \leqq 0 < \Delta_2(n_{20}^*+1,T_d)$$

and

$$\Delta_{23}(n_{30}^*+1,M) \leqq 0.$$

If

$$TC_1(n_{10}^*,T_d) \geqq \min\left\{\begin{array}{l} TC_1(n_{10}^*+1,T_1^*(n_{10}^*+1)), TC_2(n_{20}^*,M), \\ TC_2(n_{20}^*+1,T_d), TC_3(n_{30}^*,T_3^*(n_{30}^*)), \\ TC_3(n_{30}^*+1,T_3^*(n_{30}^*+1)) \end{array}\right\}, \quad (100)$$

then

$$TC(n^*,T^*) = \min\left\{\begin{array}{l} TC_1(n_{10}^*+1,T_1^*(n_{10}^*+1)),TC_2(n_{20}^*,M), \\ TC_2(n_{20}^*+1,T_d),TC_3(n_{30}^*,T_3^*(n_{30}^*)), \\ TC_3(n_{30}^*+1,T_3^*(n_{30}^*+1)) \end{array}\right\}. \tag{101}$$

Otherwise, (n^*,T^*) does not exist.

45. Suppose

$$\Delta_1(n_{10}^*+1,T_d) \leqq 0,$$

$$\Delta_2(n_{20}^*,T_d) \leqq \Delta_{23}(n_{20}^*,M) \leqq 0 < \Delta_2(n_{20}^*+1,T_d)$$

and

$$\Delta_{23}(n_{30}^*+1,M) \leqq 0.$$

If

$$\min\{TC_1(n_{10}^*,T_d),TC_1(n_{10}^*+1,T_d)\} \geqq \min\left\{\begin{array}{l} TC_2(n_{20}^*,M),TC_2(n_{20}^*+1,T_d), \\ TC_3(n_{30}^*,T_3^*(n_{30}^*)), \\ TC_3(n_{30}^*+1,T_3^*(n_{30}^*+1)) \end{array}\right\}, \tag{102}$$

then

$$T(n^*,T^*) = \min\left\{\begin{array}{l} TC_2(n_{20}^*,M),TC_2(n_{20}^*+1,T_d), \\ TC_3(n_{30}^*,T_3^*(n_{30}^*)),TC_3(n_{30}^*+1,T_3^*(n_{30}^*+1)) \end{array}\right\}. \tag{103}$$

Otherwise, (n^*,T^*) does not exist.

46. If

$$\Delta_1(n_{10}^*,T_d) > 0,\ \ \Delta_2(n_{20}^*+1,T_d) \leqq 0 < \Delta_{23}(n_{20}^*,M)$$

and

$$\Delta_{23}(n_{30}^*+1,M) \leqq 0,$$

then

$$TC(n^*,T^*) = \min\left\{\begin{array}{l} TC_1(n_{10}^*,T_1^*(n_{10}^*)),TC_1(n_{10}^*+1,T_1^*(n_{10}^*+1)),TC_2(n_{20}^*,T_2^*(n_{20}^*)), \\ TC_2(n_{20}^*+1,T_2^*(n_{20}^*+1)),TC_3(n_{30}^*,T_3^*(n_{30}^*)),TC_3(n_{30}^*+1,T_3^*(n_{30}^*+1)) \end{array}\right\}. \tag{104}$$

47. Suppose that

$$\Delta_1(n_{10}^*,T_d) \leqq 0 < \Delta_1(n_{10}^*+1,T_d),$$

$$\Delta_2(n_{20}^*+1,T_d) \leqq 0 < \Delta_{23}(n_{20}^*,M)$$

and

$$\Delta_{23}(n_{30}^*+1,M) \leqq 0.$$

If

$$TC_1(n_{10}^*,T_d) \geqq \min\left\{\begin{array}{l} TC_1(n_{10}^*+1,T_1^*(n_{10}^*+1)),TC_2(n_{20}^*,T_2^*(n_{20}^*)), \\ TC_2(n_{20}^*+1,T_2^*(n_{20}^*+1)),TC_3(n_{30}^*,T_3^*(n_{30}^*)), \\ TC_3(n_{30}^*+1,T_3^*(n_{30}^*+1)) \end{array}\right\}, \tag{105}$$

