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Solitary Solutions for some Nonlinear Evolution Equations using Bernoulli Method

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Abstract: New solitary solutions of some nonlinear partial differential equations are constructed using the generalized Bernoulli method. The main idea of this method is to make use of Bernoulli differential equation which has a simple exponential solution. The ZK-BBM (Zakharov-Kuznetsov-Benjamin–Bona–Mahony), a nonlinear dispersive, and the general Burgers-Fisher equations are solved and numerically investigated. The ZK equation that describes two-dimensional, magnetized, collisionless pair ions plasma is also presented as a problem of physical interest. Comparisons with G'/G-expansion method are given for the sake of assessments of Bernoulli method. Successfully, this method not only gives new solitary solutions of the problems under consideration but also recovers some solutions that had been obtained by other methods for the same problems.

Keywords: Solitary solutions, Bernoulli method, G'/G expansion method; Collisionless three-component plasma

1 Introduction

The nonlinear phenomena exist in many fields such as fluid mechanics, plasma physics, optical fibers, biology, solid state physics, chemical kinematics, and so on. It is well known that many nonlinear partial differential equations (NLPDEs) are widely used to describe these complex phenomena. So, the powerful and efficient methods to find analytic solutions of nonlinear equations have acquired a great deal of interest by many researchers. In addition, searching for exact solutions of these equations has become more attractive due to the availability of computer symbolic systems like Maple and Mathematica. These packages allow researchers to perform some complicated and tedious algebraic calculation on computer as well as help to find new exact solutions of NLPDEs.

Many efficient methods for obtaining analytical traveling wave solutions have been presented. Some of these methods are the generalized expansion method [1], sine-cosine method [2,3,4] F-expansion method [5]. More popular and efficient methods, we can also point to, are tanh function method [6,7,8] extended tanh function method [9,10,11], modified extended tanh-function method (METFM) [12,13], and $(G'/G)-$ expansion method [14,15,16]. Recently, the METFM combined with the random variable transformation technique (RVT)[17,18] have been proposed by Hussein and Selim [19] to solve the nonlinear stochastic generalized shallow water wave equation.

More recent, a new effective and straightforward method to construct exact solitary solutions to the NLPDEs is the generalized Bernoulli-method [20,21,22, 23] It has been implemented to get new traveling wave solutions for some special NLPDEs. The main idea of this method is to take full advantage of the nonlinear ordinary differential equation (ODE) of Bernoulli type, which has a simple exponential solution, to get new solitary solutions of NLPDEs.

In this paper, the Bernoulli method is applied to solve some important nonlinear evolution equations namely; The ZK-BBM equation, a nonlinear dispersive equation, and the general Burgers-Fisher equation. These equations have a wide range of applications in plasma physics, fluid mechanics, solid state physics, capillary-gravity waves, and chemical physics. As a real physical problem with specific application in plasma physics, the Zakharov-Kuznetsov (ZK) equation is solved and numerically presented. This equation describes magnetized, collisionless plasma consisting of positive, negative ions, and electrons. New exact solitary solutions for these problems are derived successfully. The more

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popular $G'/G-$expansion method is used to solve the same problems for the sake of comparison. The comparison shows that the Bernoulli method is a powerful and efficient method for solving such problems.

2 The Bernoulli Method

In this section we describe the Bernoulli method for finding travelling wave solutions of NLPDE. Consider a general (1+2) NLPDE in the following form

$$\Gamma(\phi,\phi_x,\phi_y,\phi_t,\phi_{tx},\phi_{ty},\phi_{xy},\phi_{xx},\phi_{yy},\phi_{xyt},.......)=0, \tag{1}$$

where $\phi=\phi(x,y,t)$ is an unknown function and Γ is a polynomial in ϕ and its derivatives.
Substituting the traveling solution $\Phi(\xi)=\phi(x,y,t)$, where $\xi=k(x+y\pm ct)$, in Eq.(1) leads to the following ODE

$$F(\Phi,\ \Phi',\ \Phi'',\ \Phi''',...)=0, \tag{2}$$

where $' := d/d\xi$.

The next crucial step is that, the solution of Eq.(2) is expressed in the following ansatz:

$$\Phi(\xi)=a_0+\sum_{i=1}^{M}a_iG^i(\xi), \tag{3}$$

where a_0 and $a_i\neq 0,\ i=1,2,3,.....M$, are constants to be determined later.

