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On The Convolution Property of a Heavy Tailed Stable Distribution

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Abstract: In this Article, we present an explicit direct proof of the convolution property for a heavy tailed stable distribution. The distribution arises and is of interest in a variety of the contexts in many disciplines: in probability and statistics, in electrical engineering, computer vision, image and signal processing and in many physical and economic processes. We shall refer to this as Lèvy's distribution in the sequel. The particular convolution property for the distribution, which entails its stability, shows that the sample mean based on a random sample of n observations from this distribution has the same distribution as that of n times a single observation. The sample mean, thus, is more variable than a single observation and increases by an order of n as the sample size n increases. The central limit theorem, evidently, does not hold for this distribution. We also give an alternative proof for the above property based on Laplace transforms. These proofs do not seem to be available in standard text books. The only proofs available use advanced arguments involving the Brownian motion process. In addition, for better understanding of Lèvy's and other stable distributions, some contextually relevant basic properties of stable distributions are also discussed and elaborated on. Stable distributions are the limiting distributions, under appropriate conditions, of normed sums of independent random variables. Their study should be of interest per se. These proofs in their detailed presentation along with an introductory discussion of stable distributions should help to fill up a notable gap in the available text-book literature. The article should be of interest from a pedagogical standpoint for seniors, first year graduate students and beginning researchers in statistics and probability.

Keywords: Convolution, Lèvy's distribution, Heavy tailed distributions, Inverse gamma distribution, Stable distributions

1 Introduction

Consider a random variable $(r.v.)X$ with density

$$f_X(x) = \frac{1}{\sqrt{2\pi x^3}} e^{-\frac{1}{(2x)}}, x > 0. \tag{1}$$

The random variable X is distributionally equivalent to the random variable $\frac{1}{Z^2}$ where $Z \sim N(0,1)$, the normal random variable with mean 0 and variance 1. The mean and variance do not exist for this distribution. The density given by equation (1) is an example of a heavy-tailed distribution; see the plot below in Fig 1.The density is a special case of an inverse-gamma family of densities (Johnson et al. 1995, p. 401; Casella and Berger 2002, p.51)[2, 3] given, for parameters $\alpha > 0, \beta > 0$, by

$$f_x(x;\alpha,\beta) = \left(\frac{\beta^\alpha}{\Gamma(\alpha)}\right) x^{-\alpha-1} e^{-(\frac{\beta}{x})}, x > 0, \tag{2}$$

with the shape and scale parameters α and β, respectively, each set equal to $\frac{1}{2}$. It is easy to see that if a r.v. Y follows the gamma density $g_y(y;\alpha,\beta) = \left(\frac{\beta^\alpha}{\Gamma(\alpha)}\right) x^{\alpha-1} e^{-\beta x}, x > 0$, then the density (2), as the name inverse gamma suggests, is that of the r.v. $X = \frac{1}{Y}$. The mean and variance of density (2) are given, respectively, by $\frac{\beta}{(\alpha-1)}$ and $\left[\frac{\beta^2}{(\alpha-1)^2(\alpha-2)}\right]$ which exist

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only if $\alpha > 1$ and $\alpha > 2$, respectively. For the density (1) with $\alpha = \beta = \frac{1}{2}$ in equation (2), thus, the mean and variance do not exist, and so the central limit theorem(CLT) for this distribution does not hold. The most basic CLT may be stated as follows:

CLT (Bagui et al. 2013) [1]: Let $\{X_n : n \geq 1\}$ be a sequence of independent and identically distributed (i.i.d.) random variables with mean $\mu, -\infty < \mu < \infty$, and variance $\sigma^2, 0 < \sigma^2 < \infty$ and set $S_n = \sum_{i=1}^{n} X_i, \bar{X}_n = [\frac{S_n}{n}]$ and

$$Z_n = \frac{S_n - n\mu}{\sigma\sqrt{n}} = \frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma}.$$

Then $Z_n \xrightarrow{d} Z \sim N(0,1)$ as $n \to \infty$. (The notation $\xrightarrow{d}$ stands for "convergence in distribution", $\sim$ stands for "distributed as" and $N(0,1)$ for a normal distribution with mean 0 and variance 1 .) $\square$

In fact, the density (1) is an extreme example in which not only that the central limit theorem does not hold, but for which the tail heaviness of the distribution leads the sample mean to being more variable than a single observation. Specifically, for this distribution the sample mean $\bar{X}_n = [\frac{S_n}{n}]$ has the same distribution as that of nX_1. To show that $\bar{X}_n$ is distributionally equivalent to nX_1, i.e., $\bar{X}_n \stackrel{d}{=} nX_1$ - which incidentally implies that the distribution (1) is strictly stable with characteristic exponent $\alpha = \frac{1}{2}$ (see Section 4 below for definition; cf. Feller (1971) p. 170)[5] - we shall prove in Section 2 the above-referred convolution property for a general family of densities (defined by equation (4) below) which contains the density (1). The main aim of this note, as stated earlier, is to present an explicit direct proof of this convolution property and also a discussion of stable distributions, not readily available in standard texts.