then

$$TC(n^*,T^*) = \min\left\{\begin{array}{l} TC_1(n_{10}^*+1,T_1^*(n_{10}^*+1)),TC_2(n_{20}^*,T_2^*(n_{20}^*)), \\ TC_2(n_{20}^*+1,T_2^*(n_{20}^*+1)),TC_3(n_{30}^*,T_3^*(n_{30}^*)), \\ TC_3(n_{30}^*+1,T_3^*(n_{30}^*+1)) \end{array}\right\}. \tag{106}$$

Otherwise, (n^*,T^*) does not exist.

48. Suppose that

$$\Delta_1(n_{10}^*+1,T_d) \leqq 0, \quad \Delta_2(n_{20}^*+1,T_d) \leqq 0 < \Delta_{23}(n_{20}^*,M)$$

and

$$\Delta_{23}(n_{30}^*+1,M) \leqq 0.$$

If

$$\min\{TC_1(n_{10}^*,T_d),TC_1(n_{10}^*+1,T_d)\} \geqq \min\left\{\begin{array}{l} TC_2(n_{20}^*,T_2^*(n_{20}^*)),TC_2(n_{20}^*+1,T_2^*(n_{20}^*+1)), \\ TC_3(n_{30}^*,T_3^*(n_{30}^*)),TC_3(n_{30}^*+1,T_3^*(n_{30}^*+1)) \end{array}\right\}, \tag{107}$$

then

$$T(n^*,T^*) = \min\left\{\begin{array}{l} TC_2(n_{20}^*,T_2^*(n_{20}^*)),TC_2(n_{20}^*+1,T_2^*(n_{20}^*+1)), \\ TC_3(n_{30}^*,T_3^*(n_{30}^*)),TC_3(n_{30}^*+1,T_3^*(n_{30}^*+1)) \end{array}\right\}. \tag{108}$$

Otherwise, (n^*,T^*) does not exist.

49. If

$$\Delta_1(n_{10}^*,T_d) > 0, \quad \Delta_2(n_{20}^*+1,T_d) \leqq 0,$$

$$\Delta_{23}(n_{20}^*,M) \leqq 0 < \Delta_{23}(n_{20}^*+1,M)$$

and

$$\Delta_{23}(n_{30}^*+1,M) \leqq 0,$$

then

$$TC(n^*,T^*) = \min\left\{\begin{array}{l} TC_1(n_{10}^*,T_1^*(n_{10}^*)),TC_1(n_{10}^*+1,T_1^*(n_{10}^*+1)),TC_2(n_{20}^*,M), \\ TC_2(n_{20}^*+1,T_2^*(n_{20}^*+1)),TC_3(n_{30}^*,T_3^*(n_{30}^*)),TC_3(n_{30}^*+1,T_3^*(n_{30}^*+1)) \end{array}\right\}. \tag{109}$$

50. Suppose that

$$\Delta_1(n_{10}^*,T_d) \leqq 0 < \Delta_1(n_{10}^*+1,T_d), \quad \Delta_2(n_{20}^*+1,T_d) \leqq 0,$$

$$\Delta_{23}(n_{20}^*,M) \leqq 0 < \Delta_{23}(n_{20}^*+1,M)$$

and

$$\Delta_{23}(n_{30}^*+1,M) \leqq 0.$$

If

$$TC_1(n_{10}^*,T_d) \geqq \min\left\{\begin{array}{l} TC_1(n_{10}^*+1,T_1^*(n_{10}^*+1)),TC_2(n_{20}^*,M)), \\ TC_2(n_{20}^*+1,T_2^*(n_{20}^*+1)),TC_3(n_{30}^*,T_3^*(n_{30}^*)), \\ TC_3(n_{30}^*+1,T_3^*(n_{30}^*+1)) \end{array}\right\}, \tag{110}$$

then

$$TC(n^*,T^*) = \min\left\{\begin{array}{l} TC_1(n_{10}^*+1,T_1^*(n_{10}^*+1)),TC_2(n_{20}^*,M)), \\ TC_2(n_{20}^*+1,T_2^*(n_{20}^*+1)),TC_3(n_{30}^*,T_3^*(n_{30}^*)), \\ TC_3(n_{30}^*+1,T_3^*(n_{30}^*+1)) \end{array}\right\}. \quad (111)$$