The parameter M can be determined from the homogenous balance between the highest order linear term and the nonlinear term in the ODE (2). More precisely, we define the degree of $\Phi(\xi)$ as $D[\Phi(\xi)]=M$ which gives rise to the degree of other expressions as follows

$$D[\frac{d^q\Phi(\xi)}{d\xi^q}]=M+q, \tag{4}$$

$$D\left[\Phi^p(\xi)\left(\frac{d^q\Phi(\xi)}{d\xi^q}\right)^s\right]=M\,p+s(q+M). \tag{5}$$

Equating the R.H.S of Eqs.(4) and (5) , we can get the value of M.

The function $G(\xi)$ in Eq.(3) is considered to be the solution of the following ODE of Bernoulli type

$$G'+\lambda G=\mu G^2, \tag{6}$$

where λ and μ are non-zero parameters to be determined. The Bernoulli ODE eq. (6), has the following general solution

$$G(\xi)=\frac{1}{\frac{\mu}{\lambda}+b\,e^{\lambda\xi}}, \tag{7}$$

where b is an arbitrary constant not equal to zero.

To assess the power of the Bernoulli method we will compare the obtained solutions by this method with the solutions obtained by $G'/G-$expansion method [14]. According to $G'/G-$expansion method the solution of Eq.(2) is expressed in a similar way to Eq.(3) as

$$\Phi(\xi)=a_0+\sum_{i=1}^{M}a_i\left(\frac{G'(\xi)}{G(\xi)}\right)^i, \tag{8}$$

where $G(\xi)$ is the solution of the following ODE

$$G''+\lambda G'+\mu G=0. \tag{9}$$

The exact solutions of the linear ODE, Eq.(9), are:

i.for $\Delta=\lambda^2-4\mu>0$,

$$\frac{G'(\xi)}{G(\xi)}=\frac{\sqrt{\Delta}}{2}\left(\frac{c_1\cosh(\frac{\sqrt{\Delta}}{2}\xi)+c_2\sinh(\frac{\sqrt{\Delta}}{2}\xi)}{c_1\sinh(\frac{\sqrt{\Delta}}{2}\xi)+c_2\cosh(\frac{\sqrt{\Delta}}{2}\xi)}\right)-\frac{\lambda}{2}, \tag{10}$$

ii.for $\Delta=\lambda^2-4\mu<0$,

$$\frac{G'(\xi)}{G(\xi)}=\frac{\sqrt{\Delta}}{2}\left(\frac{c_1\cos(\frac{\sqrt{-\Delta}}{2}\xi)-c_2\sin(\frac{\sqrt{-\Delta}}{2}\xi)}{c_1\sin(\frac{\sqrt{-\Delta}}{2}\xi)+c_2\cos(\frac{\sqrt{-\Delta}}{2}\xi)}\right)-\frac{\lambda}{2}, \tag{11}$$

iii.for $\Delta=\lambda^2-4\mu=0$,

$$\frac{G'(\xi)}{G(\xi)}=\left(\frac{c_2}{c_1+c_2\xi}\right)-\frac{\lambda}{2}. \tag{12}$$

Substituting Eq.(3) in Eq.(2) in the case of Bernoulli method and using Eq.(6), the left-hand side of Eq.(2) is converted into a polynomial in G. In a similar way, substituting Eq.(8) in the case of $G'/G-$expansion method, in Eq.(2) and using Eq.(9), the left-hand side of Eq.(2) is converted into a polynomial in (G'/G). Equating each coefficient of these polynomials to zero, yields a system of algebraic equations in a_0, a_i, c, μ and λ. By solving this system of equations with the aid of Mathematica package and using the solution of Bernoulli equation, Eq.(7) or Eqs.(10-12) in the case of $G'/G-$expansion method, we can construct the traveling wave solutions of the NLPDE (2). In the following sections, the Bernoulli method is used to solve some NLPDEs and the solutions are compared with the $G'/G-$expansion solutions.

3 Examples of NLPD equations

3.1 (2+1)-dimensional ZK-BBM equation

We start with the generalized model of ZK-BBM equation [25]:

$$u_t+u_x+a(u^n)_x+d(u_{xt}+u_{yy})_x=0, \tag{13}$$

where a, and d are constants. Using the wave variable $\xi = x+y-ct$ with $U(\xi) = u(x,y,t)$, Eq.(13) is converted into

$$(1-c)U' + a(U^n)' + d(1-c)U''' = 0. \qquad (14)$$

Integrating Eq.(14), assuming $\frac{d^2U}{d\xi^2} \to 0$, $\frac{dU}{d\xi} \to 0$ and $U \to 0$ as $\xi \to \pm\infty$, we get

$$(1-c)U + aU^n + d(1-c)U'' = 0, \qquad (15)$$

Balancing the highest order linear term, U'',with the nonlinear term, U^n, gives $M = \frac{2}{n-1}$. It is suitable to consider M as a positive integer to get a closed analytical solution. If we take $n = 3$ then $M = 1$ and