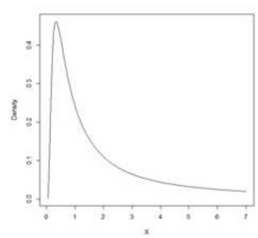

Fig. 1: Plot of the density (1)

2 The Convolution Property

To prove the above-referred convolution property of density (1), let us consider a more general family of densities of r.v.'s defined by $X_{(\tau)} = \frac{\tau^2}{Z^2}$, where $Z \sim N(0,1)$ with (cumulative) distribution function $\Phi(t) = P(Z \leq t) = \int_{-\infty}^{t} \left(\frac{1}{\sqrt{2\pi}}\right) e^{-\left(\frac{x^2}{2}\right)} dx$ and $0 < \tau < \infty$. The considered family is just the family (2) with α and β replaced with $\left(\frac{1}{2}\right)$ and $\left(\frac{\tau^2}{2}\right)$, respectively (see equation (4) below). The cumulative distribution function of $X_{(\tau)}$ calculates to

$$F_{(\tau)}(x) = P[X_{(\tau)} \leq x] = P\left[Z^2 \geq \frac{\tau^2}{x}\right] = P\left[Z \geq \frac{\tau}{\sqrt{x}}\right] + P\left[Z \leq -\frac{\tau}{\sqrt{x}}\right] = 2\left[1 - \Phi\left(\frac{\tau}{\sqrt{x}}\right)\right], x > 0 \qquad (3)$$

(since $\Phi(-x) = 1 - \Phi(x)$), with the probability density function given by

$$f_{(\tau)}(x) = \frac{d}{dx}[F_{(\tau)}(x)] = \frac{\tau}{\sqrt{2\pi}(x^{\frac{3}{2}})} e^{\frac{-\tau^2}{2x}}, x > 0. \qquad (4)$$

In (4) above when we set $\tau = 1$, we get the density (1). Now denote by $\Im$ the family of densities $\{f_{(\tau)} : 0 < \tau < \infty\}$. This family of densities $\Im$ is closed under convolutions. We put this in a Theorem below:

Theorem 1. $\Im$ is closed under convolutions, i.e., $f_{(\tau)} * f_{(\lambda)} = f_{(\tau+\lambda)}$, for all $0 < \tau, \lambda < \infty$.

Proof. Let $X_{(\tau,\lambda)} = X_{(\tau)} + X_{(\lambda)}$, then the density $f_{(\tau,\lambda)}$ of $X_{(\tau,\lambda)}$ is given by, for $0 < t < \infty$,

$$f_{(\tau,\lambda)}(t) = \int_0^t f_{(\tau)}(t-y) f_{(\lambda)}(y) dy = \frac{\tau\lambda}{2\pi} \int_0^t \frac{1}{[(t-y)y]^{\frac{3}{2}}} e^{-\frac{1}{2}\left[\frac{\tau^2}{t-y}+\frac{\lambda^2}{y}\right]} dy = \frac{\tau\lambda}{2\pi t^2} \int_0^t \frac{1}{[u(1-u)]^{\frac{3}{2}}} e^{-\frac{1}{2t}\left[\frac{\tau^2}{1-u}+\frac{\lambda^2}{u}\right]} du, \tag{5}$$

the last equality following by substituting $u = \left(\frac{y}{t}\right)$ as the variable of integration on the RHS integral (since $0 < y < t$ yields $0 < u < 1$ as the limits of integrations). From (5), by multiplying before the integral with factor $e^{-\left[\frac{(\tau+\lambda)^2}{2t}\right]}$ and dividing with it inside the integral, we obtain after some simplification that, for $0 < t < \infty$,

$$f_{(\tau,\lambda)}(t) = \frac{\tau\lambda}{2\pi t^2} e^{\frac{-(\tau+\lambda)^2}{(2t)}} \int_0^t \frac{1}{[u(1-u)]^{\frac{3}{2}}} e^{\frac{-[\lambda - u(\tau+\lambda)]^2}{[(2t)u(1-u)]}} du. \tag{6}$$