Otherwise, (n^*,T^*) does not exist.

51. Suppose that

$$\Delta_1(n_{10}^*+1,T_d) \leqq 0, \quad \Delta_2(n_{20}^*+1,T_d) \leqq 0,$$

$$\Delta_{23}(n_{20}^*,M) \leqq 0 < \Delta_{23}(n_{20}^*+1,M)$$

and

$$\Delta_{23}(n_{30}^*+1,M) \leqq 0.$$

If

$$\min\{TC_1(n_{10}^*,T_d),TC_1(n_{10}^*+1,T_d)\} \geqq \min\left\{\begin{array}{l} TC_2(n_{20}^*,M),TC_2(n_{20}^*+1,T_2^*(n_{20}^*+1)), \\ TC_3(n_{30}^*,T_3^*(n_{30}^*)),TC_3(n_{30}^*+1,T_3^*(n_{30}^*+1)) \end{array}\right\}, \quad (112)$$

then

$$T(n^*,T^*) = \min\left\{\begin{array}{l} TC_2(n_{20}^*,M),TC_2(n_{20}^*+1,T_2^*(n_{20}^*+1)), \\ TC_3(n_{30}^*,T_3^*(n_{30}^*)),TC_3(n_{30}^*+1,T_3^*(n_{30}^*+1)) \end{array}\right\}. \quad (113)$$

Otherwise, (n^*,T^*) does not exist.

52. If

$$\Delta_1(n_{10}^*,T_d) > 0, \ \Delta_{23}(n_{20}^*+1,M) \leqq 0$$

and

$$\Delta_{23}(n_{30}^*+1,M) \leqq 0,$$

then

$$TC(n^*,T^*) = \min\left\{\begin{array}{l} TC_1(n_{10}^*,T_1^*(n_{10}^*)),TC_1(n_{10}^*+1,T_1^*(n_{10}^*+1)),TC_2(n_{20}^*,M), \\ TC_2(n_{20}^*+1,M),TC_3(n_{30}^*,T_3^*(n_{30}^*)),TC_3(n_{30}^*+1,T_3^*(n_{30}^*+1)) \end{array}\right\}. \quad (114)$$

53. Suppose that

$$\Delta_1(n_{10}^*,T_d) \leqq 0 < \Delta_1(n_{10}^*+1,T_d), \ \Delta_{23}(n_{20}^*+1,M) \leqq 0$$

and

$$\Delta_{23}(n_{30}^*+1,M) \leqq 0.$$

If

$$TC_1(n_{10}^*,T_d) \geqq \min\left\{\begin{array}{l} TC_1(n_{10}^*+1,T_1^*(n_{10}^*+1)),TC_2(n_{20}^*,M)), \\ TC_2(n_{20}^*+1,M),TC_3(n_{30}^*,T_3^*(n_{30}^*)), \\ TC_3(n_{30}^*+1,T_3^*(n_{30}^*+1)) \end{array}\right\}, \quad (115)$$

then

$$TC(n^*,T^*) = \min\left\{\begin{array}{l} TC_1(n_{10}^*+1,T_1^*(n_{10}^*+1)),TC_2(n_{20}^*,M)), \\ TC_2(n_{20}^*+1,M),TC_3(n_{30}^*,T_3^*(n_{30}^*)), \\ TC_3(n_{30}^*+1,T_3^*(n_{30}^*+1)) \end{array}\right\}. \quad (116)$$

Otherwise, (n^*,T^*) does not exist.