$$U(\xi) = a_0 + a_1 G. \qquad (16)$$

Substituting Eq.(16) in Eq.(15), using Eq.(6), and equating the coefficients of $G^i (i = 0, 1, 2, 3)$ to zero gives a system of algebraic equations in the parameters a_0, a_1 ,c, μ and λ. Solving this system of equations and inserting the obtained values of these parameters together with the value of G, from Eq.(7), into Eq.(16) give the following solutions

$$u_1(x,y,t) = a_0 + \frac{\sqrt{2}\sqrt{d}\mu a_0}{b\ \exp\left(-\frac{\sqrt{2}(x+y-ct)}{\sqrt{d}}\right) - \frac{\sqrt{d}\mu}{\sqrt{2}}}, \qquad (17)$$

$$u_2(x,y,t) = a_0 - \frac{\sqrt{2}\sqrt{d}\mu a_0}{b\exp\left(\frac{\sqrt{2}(x+y-ct)}{\sqrt{d}}\right) + \frac{\sqrt{d}\mu}{\sqrt{2}}}, \qquad (18)$$

where $c = \left(1 + aa_0^2\right)$.
Eqs.(17) and (18) are the solutions for ZK-BBM, Eq.(13), for $n = 3$ where a_0, b and μ are arbitrary constants. The obtained solutions in Eqs.(17) and (18) are different from that obtained in [13,25]. If we take $n = 2$ then $M = 2$ and Eq.(8) gives

$$U(\xi) = a_0 + a_1 G + a_2 G^2. \qquad (19)$$

Using Eq.(19) in a similar manner gives the following solutions

$$u_3(x,y,t) = \frac{c-1}{a} + \frac{6(c-1)\sqrt{d}\mu^2}{a\left[b\exp\left(\frac{x+y-ct}{\sqrt{d}}\right)+\sqrt{d}\mu\right]^2} - \frac{6(c-1)\sqrt{d}\mu}{a\left[b\exp\left(\frac{x+y-ct}{\sqrt{d}}\right)+\sqrt{d}\mu\right]}, \qquad (20)$$

$$u_4(x,y,t) = \frac{c-1}{a} + \frac{6(c-1)\sqrt{d}\mu^2}{3a\left[b\exp\left(-\frac{x+y-ct}{\sqrt{d}}\right)-\sqrt{d}\mu\right]^2} + \frac{6(c-1)\sqrt{d}\mu}{a\left[b\exp\left(-\frac{x+y-ct}{\sqrt{d}}\right)-\sqrt{d}\mu\right]}, \qquad (21)$$

$$u_5(x,y,t) = \frac{c-1}{a} - \frac{2(c-1)\sqrt{d}\mu^2}{3a\left[b\exp\left(\frac{x+y-ct}{\sqrt{d}}\right)+\sqrt{d}\mu\right]^2} - \frac{2(c-1)\sqrt{d}\mu}{a\left[b\exp\left(\frac{x+y-ct}{\sqrt{d}}\right)+\sqrt{d}\mu\right]}, \qquad (22)$$

and

$$u_6(x,y,t) = \frac{c-1}{a} + \frac{2(c-1)\sqrt{d}\mu^2}{3a\left[b\exp\left(-\frac{x+y-ct}{\sqrt{d}}\right)-\sqrt{d}\mu\right]^2} - \frac{2(c-1)\sqrt{d}\mu}{a\left[b\exp\left(-\frac{x+y-ct}{\sqrt{d}}\right)-\sqrt{d}\mu\right]}. \qquad (23)$$

We have restricted the treatment to the nontrivial and real solutions. Fig.1 shows the solution, $u_2(x,y,t)$, Eq.(17), where Fig.1-a illustrates the travelling wave solution at different times with $y = 1$. The 3-D graph in Fig.1-b illustrates the variations of the solution along the spatial dimensions, x and y, at a specified time.
Remark 1:
If we choose $\frac{\sqrt{d}\mu}{\sqrt{2b}} = 1$ in Eqs.(17) and (18) and make use of the relations:

$$\frac{1}{1+e^z} = \frac{1}{2}\left[1 - \tanh\frac{z}{2}\right] \text{ and } \frac{1}{1-e^z} = \frac{1}{2}\left[1 - \coth\frac{z}{2}\right], \qquad (24)$$

the special kink soliton solutions, presented in [13,25], are obtained. Namely,

$$u_{s1}(x,y,t) = a_0\coth\left[\frac{1}{\sqrt{2d}}(x+y-ct)\right], \qquad (25)$$

$$u_{s2}(x,y,t) = a_0\tanh\left[\frac{1}{\sqrt{2d}}(x+y-ct)\right]. \qquad (26)$$

Moreover, when $d < 0$ the soliton solutions can be converted to the following periodic ones:

$$u_{p1}(x,y,t) = -i\ a_0\cot\left[\frac{1}{\sqrt{-2d}}(x+y-ct)\right], \qquad (27)$$

$$u_{p2}(x,y,t) = i\ a_0\tan\left[\frac{1}{\sqrt{-2d}}(x+y-ct)\right], \qquad (28)$$

which have been obtained in [25].