Now we make a further variable of integration substitution in the RHS integral in (6) by setting

$$v = \frac{u(\tau+\lambda) - \lambda}{\sqrt{u(1-u)}}, 0 < u < 1; \tag{7}$$

clearly, v increases from $-\infty$ to 0 as u increases from 0 to $\left[\frac{\lambda}{(\tau+\lambda)}\right]$ and from 0 to $+\infty$ as u increases from $\left[\frac{\lambda}{(\tau+\lambda)}\right]$ to 1. So $v \nearrow$ from $-\infty$ to $+\infty$ as $u \nearrow$ from 0 to 1. Now to evaluate the integral on the right of (6), we solve equation (7) for u in terms of the variable v. From (7), we obtain the quadratic equation

$$[(\tau+\lambda)^2 + v^2]u^2 - [2\lambda(\tau+\lambda) + v^2]u + \lambda^2 = 0, \tag{8}$$

which yields two solutions u_1 and u_2 given by (signs $+$ and $-$, respectively)

$$u_1, u_2 = \frac{[2\lambda(\tau+\lambda) + v^2] \pm \sqrt{v^4 + 4\tau\lambda v^2}}{2[(\tau+\lambda)^2 + v^2]}. \tag{9}$$

Both solutions u_1 and u_2 in (9) are valid: This follows since firstly

$$\sqrt{v^4 + 4\tau\lambda v^2} = \sqrt{(v^2 + 2\tau\lambda)^2 - 4\tau^2\lambda^2} < v^2 + 2\tau\lambda < 2\lambda(\tau+\lambda) + v^2, \tag{10}$$

so that both solutions clearly are positive; and secondly that, in view of (10), the numerator in (9) does not exceed the expression below, namely,

$$[2\lambda(\tau+\lambda) + v^2 + (2\tau\lambda + v^2)] = 2[v^2 + 2\tau\lambda + \lambda^2] < 2[v^2 + (\tau+\lambda)^2]. \tag{11}$$

Equation (9) and the inequality (10) show that $u_1, u_2 > 0$ and equation (9) and the inequality (11) that $u_1, u_2 < 1$.
In fact, as we shall see, either of these two solutions u_1 and u_2 do enable us to evaluate the integral on the right side of (6). Now note that from (7) we obtain, after some simplification, that

$$dv = \frac{\tau u + \lambda(1-u)}{2[u(1-u)]^{\frac{3}{2}}} du, \tag{12}$$

so that using (7) and (12) in (6), we obtain

$$f_{(\tau,\lambda)}(t) = \frac{\tau\lambda}{\pi t^2} e^{\frac{-(\tau+\lambda)^2}{2t}} \int_{-\infty}^{\infty} \frac{1}{[\tau u + \lambda(1-u)]} e^{\frac{-v^2}{2t}} dv. \tag{13}$$

Now to simplify the integral in (13), we evaluate $[\tau u + \lambda(1-u)]$ in terms of v by using one of the values u_1 and u_2 in (9), say, u_1 (that the other u_2, would yield the same result becomes clear readily): Note that substituting the value1u_1, we obtain

$$\begin{aligned} \tau u + \lambda(1-u) &= \frac{\tau[2\lambda(\tau+\lambda) + v^2 + \sqrt{v^4 + \tau\lambda v^2}] + \lambda[2(\tau+\lambda)^2 + 2v^2 - 2\lambda(\tau+\lambda) - v^2 - \sqrt{v^4 + 4v\lambda}}{2[(\tau+\lambda)^2 + v^2]} \\ &= \frac{(\tau+\lambda)(v^2 + 4\tau\lambda) + (\tau-\lambda)\sqrt{v^4 + 4\tau\lambda v^2}}{2[(\tau+\lambda)^2 + v^2]} = \frac{\sqrt{v^2 + 4\tau\lambda}[(\tau+\lambda)\sqrt{v^2 + 4\tau\lambda} + (\tau-\lambda)v]}{2[(\tau+\lambda)^2 + v^2]}, \end{aligned} \tag{14}$$

so that from (14) we have, after multiplying numerator and denominator in (14) with $[\tau+\lambda\sqrt{v^2+4\tau\lambda}-(\tau-\lambda)v]$, that

$$\frac{1}{\tau u+\lambda(1-u)}=\frac{2[(\tau+\lambda)^2+v^2][(\tau+\lambda)\sqrt{v^2+4\tau\lambda}-(\tau-\lambda)v]}{\sqrt{v^2+4\tau\lambda}[(\tau+\lambda)^2(v^2+4\tau\lambda)-(\tau-\lambda)^2v^2]}$$
$$=\frac{2[(\tau+\lambda)^2+v^2][(\tau+\lambda)\sqrt{v^2+4\tau\lambda}-(\tau-\lambda)v]}{\sqrt{v^2+4\tau\lambda}[v^2+(\tau+\lambda)^2](4\tau\lambda)}=\frac{(\tau+\lambda)}{2\tau\lambda}-\frac{(\tau-\lambda)}{2\tau\lambda}\cdot\frac{v}{\sqrt{v^2+4\tau\lambda}}. \tag{15}$$