54. Suppose that

$$\Delta_1(n_{10}^*+1,T_d) \leqq 0, \ \Delta_{23}(n_{20}^*+1,M) \leqq 0$$

and

$$\Delta_{23}(n_{30}^*+1,M) \leqq 0.$$

If

$$\min\{TC_1(n_{10}^*,T_d),TC_1(n_{10}^*+1,T_d)\} \geqq \min\left\{\begin{array}{l} TC_2(n_{20}^*,M),TC_2(n_{20}^*+1,M), \\ TC_3(n_{30}^*,T_3^*(n_{30}^*)),TC_3(n_{30}^*+1,T_3^*(n_{30}^*+1)) \end{array}\right\}, \quad (117)$$

then

$$T(n^*,T^*) = \min\left\{\begin{array}{l} TC_2(n_{20}^*,M),TC_2(n_{20}^*+1,M), \\ TC_3(n_{30}^*,T_3^*(n_{30}^*)),TC_3(n_{30}^*+1,T_3^*(n_{30}^*+1)) \end{array}\right\}. \quad (118)$$

Otherwise, (n^*,T^*) does not exist.

*Proof.*Our demonstration of Theorem 1 is presented in Appendix A.

In view of Lemma 1 and the equations (11a) and (11b), the techniques of the analytical arguments applied in the proof of Theorem 1 (see Appendix A) can be used to demonstrate each of the following results.

Theorem 2. *Suppose that $T_d > M$.*

1. *If*

$$\Delta_1(n_{10}^*,T_d) > 0 \quad \text{and} \quad \Delta_3(n_{30}^*,T_d) > 0,$$

then

$$TC(n^*,T^*) = \min\left\{\begin{array}{l} TC_1(n_{10}^*,T_1^*(n_{10}^*)),TC_1(n_{10}^*+1,T_1^*(n_{10}^*+1)), \\ TC_3(n_{30}^*,T_d),TC_3(n_{30}^*+1,T_d) \end{array}\right\}. \quad (119)$$

2. *Suppose that*

$$\Delta_1(n_{10}^*,T_d) \leqq 0 < \Delta_1(n_{10}^*+1,T_d)$$

and

$$\Delta_3(n_{30}^*,T_d) > 0.$$

If

$$TC_1(n_{10}^*,T_d) \geqq \min\{TC_1(n_{10}^*+1,T_1^*(n_{10}^*+1)),TC_3(n_{30}^*,T_d),TC_3(n_{30}^*+1,T_d)\},$$

then

$$TC(n^*,T^*) = \min\{TC_1(n_{10}^*+1,T_1^*(n_{10}^*+1)),TC_3(n_{30}^*,T_d),TC_3(n_{30}^*+1,T_d)\}. \quad (120)$$

Otherwise, (n^*, T^*) *does not exist.*
3. *Suppose*

$$\Delta_1(n^*_{10}+1, T_d) \leqq 0 \qquad and \qquad \Delta_3(n^*_{30}, T_d) > 0.$$

If

$$\begin{aligned} &\min\{TC_1(n^*_{10}, T_d), TC_1(n^*_{10}+1, T_d)\} \\ &\quad \geqq \min\{TC_3(n^*_{30}, T_d), TC_3(n^*_{30}+1, T_d)\}, \end{aligned}$$

then

$$\begin{aligned} &TC(n^*, T^*) \\ &\quad = \min\{TC_3(n^*_{30}, T_d), TC_3(n^*_{30}+1, T_d)\}. \end{aligned} \tag{121}$$

Otherwise, (n^*, T^*) *does not exist.*
4. *If*

$$\Delta_1(n^*_{10}, T_d) > 0 \qquad and \qquad \Delta_3(n^*_{30}, T_d) \leqq 0 < \Delta_3(n^*_{30}+1, T_d),$$

then

$$\begin{aligned} &TC(n^*, T^*) \\ &\quad = \min\left\{ \begin{array}{l} TC_1(n^*_{01}, T^*_1(n^*_{01})), TC_1(n^*_{01}+1, T^*_1(n^*_{01}+1)), \\ TC_3(n^*_{30}, T^*_3(n^*_{30})), TC_3(n^*_{30}+1, T_d) \end{array} \right\}. \end{aligned} \tag{122}$$