The $G'/G-$expansion method can be used in a similar way to obtain the solutions of ZK-BBM. However, we have picked up one of these solutions for the sake of comparison; namely

$$u(x,y,t) = \frac{\sqrt{2}a_0}{\sqrt{d\lambda}}\tanh\left[\frac{1}{\sqrt{2d}}(x+y-ct)\right], \qquad (29)$$

where $c = 1 + \frac{2aa_0^2}{d\lambda^2}$ and λ, a_0 are arbitrary.

It is evident that the solution with the $G'/G-$expansion method, Eq.(29), is identical for the special solution obtained with Bernoulli method, Eq.(26) for an appropriate choice of the arbitrary parameters. This assesses that the Bernoulli method not only gives new solutions but also recovers some solutions evaluated by other methods.

3.2 The nonlinear dispersive equation

A nonlinear dispersive equation takes the form [2]:

$$u^n\,(u^n)_t + a\left(u^{3n}\right)_x + d\,u^n\,(u^n)_{xxx} = 0, \qquad n > 0, \qquad (30)$$

where a and d are constants. Setting $v = u^n$ and dividing both sides by v, Eq.(30) is reduced to the usual Korteweg-De Vries (KDV) equation

$$v_t + \frac{3}{2}a\left(v^2\right)_x + d\,v_{xxx} = 0, \qquad n > 0. \tag{31}$$

This equation has wide range of applications in many branches of physics. Using the transformation $\xi = x - ct$ (c is the wave speed) Eq.(31) is converted into the following ODE

$$-cV' + \frac{3}{2}a\left(V^2\right)' + dV''' = 0, \tag{32}$$

where $V(\xi) = v(x,t)$. Integrating Eq.(32) once with respect to ξ and setting the constant of integration to be zero, assuming $\frac{d^2V}{d\xi^2} \to 0$, $\frac{dV}{d\xi} \to 0$, and $V \to 0$ as $\xi \to \pm\infty$, we get

$$-cV + \frac{3}{2}aV^2 + dV'' = 0. \tag{33}$$

Balancing $V''$with V^2 leads to $M = 2$. Then, according to Eq.(3)

$$V(\xi) = a_0 + a_1 G(\xi) + a_2 G^2(\xi), \tag{34}$$

where $G(\xi)$ is given by Eq.(7). Using Bernoulli Method as in the ZK-BBM problem we can drive the exact solutions of Eq.(30) as

$$u_1(x,t) = \left[\frac{4bd\,\lambda^3\mu\exp\left[\lambda\left(x - d\lambda^2 t\right)\right]}{a\left(b\exp\left[\lambda\left(x - d\lambda^2 t\right)\right]\lambda + \mu\right)^2}\right]^{\frac{1}{n}}, \tag{35}$$

$$u_2(x,t) = \left[-\frac{2d\lambda^2}{3a} + \frac{4bd\,\lambda^3\mu\exp\left[\lambda\left(x + d\lambda^2 t\right)\right]}{a\left(b\exp\left[\lambda\left(x + d\lambda^2 t\right)\right]\lambda + \mu\right)^2}\right]^{\frac{1}{n}}, \tag{36}$$

where λ and μ are arbitrary constants.
The solutions in Eqs.(35) and (36) differ from the others in [2] and [13]. So we can say that they are new exact solutions for the current problem. Fig.2 illustrates the behavior of the solution, Eq.(35). Fig.2-a illustrates the travelling wave solution at different times while the 3-D graph in Fig.2-b illustrates the variations of the solution with x and t.

Remark 2:
If we choose λ and μ such that $\frac{\mu}{\lambda} = b$ and make use of the relation

$$\frac{e^z}{(1+e^z)^2} = \frac{1}{4}sech^2\left[\frac{z}{2}\right] \tag{37}$$

in Eqs.(35) and (36) the special soliton solutions are obtained. Namely,

$$u_{s1}(x,t) = \left[\frac{d\lambda^2}{a}\text{sech}^2\left[\frac{\lambda}{2}\left(x - d\lambda^2 t\right)\right]\right]^{\frac{1}{n}}, \tag{38}$$

$$u_{s2}(x,t) = \left[-\frac{2d\lambda^2}{3a} + \frac{d\lambda^2}{a}\text{sech}^2\left[\frac{\lambda}{2}\left(x + d\lambda^2 t\right)\right]\right]^{\frac{1}{n}}, \tag{39}$$