Now substituting (15) in (13), we obtain

$$f_{(\tau,\lambda)}(t)=\frac{(\tau+\lambda)}{2\pi t^2}e^{\frac{-(\tau+\lambda)^2}{(2t)}}\left[\int_{-\infty}^{\infty}e^{\frac{-v^2}{(2t)}}dv-\frac{(\tau-\lambda)}{(\tau+\lambda)}\int_{-\infty}^{\infty}\frac{v}{\sqrt{v^2+4\tau\lambda}}e^{\frac{-v^2}{(2t)}}dv\right]; \tag{16}$$

while the first integral on the RHS of (16) equals $\sqrt{2\pi t}$ using the property of normal density, the second integral vanishes since its range is $(-\infty,\infty)$ and the integrand is an odd function. Thus from (16), we conclude that

$$f_{(\tau,\lambda)}(t)=\frac{(\tau+\lambda)}{\sqrt{2\pi}(t^{\frac{3}{2}})}e^{\frac{-(\tau+\lambda)^2}{2t}}=f_{(\tau+\lambda)(t)}\text{ for }0<t<\infty, \tag{17}$$

the last equality in (17) following by definition, with $f_{(\tau,\lambda)}\in\Im$. Thus we have proved that $\Im$ is closed under convolutions. □

Theorem 2. Let $X,X_1,X_2,\cdots,X_n$ be an i.i.d sample from the density $f_{(1)}$. Set $S_n=\sum_{i=1}^{n}X_i$ and $\bar{X_n}=\left[\frac{S_n}{n}\right]$. Then, $\bar{X_n}$ is distributionally equivalent to nX, or equivalently that $S_n\overset{d}{=}n^2X$ for all n.

Proof. The preceding Theorem 1, coupled with a simple induction argument and equation (4), yields that the density of $S_n=\sum_{i=1}^{n}X_i$ is $f_{(n)}$ (namely, the n- fold convolution of $f_{(1)}$ with itself, i.e., $f_{(1)}^{*n}$ equals $f_{(n)}$), given by (4) with $\tau=n$. In view of (3), by definition it implies that $S_n\overset{d}{=}n^2X$, where $X=\left(\frac{1}{Z^2}\right)$ or equivalently that $\bar{X_n}\overset{d}{=}nX$ for all n. To see this, we use equation (3) to derive $P[\left(\frac{\bar{X_n}}{n}\right)\leq x]=P[S_n\leq n^2x]=F_{(n)}(n^2x)=2\left[1-\Phi\left(\frac{n}{\sqrt{n^2x}}\right)\right]=2\left[1-\Phi\left(\frac{1}{\sqrt{x}}\right)\right]=F_{(1)}(x),x>0$. The last equation implies that $\left[\frac{\bar{X_n}}{n}\right]\overset{d}{=}X$, or equivalently that $S_n\overset{d}{=}n^2X$. This completes the proof. □

3 Proofs of Theorems 1 and 2 based on Laplace transforms

We shall now furnish alternative proofs of Theorem 1 and Theorem 2 based on the Laplace transformation technique.

Definition. If F denotes a probability distribution function concentrated on $\mathbb{R}^+=[0,\infty)$, the Laplace transform φ of F is the function defined for $t\geq 0$ by $\varphi(t)=\int_0^\infty e^{-tx}dF(x)$.

Proposition 1. Distinct probability distributions on $\mathbb{R}^+$ have distinct Laplace transforms.

Proof. See Feller (1971); XIII p. 430 [5]. □

We first derive the Laplace transform of $f_{(1)}$ given by (1):

Lemma 1. The Laplace transform of density $f_{(1)}$ of (1) is given by $\varphi(t)=e^{-\sqrt{2t}}$ for $t\geq 0$.

Proof. First note that for $t\geq 0$

$$\varphi_{(1)}(t)=\int_0^\infty e^{-tx}f_{(1)}(x)dx=\frac{1}{\sqrt{2\pi}}\int_0^\infty x^{\frac{-3}{2}}e^{\frac{-(1+2tx^2)}{2x}}dx$$
$$=\frac{e^{-\sqrt{2t}}}{\sqrt{2\pi}}\int_0^\infty x^{\frac{-3}{2}}e^{-\frac{1}{2}\left(\frac{\sqrt{2t}x-1}{\sqrt{x}}\right)^2}dx. \tag{18}$$