5. *Suppose that*

$$\Delta_1(n^*_{10}, T_d) \leqq 0 < \Delta_1(n^*_{10}+1, T_d)$$

and

$$\Delta_3(n^*_{30}, T_d) \leqq 0 < \Delta_3(n^*_{30}+1, T_d).$$

If

$$\begin{aligned} &TC_1(n^*_{10}, T_d) \\ &\quad \geqq \min\{TC_1(n^*_{10}+1, T^*_1(n^*_{10}+1)), TC_3(n^*_{30}, T^*_3(n^*_{30})), TC_3(n^*_{30}+1, T_d)\}, \end{aligned}$$

then

$$\begin{aligned} &TC(n^*, T^*) \\ &\quad = \min\{TC_1(n^*_{10}+1, T^*_1(n^*_{10}+1)), TC_3(n^*_{30}, T^*_3(n^*_{30})), TC_3(n^*_{30}+1, T_d)\}. \end{aligned} \tag{123}$$

Otherwise, (n^*, T^*) *does not exist.*
6. *Suppose that*

$$\Delta_1(n^*_{10}+1, T_d) \leqq 0$$

and

$$\Delta_3(n^*_{30}, T_d) \leqq 0 < \Delta_3(n^*_{30}+1, T_d).$$

If

$$\begin{aligned} &\min\{TC_1(n^*_{10}, T_d), TC_1(n^*_{10}+1, T_d)\} \\ &\quad \geqq \min\{TC_3(n^*_{30}, T^*_3(n^*_{30})), TC_3(n^*_{30}+1, T_d)\}, \end{aligned}$$

then

$$T(n^*, T^*) = \min\{TC_3(n^*_{30}, T^*_3(n^*_{30})), TC_3(n^*_{30}+1, T_d)\}. \tag{124}$$

Otherwise, (n^*, T^*) *does not exist.*
7. *If*

$$\Delta_1(n^*_{10}, T_d) > 0 \qquad and \qquad \Delta_3(n^*_{30}, T_d) \leqq 0,$$

then

$$\begin{aligned} &TC(n^*, T^*) \\ &\quad = \min\left\{ \begin{array}{l} TC_1(n^*_{10}, T^*_1(n^*_{10})), TC_1(n^*_{10}+1, T^*_1(n^*_{10}+1)), \\ TC_3(n^*_{30}, T^*_3(n^*_{30})), TC_3(n^*_{30}+1, T^*_3(n^*_{30}+1)) \end{array} \right\}. \end{aligned} \tag{125}$$

8. *Suppose that*

$$\Delta_1(n^*_{10}, T_d) \leqq 0 < \Delta_1(n^*_{10}+1, T_d)$$

and

$$\Delta_3(n^*_{30}+1, T_d) \leqq 0.$$

If

$$\begin{aligned} &TC(n^*_{10}, T_d) \\ &\quad \geqq \min\left\{ \begin{array}{l} TC_1(n^*_{10}+1, T^*_1(n^*_{10}+1)), TC_3(n^*_{30}, T^*_3(n^*_{30})), \\ TC_3(n^*_{30}+1, T^*_3(n^*_{30}+1)) \end{array} \right\}, \end{aligned}$$

then

$$\begin{aligned} &TC(n^*, T^*) \\ &\quad = \min\left\{ \begin{array}{l} TC_1(n^*_{10}+1, T^*_1(n^*_{10}+1)), TC_3(n^*_{30}, T^*_3(n^*_{30})), \\ TC_3(n^*_{30}+1, T^*_3(n^*_{30}+1)) \end{array} \right\}. \end{aligned} \tag{126}$$