Moreover, when λ is an imaginary number, the soliton solutions can be converted to periodic solutions as

$$u_{p1}(x,t) = \left[-\frac{d\,\delta^2}{a}\sec^2\left[\frac{\delta}{2}\left(x + d\,\delta^2 t\right)\right]\right]^{\frac{1}{n}}, \tag{40}$$

$$u_{p2}(x,t) = \left[\frac{2d\,\delta^2}{3a} - \frac{d\delta^2}{a}\sec^2\left[\frac{\delta}{2}\left(\mathrm{x} - \mathrm{d}\delta^2\mathrm{t}\right)\right]\right]^{\frac{1}{n}}, \tag{41}$$

where $\lambda = i\,\delta$.
Special soliton and periodic solutions, Eqs. (38) to (41) are identical to those obtained in [2].

Again the $G'/G-$expansion method can be used to obtain the solutions of Eq.(30). Different solutions are obtained and one of them is taken for the sake of comparison; namely

$$u(x,t) = \left[\frac{c}{a}\text{sech}^2\left[\frac{1}{2}\sqrt{\frac{c}{d}}(x - ct)\right]\right]^{\frac{1}{n}}, \tag{42}$$

which is the same as solution (38) considering $c = d\lambda^2$.This assesses again that the Bernoulli method not only gives new solutions but also recovers some solutions evaluated by another methods.

3.3 The general Burgers–Fisher equation

Another illustrative example of the Bernoulli method is the solution of the general Burgers-Fisher equation given by [26]

$$u_t + pu^s u_x - u_{xx} - qu(1 - u^s) = 0, \tag{43}$$

where p, q and s are some parameters. This equation has a wide range of applications in plasma physics, fluid mechanics, capillary-gravity waves, nonlinear optics and chemical physics. If we put $s = p = q = 1$in Eq.(43) we get the usual Burgers-Fisher equation.

Setting $v = u^s$, Eq.(43) becomes

$$vv_t + pv^2 v_x - vv_{xx} + \left(1 - \frac{1}{s}\right)v_x^2 - qsv^2(1 - v) = 0. \tag{44}$$

Using the wave variable $\xi = k(x - ct)$ (k and c are the wave number and the wave speed respectively) and setting $V(\xi) = v(x,t)$, Eq.(44) is converted into

$$-kcVV' + pkV^2V' - k^2\left(1 - \frac{1}{s}\right)V'^2 - qs(1 - V)V^2 = 0. \tag{45}$$

Balancing $V^2V'$with V'^2gives $M=1$. Then, according to Eq.(3)

$$V(\xi)=a_0+a_1G. \tag{46}$$

Using the Bernoulli method used in the previous section three solutions of Eq.(43) can be derived as

$$u_1(x,t)=\left[1+\frac{a_1}{b\exp\left(-\frac{ps(-\frac{(p^2+q(1+s)^2)t}{p(1+s)}+x)}{1+s}\right)-a_1}\right]^{\frac{1}{s}}, \tag{47}$$

$$u_2(x,t)=\left[\frac{a_1}{b\exp\left(\frac{ps(-\frac{(p^2+q+2qs+qs^2)t}{p(1+s)}+x)}{1+s}\right)+a_1}\right]^{\frac{1}{s}}, \tag{48}$$

where a_1 and b are arbitrary constants.

It is noted that the new solutions in Eqs.(47) and (48) are not obtained by Wazwaz [8], using tanh method, as well as by El-wakil and Abdou [13], using METFM. In addition, these solutions are different from that obtained by Chen and Zhang [26]. Fig.3 illustrates the behavior of the solution, Eq.(47), where Fig.3-a illustrates the travelling wave solution at different times and the 3-D graph in Fig.3-b illustrates the variations of the solution with x and t.

Remark 3:

If we take $\frac{b}{a_1}=-1$ and make use of the relations in Eq.(24), solutions in Eq.(47) and (48) can be expressed as

$$u_{p1}(x,t)=\left[\frac{1}{2}(1+\tanh[\frac{ps(\frac{(p^2+q(1+s)^2)t}{p(1+s)}-x)}{2(1+s)}])\right]^{\frac{1}{s}}. \tag{49}$$

$$u_{p2}(x,t)=\left[\frac{1}{2}(1+\coth[\frac{ps(\frac{(p^2+q(1+s)^2)t}{p(1+s)}-x)}{2(1+s)}])\right]^{\frac{1}{s}}, \tag{50}$$

These special solutions are the same as that obtained in [8] and agree with some of those obtained in [26].