Now if we set $y=\left[\frac{(\sqrt{2t}x-1)}{\sqrt{x}}\right]$ as the transformed variable of integration in the preceding integral, we note that as $x\nearrow$ from 0 to ∞,$y\nearrow$ from $-\infty$ to ∞ and the differential dy evaluates to $dy=\left[\frac{(1+x\sqrt{2t})}{2x^{\frac{3}{2}}}\right]dx$. Also solving the quadratic equation $\sqrt{2t}x - y\sqrt{x} - 1 = 0$ for $\sqrt{x}$, we obtain the only valid solution as $\sqrt{x} = \left[\frac{(y+\sqrt{y^2+4\sqrt{2t}})}{2\sqrt{2t}}\right]$ (since $x > 0$), with the last equality yielding (upon squaring) $4tx = y^2 + 2\sqrt{2t} + y\sqrt{y^2+4\sqrt{2t}}$, which leads to

$$4tx + 2\sqrt{2t} = y^2 + 4\sqrt{2t} + y\sqrt{y^2+4\sqrt{2t}} = \left[y^2+4\sqrt{2t}\right]^{\frac{1}{2}}\left[(y^2+4\sqrt{2t})^{\frac{1}{2}} + y\right]. \tag{19}$$

Now note from (18) that, using the x - expressions for the transformed variable of integration y and its differential dy from above, we can write $\varphi_{(1)}(t)$ as

$$\varphi_{(1)}(t) = \left(\frac{1}{\sqrt{2\pi}}\right)e^{-\sqrt{2t}}\int_{-\infty}^{\infty}\frac{2}{1+x\sqrt{2t}}e^{-\frac{1}{2}y^2}dy$$
$$= \left(\frac{1}{\sqrt{2\pi}}\right)e^{-\sqrt{2t}}\int_{-\infty}^{\infty}\frac{4\sqrt{2t}}{\left[4tx+2\sqrt{2t}\right]}e^{-\frac{1}{2}y^2}dy,$$

with the last equality obtained by multiplying the numerator and the denominator with $(2\sqrt{2t})$. Now substituting the expression for $[4tx+2\sqrt{2t}]$ from (19) in the preceding integral, we obtain

$$\varphi_{(1)}(t) = \frac{1}{\sqrt{2\pi}}e^{-\sqrt{2t}}\int_{-\infty}^{\infty}\frac{4\sqrt{2t}}{4tx+2\sqrt{2t}}e^{-\frac{y^2}{2}}dy$$
$$= \frac{1}{\sqrt{2\pi}}e^{-\sqrt{2t}}\int_{-\infty}^{\infty}\frac{(4\sqrt{2t})e^{\frac{-y^2}{2}}}{[y^2+4\sqrt{2t}]^{\frac{1}{2}}[(y^2+4\sqrt{2t})^{\frac{1}{2}}+y]}dy$$
$$= \frac{1}{\sqrt{2\pi}}e^{-\sqrt{2t}}\int_{-\infty}^{\infty}\frac{4\sqrt{2t}[(y^2+4\sqrt{2t})^{\frac{1}{2}}-y}{[y^2+4\sqrt{2t}]^{\frac{1}{2}}4\sqrt{2t}}e^{\frac{-y^2}{2}}dy$$
$$= \frac{1}{\sqrt{2\pi}}e^{-\sqrt{2t}}\int_{-\infty}^{\infty}e^{\frac{-y^2}{2}}dy - \frac{1}{\sqrt{2\pi}}e^{-\sqrt{2t}}\int_{-\infty}^{\infty}\frac{y}{[y^2+4\sqrt{2t}]^{\frac{1}{2}}}e^{\frac{-y^2}{2}}dy$$
$$= e^{-\sqrt{2t}}, \tag{20}$$

where for evaluating the second equality on the right in (20), we have multiplied the integrand's numerator and denominator with $[(y^2+4\sqrt{2t})^{\frac{1}{2}}-y]$, the last equality following since $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{\frac{-y^2}{2}}dy = 1$ and the latter integral being identically equal to zero- since, as in the proof of Theorem 1, the integrand here is again an odd function. This proves Lemma 1. □

We now present the alternative proofs:

Proof of Theorem 1 based on Lemma 1. First note that, by Lemma 1, the Laplace transform of $X_{(\tau,\lambda)} = X_{(\tau)} + X_{(\lambda)}$, with $X_{(\tau)}$ and $X_{(\lambda)}$ independent, is given for $t \geq 0$ by

$$\varphi_{(\tau,\lambda)}(t) = E[e^{-t(X_{(\tau)}+X_{(\lambda)})}] = E[e^{-tX_{(\tau)}}]E[e^{-tX_{(\lambda)}}]$$
$$= e^{-\tau\sqrt{2t}}e^{-\lambda\sqrt{2t}} = e^{-(\tau+\lambda)\sqrt{2t}}, \tag{21}$$

and the Laplace transform of $X_{(\tau+\lambda)}$ evaluates by Lemma 1 to

$$\varphi_{(\tau+\lambda)}(t) = E[e^{-tX_{(\tau+\lambda)}}] = E[e^{-t(\tau+\lambda)^2X_{(1)}}] = e^{-(\tau+\lambda)\sqrt{2t}} \tag{22}$$

for $t \geq 0$. From (21) and (22), in view of Proposition 1, the conclusion of Theorem 1 follows, namely that $f_{(\tau)} * f_{(\lambda)} = f_{(\tau+\lambda)}$ for all $\tau, \lambda > 0$. The proof is complete. □