Otherwise, (n^*, T^*) *does not exist.*
9. *Suppose that*

$$\Delta_1(n^*_{10}+1, T_d) \leqq 0 \qquad and \qquad \Delta_3(n^*_{30}+1, T_d) \leqq 0.$$

If

$$\begin{aligned} &\min\{TC_1(n^*_{10}, T_d), TC_1(n^*_{10}+1, T_d)\} \\ &\quad \geqq \min\{TC_3(n^*_{30}, T^*_3(n^*_{30})), TC_3(n^*_{30}+1, T^*_3(n^*_{30}+1))\}, \end{aligned}$$

then

$$T(n^*, T^*) = \min\{TC_3(n^*_{30}, T^*_3(n^*_{30})), TC_3(n^*_{30}+1, T^*_3(n^*_{30}+1))\}. \tag{127}$$

Otherwise, (n^*, T^*) *does not exist.*

6 An Illustrative Numerical Example

Example. Let $A = \$5$ per order, $p = \$1$ per unit, $c = \$0.5$ per unit, $h_b = \$0.5$ per unit per year, $D = 3,600$ units per year, $M = 30/365$ years, $h_v = \$0.25$ per unit per year, $F = \$10$ per order, $S = \$120$ per production run, $I_d = I_v = 0.04$, $I_c = 0.06$, $Q_d = 200$ units per year and $R = 7,200$ units per year. Now

$$0.04 = pI_d > cIc = 0.03 \qquad \text{and} \qquad T_d = Q_d/D < M.$$

Then we have

$$n_1 = 5.8240, \quad n_2 = 5.8787, \quad n_3 = 5.8570,$$

$$n_{10}^{*}=n_{20}^{*}=n_{30}^{*}=5$$

and

$$2(A+F)+DM^{2}(cI_c-pI_d)=29.7568024>0.$$

According to the equations (1a) to (1c), (25), (31) and (34), we can obtain

$$TC_1(5,T_d)=817.5,\quad TC_1(6,T_d)=758,\quad TC_1(5,M)=812.5822,$$

$$TC_1(6,M)=753.0822,\quad TC_3(5,T_3^*(5))=TC_3(5,0.13674988)=565.6471,$$

$$TC_3(6,T_3^*(6))=TC_3(6,0.123037372)=563.9973,\qquad \Delta_1(n_{10}^*+1,T_d)<0$$

and

$$\Delta_{23}(n_{20}^*+1,M)=\Delta_{23}(n_{30}^*+1,M)=\Delta_{23}(6,M)<0.$$

Hence, clearly, we find that

$$\begin{aligned}&\min\{TC_1(n_{10}^*,T_d),TC_1(n_{10}^*+1,T_d)\}\\&\quad=\min\{817.5,758\}=758>563.9973\\&\quad=\min\{TC_2(5,M),TC_2(6,M),TC_3(5,T_3^*(5)),TC_3(6,T_3^*(6))\}.\end{aligned}$$

The assertion **54** of Theorem 1 reveals that

$$(n^*,T^*)=(6,T_3^*(6))=(6,0.123037372)$$

and

$$TC(n^*,T^*)=563.9973,$$

which is consistent with the optimal solution and value in Table 1 of Teng *et al.* [12]. In addition, if we apply Theorems 1 and 2 in this paper in order to find the optimal solutions and values of those examples presented in Tables 1, 2 and 3 of Teng *et al.* [12], we find that all of the optimal solutions and values obtained by applying our Theorems 1 and 2 are precisely the same as those presented in Teng *et al.* [12].