Again, for more assessment of the proposed method the $G'/G-$expansion method can be used to obtain a number of solutions of the Burgers-Fisher equation, Eq.(43). The following solution is chosen to be compared with Bernoulli method's solutions;

$$u(x,t)=[\tfrac{1}{2}\left(1+\tanh\left[\frac{s\left(p^2t+q(1+s)^2t-p(1+s)x\right)}{2(1+s)^2}\right]\right)]^{\frac{1}{s}}, \tag{51}$$

which is the same solution as in Eq.(49).

4 Real Physical Problem

As a real physical problem, the solution of ZK equation for the electrostatic potential u that takes the following form

$$u_t+\frac{A}{2}(u^2)_x+(Bu_{xx}+Cu_{yy})_x=0, \tag{52}$$

is investigated. This equation is another form of ZK-BBM equation, Eq.(13), and was studied in details by Abdelsalam and Selim [14]. It describes two-dimensional, magnetized, collisionless three-component plasma consisting of positive ions, negative ions, and electrons with densities n_+, n_- and n_e respectively. The coefficient of the nonlinear term, A, the coefficient of the dispersion term, B, and the coefficient of the combined term, C, are constants, depend on the parameters of plasma systems, given by $A=B(\frac{3}{\lambda_1^4}-\frac{3\beta Q^2}{\lambda_1^4}-\frac{3\alpha}{4})$, $B=(\frac{\lambda_1^3}{2+2\beta Q})$, and $C=B(1+\frac{1}{\omega_{c+}^2}+\frac{\beta Q}{\omega_{c-}^2})$. The wave speed λ_1 is given by $\lambda_1=\sqrt{\frac{2(1+\beta Q)}{3\alpha}}$, where $\alpha=n_{e0}/n_{+0}$ and $\beta=n_{-0}/n_{+0}$. n_{+0}, n_{-0} and n_{e0} are the unperturbed positive, negative ions, and electrons densities respectively. $Q=m_+/m_-$ is the positive ion to the negative ion mass ratio. $\omega_{c\pm}$ are the positive/negative ions cyclotron frequencies.

Using the traveling wave transformation $\xi=L_xx+L_yy-vT$,where v represents a constant speed (scaled by the ion-acoustic speed), in Eq.(52) we get

$$-vU_\xi+A_0UU_\xi+B_0U_{\xi\xi\xi}=0, \tag{53}$$

L_x and L_y are the directional cosines of the wave vector along x and y-axes so that $L_x^2+L_y^2=1$, $A_0=L_xA$ and $B_0=L_x(BL_x^2+CL_y^2)$.

Eq.(53) is solved by $G'/G-$expansion method in Ref. [14] and numerical results are investigated. In what follows, Eq. Eq.(53) will be solved by Bernoulli method and we compare graphically with the results of $G'/G-$expansion method [14].

Integrating Eq..(53), assuming $\frac{d^2U}{d\xi^2}\to 0$, $\frac{dU}{d\xi}\to 0$, and $U\to 0$ as $\xi\to\pm\infty$, we get

$$-vU+\frac{A_0}{2}U^2+B_0U_{\xi\xi}=0. \tag{54}$$

Balancing the highest order linear term, U'',with the nonlinear term, U^2,gives$M=2$ and hence Eq.(6) is reduced to

$$U(\xi)=a_0+a_1G+a_2G^2, \tag{55}$$

Substituting Eq..(55) into Eq..(54), using Eq..(6), and equating the coefficients of $G^i(i=0,1,2,3)$ to zero give a system of algebraic equations in the parameters a_0, a_1, a_2,$\mu$and λ. Solving this system of equations and inserting the obtained values of the parameters together with the value of G from Eq.(7) into Eq..(55) to give

$$u_1(x,y,t) = -\frac{12b\,\exp\left(\frac{\sqrt{v}(-tv+xL_x+yL_y)}{\sqrt{B_0}}\right)v^{3/2}\mu\sqrt{B_0}}{A_0\left(b\sqrt{v}-\exp\left(\frac{\sqrt{v}(-tv+xL_x+yL_y)}{\sqrt{B_0}}\right)\mu\sqrt{B_0}\right)^2}, \quad (56)$$

$$u_2(x,y,t) = \frac{12b\,\exp\left(\frac{\sqrt{v}(-tv+xL_x+yL_y)}{\sqrt{B_0}}\right)v^{\frac{3}{2}}\mu\sqrt{B_0}}{A_0\left(b\,\exp\left(\frac{\sqrt{v}(-tv+xL_x+yL_y)}{\sqrt{B_0}}\right)\sqrt{v}+\mu\sqrt{B_0}\right)^2}, \quad (57)$$

where μ is an arbitrary constant.