Proof of Theorem 2 based on Lemma 1. Let $X, X_1, X_2, \cdots, X_n$ be an i.i.d. sample from the distribution $F_{(1)}$. First note that the Laplace transform of $\sum_{i=1}^{n} X_i$ is given by

$$\varphi_n(t) = E\left(e^{-t\sum_{i=1}^{n}X_i}\right) = \prod_{i=1}^{n}E\left(e^{-tX_i}\right) = \varphi_1^n(t) = e^{-n\sqrt{2t}}, \tag{23}$$

the last equality following by Lemma 1. Also the Laplace transform of $Y_n = n^2 X_1$ by Lemma 1 is given by

$$\varphi_n^*(t) = E[e^{-tn^2 X}] = \varphi_1(n^2 t) = e^{-\sqrt{2n^2 t}} = e^{-n\sqrt{2t}}. \tag{24}$$

The proof of Theorem 2 now follows from (23) and (24) in view of Proposition 1. The proof is complete. □

4 Stable Distributions

For the sake of completion, and a better understanding of the relevant aspects of $F_{(1)}$ as a heavy tailed stable distribution, we present below two equivalent definitions and some basic propositions concerning stable distributions.

Definition 1. A non-degenerate distribution F is said to be stable (in the broad sense) if, for each positive integer n and any i.i.d. sample $X_1, X_2, \cdots, X_n$ and a r.v. X from F, there exist constants $c_n > 0, \gamma_n$ such that

$$S_n \stackrel{d}{=} c_n X + \gamma_n, \tag{25}$$

$n = 1, 2, \cdots$. F is said to be strictly stable if (25) holds with $\gamma_n = 0$. □

We now state some basic results concerning stable distributions. While a deeper discussion of Stability does need the use of advanced probability concepts and tools, for the discussion of some basic properties of stable distributions here, we shall confine ourselves to only elementary methodology and reasoning:

Proposition 2. If F is a stable distribution, whether in the broad or strict sense, according to Definition 1, then

(a) The norming constants $c_n, n = 1, 2, ...$, in (25) are of the form $c_n = n^{\frac{1}{\alpha}}$ for some $0 < \alpha \le 2$ (the constant $\alpha, 0 < \alpha \le 2$, is called the characteristic exponent (c.e.) of F);
(b) A stable distribution, by virtue of its definition, is a continuous one;
(c) If F is strictly stable with c.e. α (i.e., $\gamma_n = 0$, in Definition 1), then for any constants $s, t > 0$ and r.v.'s X, X_1, X_2 - with X_1 and X_2 independent - from F, we have

$$s^{\frac{1}{\alpha}} X_1 + t^{\frac{1}{\alpha}} X_2 \stackrel{d}{=} (s+t)^{\frac{1}{\alpha}} X. \tag{26}$$

Proof. For the proof of Proposition 2(a), we refer the reader to that of Theorem 1 on p.170 of Feller (1971)[5] where an excellent exposition is available. For the proofs of parts 2(b) and 2(c) , however, we shall add some elaborations (see Exercise 2 on p. 215 and Theorem 3 on p. 172 of this book). Towards this end, for given integers $m, n > 0$, let $X_i, i = 1, 2, ..., m, m+1, ...n, X^{(1)}, X^{(2)}$ and X be i.i.d. r.v.'s from a strictly stable distribution F. Then denoting $S_m = \sum_{i=1}^{m} X_i, S_n^{(m)} = \sum_{i=1}^{n} X_{m+i}$ and similarly S_{m+n}, and noting that $S_m + S_n^{(m)} = S_{m+n}$, we obtain from equation (25) of Definition 1 and Proposition 2(a) that $m^{\frac{1}{\alpha}} X^{(1)} + n^{\frac{1}{\alpha}} X^{(2)} \stackrel{d}{=} (m+n)^{\frac{1}{\alpha}} X$, or equivalently that

$$\left(\frac{m}{m+n}\right)^{\frac{1}{\alpha}} X^{(1)} + \left(\frac{n}{m+n}\right)^{\frac{1}{\alpha}} X^{(2)} \stackrel{d}{=} X. \tag{27}$$

We first prove Proposition 2(b): For this, we may assume wlog that F is symmetric. This is because the symmetrized distribution $\hat{F}$ - the distribution of $[X' - X'']$ when r.v.'s X', X'' are i.i.d. from F, which itself is evidently stable and symmetric and therefore strictly stable - is continuous if and only if F is continuous. Thus, the proof of continuity of a strictly stable distribution implies that of the symmetric $\hat{F}$, which in turn implies that of the (broadly) stable distribution F.