7 The Shortcomings of Theorems 3 and 4 in Teng *et al.* [12]

Shortcoming 1. Let $A=\$8$ per order, $p=\$50$ per unit, $c=\$10$ per unit, $h_b=\$0.5$ per unit per year, $D=3,600$ units per year, $M=30/365$ years, $h_v=\$0.25$ per unit per year, $F=\$10$ per order, $S=\$200$ per production run, $I_d=I_v=0.04$, $I_c=0.06$, $R=7,200$ units per year and $Q_d=200$ units per year. Now

$$2=pI_d>cI_c=0.6$$

and

$$T_d=200/3,600<30/365=M.$$

Then we have

$$n_1=9.8882,\quad n_2=14.9071,\quad n_3=42.4614,\quad n_{10}^*=9,$$

$$n_{20}^*=14,n_{30}^*=42$$

and

$$2(A+F)+DM^{2}(cI_c-pI_d)=1.9523>0.$$

According to the equations (1a) to (1c), (25), (31) and (34), we can obtain

$$\Delta_1(10,T_d)<0,\quad \Delta_2(15,T_d)<0<\Delta_{23}(14,M)<\Delta_{23}(42,M),$$

$$TC_1(9,T_d)=946.5,\qquad TC_1(10,T_d)=919,$$

$$TC_2(14,T_2^*(14))=TC_2(14,0.064964257)=520.5285,$$

$$TC_2(15,T_2^*(15))=TC_2(15,0.063078015)=520.0541,$$

$$TC_3(42,M)=950.0872\qquad\text{and}\qquad TC_3(43,M)=967.2330.$$

Hence we get

$$\begin{aligned}&\min\{TC_1(n_{10}^*,T_d),TC_1(n_{10}^*+1,T_d)\}\\&\quad=919>520.0541\\&\quad=\min\{TC_2(14,T_2^*(14)),TC_2(15,T_2^*(15)),TC_3(42,M),TC_3(43,M)\}.\end{aligned}$$

Theorem 1(12) reveals that

$$(n^*,T^*)=(15,0.063078015)$$

and

$$TC(n^*,T^*)=520.0541.$$

If Theorem 3 in Teng *et al.* [12] is used to locate the optimal solution of this example, then Theorem 3 in Teng *et al.* [12] should be executed six times for $n=9,10,14,15,42$ and 43. In fact, using Theorem 3 in Teng *et al.* [12] can not decide where the optimal solution (n^*,T^*) is located. Consequently, when we apply Theorem 3 in Teng *et al.* [12] to locate the optimal solution, the searching process is rather cumbersome.

Shortcoming 2. The validities of Theorems 3 and 4 in the paper by Teng *et al.* [12] are based upon the assumption that $pI_d\geqq cI_c$. If

$$pI_d<cI_c,$$

the validities of Theorems 3 and 4 in Teng *et al.* [12] are questionable. However, Theorems 1 and 2 in this paper are always true without the assumption that

$$pI_d\geqq cI_c.$$

Hence, naturally, Theorems 1 and 2 of this paper extend Theorems 3 and 4 in Teng *et al.* [12].

Shortcoming 3. Equations (51a) to (51c) demonstrate that the equation (54) holds true. However, Teng *et al.* [12] do not give the concrete proof for the validity of the equation (54).

Shortcoming 4. Equation (5) implies that there are the following three cases that would occur:
Case 1. $TC_1(n,T_d)>TC_2(n,T_d)$;
Case 2. $TC_1(n,T_d)=TC_2(n,T_d)$

Case 3. $TC_1(n,T_d) < TC_2(n,T_d)$.
Teng *et al.* [12] assumed that

$$TC_1(n,T_d) > TC_2(n,T_d).$$

Therefore, Theorems 3 and 4 in Teng *et al.* [12] always assure that $T^*(n)$ exists. However, if

$$TC_1(n,T_d) \leqq TC_2(n,T_d),$$

then Theorems 3(4) and 4(1) in Teng *et al.* [12] are not necessarily true. Obviously, if

$$TC_1(n,T_d) < TC_2(n,T_d),$$

then $T^*(n)$ does not exist.