Remark 4:

If we choose $\frac{\mu}{b}\sqrt{\frac{B_0}{\upsilon}} = 1$ and make use of the relation in Eq.(42) and the following relation

$$\frac{e^z}{(e^z-1)^2} = \frac{1}{4}\csc h^2[\frac{z}{2}], \quad (58)$$

solutions (56) and (57) become

$$u_{s1}(x,y,t) = -\frac{3v}{A_0}\mathrm{csch}^2[\frac{1}{2}\sqrt{\frac{v}{B_0}}(-tv+xL_x+yL_y)], \quad (59)$$

$$u_{s2}(x,y,t) = \frac{3v}{A_0}\mathrm{sech}^2[\frac{1}{2}\sqrt{\frac{v}{B_0}}(-tv+xL_x+yL_y)], \quad (60)$$

Some of the solutions of Eq.(53) with $G'/G-$expansion method are [14]:

$$u_1(x,y,t) = -\frac{3v}{A_0}\mathrm{csch}[\frac{1}{2}\sqrt{\frac{v}{B_0}}(-tv+xL_x+yL_y)]^2, \quad (61)$$

$$u_2(x,y,t) = \frac{3v}{A_0}\mathrm{sech}[\frac{1}{2}\sqrt{\frac{v}{B_0}}(-tv+xL_x+yL_y)]^2, \quad (62)$$

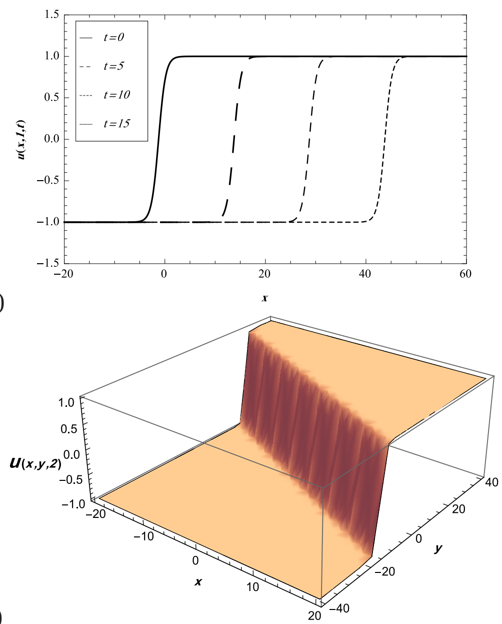
or

$$u_2(x,y,t) = \phi_m\mathrm{sech}[\frac{1}{W}(-tv+xL_x+yL_y)]^2, \quad (63)$$

where W and ϕ_m are the wave width and amplitude of ion acoustic solitary wave respectively. It is obvious that the solutions obtained using $G'/G-$expansion, Eq.(61) and (62) , are identical to the special solutions that obtained by the Bernoulli method, Eq.(59) and (60)

For the remaining solutions obtained with $G'/G-$expansion method, the reader can refer to Ref.[14]. However, we pay attention to the soliton solution, Eq.(62), because it is usually the required solution.

Fig.4 shows the variation of the solitary solution $u(\xi)$ as given by Eq.(60), obtained using the Bernoulli method (or equivalently Eq.(62) using $G'/G-$expansion method) at different positive to negative ion ratio, $\beta = n_{-0}/n_{+0}$, and the positive ions cyclotron frequency,ω_{c+}. The positive ions cyclotron frequency,$\omega_{c+}$is related to the external magnetic field, Ω_0, according to the relation, $\omega_{c+} = (e\Omega_0/m_+c)$. It is clear from Figs.4-a that the increase of β, increases the solitary wave amplitude and width. In Figs.4-b it is noted that the increase of the external magnetic field decreases the solitary width and has no effect on the solitary amplitude.



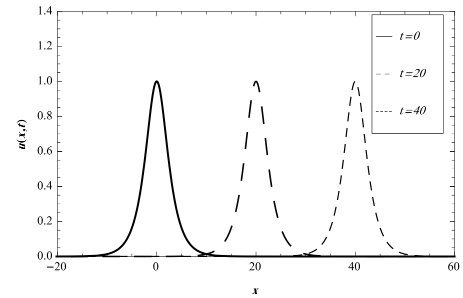
(a)

(b)

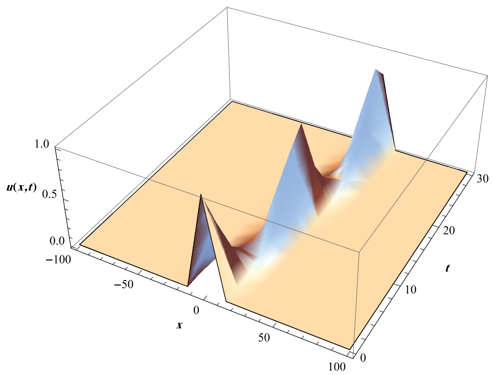
Fig. 1: The evolution of the exact solution, Eqs.(17) and (18) , with $a_0 = 1, a = 2,\ d = 1$ and $b = 1$ a) Variations of the solution at different time values with $y = 1$, b) 3D graph for the solution at $t = 2$.