Suppose now that the (wlog assumed) symmetric F is not continuous and has an atom at an arbitrary point $t \neq 0$ with a positive (probability) weight $p(0 < p < 1)$. Then the LHS of equation (27) tells us that the point $t_{mn} = \left[\left(\frac{m}{m+n}\right)^{\frac{1}{\alpha}} + \left(\frac{n}{m+n}\right)^{\frac{1}{\alpha}}\right] \cdot t$ must also be an atom of F -the distribution of the RHS - with a (probability) weight $\ge P[X^{(1)} = t] \cdot P[X^{(2)} = t] = p^2$, for each pair $(m, n), m, n = 1, 2, \cdots$. This is impossible, since the total (probability) weight cannot exceed one. In case the assumed discontinuous F has only a unique atom at the origin $t = 0$ of (probability) weight $p > 0$, then the RHS and LHS of the same equation imply different (probability) weights p and p^2, respectively, at the origin, leading again to a contradiction. A stable distribution, thus, has to be a continuous one.

We elaborate now on the proof of part 2(c). From equation (27) above, which is valid for all strictly stable distributions F, it follows at once by dividing the equation with $\left[\frac{n}{(m+n)}\right]^{\frac{1}{\alpha}}$ on both sides that $\left(\frac{m}{n}\right)^{\frac{1}{\alpha}} X^{(1)} + X^{(2)} \stackrel{d}{=} \left[1 + \left(\frac{m}{n}\right)\right]^{\frac{1}{\alpha}} X$, or equivalently that $\left(\frac{s}{t}\right)^{\frac{1}{\alpha}} X^{(1)} + X^{(2)} \stackrel{d}{=} \left[1+\left(\frac{s}{t}\right)\right]^{\frac{1}{\alpha}} X$ for any real numbers $s,t > 0$ as long as $\left(\frac{s}{t}\right)$ is a rational number. Since the set of all rational numbers is dense on the real line, by coupling this fact with the continuity of F, it follows that the last equation and, therefore, the equation (26) of proposition 2(c) holds for all real $s,t,>0$ and all strictly stable F. The proof is complete. □

Remark 1. In view of Definition 1 of stability and Proposition 2(a), a non-degenerate distribution F is strictly stable with $\alpha(0<\alpha\leq 2)$ as c.e. if and only if, given an i.i.d. sample $\{X_1,X_2,...,X_n\}$ and a r.v. X from it, $S_n \stackrel{d}{=} n^{\frac{1}{\alpha}}X$ for all n. Upon comparing this last equation with the result of Theorem 2, viz. that for $F_{(1)}$, the preceding result holds as $S_n \stackrel{d}{=} n^2 X_1$ for $n = 1,2...$, the strictly stability of $F_{(1)}$ with c.e. $\alpha = \frac{1}{2}$ follows forthwith. □

We now state a Proposition dealing with conversion of "broad" to "strict" stability:

Proposition 3. If F is a stable distribution with c.e. α for some $0<\alpha\leq 2)$, then retaining the notations of Definition 1 and Proposition 2, it follows that

(a) When $\alpha \neq 1$, we can select a centering constant b such that the distribution $F(x-b), -\infty < x < \infty$, is strictly stable; or equivalently, that the equation (25) in Definition 1 of stability is satisfied with S_n, X, and γ_n in (25) replaced, respectively, with S'_n, X', and 0, where $S'_n = \sum_{j=1}^{n} X'_j$ with $X'_j = X_j + b$ and $X' = X + b$;

(b) When $\alpha = 1$, there exists a constant γ such that the following analogue of equation (26) in Proposition 2(c) holds, namely, that for all $s,t>0$

$$s(X_1+\gamma lns)+t(X_2+\gamma lnt) \stackrel{d}{=} (s+t)[X+\gamma ln(s+t)]. \tag{28}$$

Proof. For the proofs of Propositions 3(a) and 3(b), we refer the reader to Theorem 2 on p. 171 and Exercise 4 on p. 215, respectively, of Feller (1971)[5]. However, some elaboration for the proofs seems in order.