8 Concluding Remarks and Observations

The main results of Theorems 3 and 4 in Teng *et al.* [12] are to determine the optimal replenishment cycle $T^*(n)$ when n is given. In essence, Theorems 3 and 4 in Teng *et al.* [12] can not *directly* provide the optimal solution (n^*,T^*) of $TC(n,T)$. In fact, Theorems 3 and 4 in Teng *et al.* [12] do not execute the equation (7) to obtain the optimal solution (n^*,T^*). In order to obtain the optimal solution (n^*,T^*), Theorems 3 and 4 in Teng *et al.* [12] should, in general, be executed to obtain the optimal solution $(n,T^*(n))$, (n,T_d) or (n,M) when

$$n = n_{10}^*, \quad n_{10}^*+1, \quad n_{20}^*, \quad n_{20}^*+1, \quad n_{30}^* \quad \text{and} \quad n_{30}^*+1,$$

respectively. Subsequently, the process in Teng *et al.* [12] should make a lot of comparisons among all values of

$$TC(n,T^*(n)), \quad TC(n,T_d) \quad \text{and} \quad TC(n,M)$$

for all

$$n = n_{10}^*, \quad n_{10}^*+1, \quad n_{20}^*, \quad n_{20}^*+1, \quad n_{30}^* \quad \text{and} \quad n_{30}^*+1$$

to determine the optimal solution (n^*,T^*) of $TC(n,T)$. In general, the process of locating the optimal solution (n^*,T^*) is rather cumbersome. However, Theorems 1 and 2 in this paper *directly* execute the equation (7) to provide concrete solution procedures for locating the optimal solution (n^*,T^*) of $TC(n,T)$. In fact, the executions of Theorems 1 and 2 of this paper are rather simple to locate the optimal solution (n^*,T^*) of $TC(n,T)$ in comparison with those of Theorems 3 and 4 in Teng *et al.* [12]. Based upon the above arguments, this paper does not only remove the aforementioned shortcomings, but it also provides the complete and concrete solution procedures for Teng *et al.* [12]. Obviously, therefore, this paper significantly improves the work presented by Teng *et al.* [12].

Appendix A

Here we present the detailed proof of the assertion **1** of Theorem 1 as follows:

1. If

$$\Delta_1(n_{10}^*,T_d) > 0, \qquad \Delta_2(n_{20}^*,T_d) > 0$$

and

$$\Delta_{23}(n_{30}^*,M) > 0,$$

then Lemma 1 [(A),(C) and (F)] and the equations (35) to (38) imply that

(1.1) $TC_1(n_{10}^*,T)$ is decreasing on $(0,T_1^*(n_{10}^*)]$ and increasing on $[T_1^*(n_{10}^*),T_d)$.
(1.2) $TC_1(n_{10}^*+1,T)$ is decreasing on $(0,T_1^*(n_{10}^*+1)]$ and increasing on $[T_1^*(n_{10}^*+1),T_d)$.
(1.3) $TC_2(n_{20}^*,T)$ is increasing on $[T_d,M]$ and $TC_3(n_{20}^*,T)$ increasing on $[M,\infty)$.
(1.4) $TC_2(n_{20}^*+1,T)$ is increasing on $[T_d,M]$ and $TC_3(n_{20}^*+1,T)$ increasing on $[M,\infty)$.
(1.5) $TC_3(n_{30}^*,T)$ is increasing on $[M,\infty)$.
(1.6) $TC_3(n_{30}^*+1,T)$ is increasing on $[M,\infty)$.

In view of **(1.1)** to **(1.6)**, the equations (1a) to (1c), (5) and (54), we obtain

$$TC(n^*,T^*) = \min\left\{\begin{array}{l} TC_1(n_{10}^*,T_1^*(n_{10}^*)), TC_1(n_{10}^*+1,T_1^*(n_{10}^*+1)), \\ TC_2(n_{20}^*,T_d), TC_2(n_{20}^*+1,T_d), \\ TC_3(n_{30}^*,M), TC_3(n_{30}^*+1,M) \end{array}\right\}.$$

Similarly, by applying Lemma 1, and the equations (1a) to (1c), (5) and (54), the techniques of the arguments used in proving the assertion **1** of Theorem 1 can be employed to demonstrate that the assertions **2** to **54** of Theorem 1 hold true.

Finally, by incorporating all of the above-mentioned arguments, we complete the proof of Theorem 1.

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