5 Conclusions and discussion

We have implemented the Bernoulli method to solve some NLPDEs. These NLPDEs are important models in mathematical physics. The considered problems are the ZK-BBM equation, a nonlinear dispersive equation, the general Burgers-Fisher equation and the ZK equation that describes two-dimensional, magnetized, collisionless three-component plasma. Applying Bernoulli method we have obtained new exact solutions not obtained in the published articles for the same problems. For special

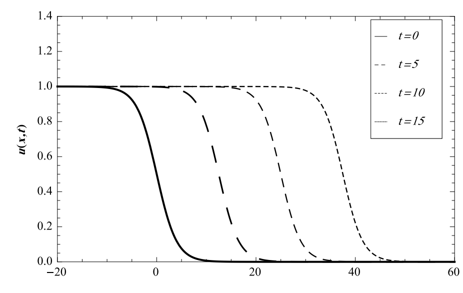


(a)

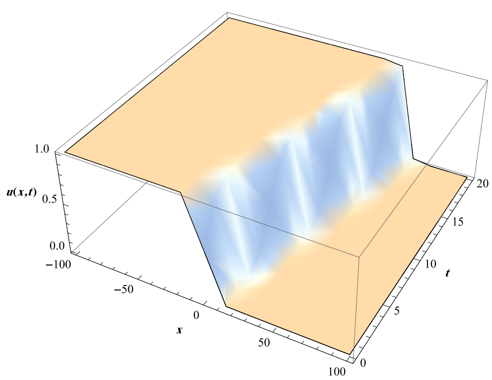


(b)

Fig. 2: The evolution of the exact solitary solution, Eq.(35), with $a=1, d=1,\ \mu=\lambda=b=1$ and $n=2$. a) Variations of the solution at different time values, b) 3D graph for the solution.

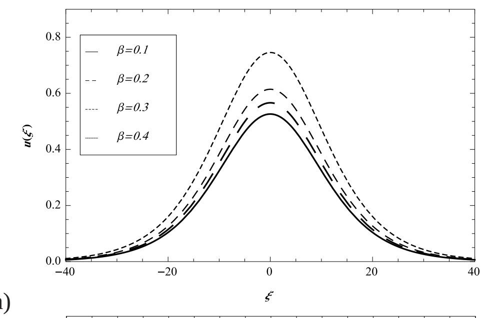


(a)

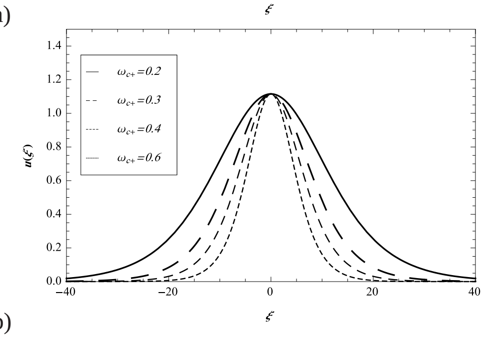


(b)

Fig. 3: The evolution of the exact solutions, Eq.(47), with $p=1, q=1, b=1,\ a_1=-1$ and $s=1$. a) Variations of the solution at different time values, b) 3-D graph for the solution.



(a)



(b)

Fig. 4: The variation of the solitary solution of ZK equation, as given by Eqs. (60), obtained using Bernoulli method at a) different positive to negative ion ratio, β, and b) at different $\omega_{c+}$with $\mu=1,\ b=1, L_x=0.3,\ Q=0.9,, \omega_{c-}=0.6, \nu=0.1$ ($\omega_{c+}=0.3$ in a and $\beta=0.3$ in b).

cases of the arbitrary parameters some soliton and periodic solutions found in the literature are recovered. For more assessment of the method, a comparison of selected solutions obtained with Bernoulli method to that obtained with $G'/G-$expansion method is presented. This comparison shows that, The selected solutions obtained with $G'/G-$expansion method are identical to some special solutions obtained with Bernoulli method.

Finally we can conclude that Bernoulli method is reliable and readily applicable to a large number of NLPDs. Moreover, we think that it is simpler and more effective than other methods. Hence, we are investigating how to improve Bernoulli method to treat other complicated types of NLPDEs in the future.

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