Let F be a stable distribution with c.e. $\alpha, 0<\alpha\leq 2$. Then if $S_{nm} = \sum_{j=1}^{n} S_{mj} = \sum_{j=1}^{n}\left(\sum_{i=1}^{m} X_{ij}\right)$, where r.v.'s $X_{ij}, i = 1,2,...,m$ and $j = 1,2,...,n$ are i.i.d. r.v.'s from F, it follows from equation (25) of Definition 1 that, if r.v.'s $X_1, X_2, \cdots, X_{m\vee n}, X$ are also i.i.d. from F, we have

$$S_{mn} \stackrel{d}{=} \sum_{j=1}^{n}(c_m X_j + \gamma_m) = c_m S_n + n\gamma_m \stackrel{d}{=} c_m c_n X + (c_m\gamma + n\gamma_m) = c_m c_n X + \gamma_{mn}(say), \tag{29}$$

where $\gamma_{mn} = c_m\gamma_n + n\gamma_m$. We first prove Proposition 3(a) and assume $\alpha \neq 1$. Then, interchanging the role of m and n in (29), we also obtain $S_{nm} \stackrel{d}{=} c_m c_n X + \gamma_{nm}$. Since $S_{nm} = S_{mn}$, it follows from comparing the two equations that $\gamma_{mn} = \gamma_{nm}$ or equivalently that $\left[\frac{\gamma_n}{c_n - n}\right] = \left[\frac{\gamma_m}{c_m - m}\right]$, implying that $\left[\frac{\gamma_n}{c_n - n}\right]$ does not depend on n and, therefore, equals a constant (say) b. This means that $\gamma_n = b(c_n - n)$, which transforms equation (25) to $S_n \stackrel{d}{=} c_n X + b(c_n - n)$, or equivalently to $S'_n \stackrel{d}{=} c_n X'$, where $S'_n = \sum_{i=1}^{n} X'$ with $X'_i = X_i + b$ and $X' = X + b$. Thus, the distribution $F(x-b), -\infty < x < \infty$, of $X'_i = X_i + b$ is strictly stable according to Definition 1. This proves 3(a).

To prove Proposition 3(b) for the case when $\alpha = 1$, note that equation $\gamma_{mn} = \gamma_{nm}$ is now an identity and is, therefore vacuous. However, we can still solve for the constant γ_n in equation (25) by solving the equation $\gamma_{mn} = m\gamma_n + n\gamma_m$ from (29) when $\alpha = 1$: Now note that by successively setting $m = n^k, k = 1,2,...,(\nu-1)$ in the preceding equation, we obtain that $\gamma_{n^2} = 2n\gamma_n, \gamma_{n^3} = n\gamma_{n^2} + n^2\gamma_n = 2n^2\gamma_n + n^2\gamma_n = 3n^2\gamma_n, ..., \gamma_{n^\nu} = \nu n^{\nu-1}\gamma_n$ for any integer $\nu > 0$. These considerations show that $\left(\frac{\gamma_{n^\nu}}{\nu n^\nu}\right)$ remains constant, equal to the same $p_n, -\infty < p_n < \infty$ (say), for all values $\nu = 1,2,\cdots$. The only solution for this constant is $p_n = \gamma lnn$ for some fixed $\gamma, -\infty < \gamma < \infty$, which also satisfies the equation $\gamma_{mn} = m\gamma_n + n\gamma_m$, or equivalently, $\left(\frac{\gamma_{mn}}{mn}\right) = \left[\left(\frac{\gamma_m}{m}\right) + \left(\frac{\gamma_n}{n}\right)\right]$. It follows that $\gamma_{n^\nu} = \gamma\nu n^\nu lnn = \gamma n^\nu lnn^\nu$ for all ν, so that $\gamma_n = \gamma n lnn$. This last equation transforms equation (25) in Definition 1 to

$$S_n \stackrel{d}{=} n(X+\gamma lnn). \tag{30}$$

The extension of the strict stability type equation (30) to equation (28) in Proposition 3(b) - the analogue of equation (26) in Proposition 2(c) for strictly stable distributions - can be achieved by its application as follows. For given integers $m,n>0$, we have in view of equation (30) that

$$S_{m+n} = (m+n)[X+\gamma ln(m+n)], \tag{31}$$

as well as

$$S_{m+n} = S_m + S_n \stackrel{d}{=} m(X_1 + \gamma lnm) + n(X_2 + \gamma lnn); \tag{32}$$

from (31) and (32), we obtain

$$m(X_1 + \gamma lnm) + n(X_2 + \gamma lnn) \stackrel{d}{=} (m+n)[X + \gamma ln(m+n)], \tag{33}$$

where the independent r.v.'s X_1, X_2, X and all those independent X's constituting S_{m+n} are distributed according to F. The same reasoning now as employed for Proposition 2(c) - namely, the continuity of stable distributions and that, in view of the preceding equation (33), the equation (28) of Proposition 3(b) holds whenever $\left(\frac{s}{t}\right)$ is rational - ensures that (28) holds for all real s, t that are positive. This completes the proof of Proposition 3(b). □

We now state the "equivalent" Definition 2 for stable distributions (cf. Feller (1971), Problem 1, p. 215)[5]. Its equivalence to Definition 1 can be established using the result of Proposition 2(c) above.

Definition 2. A non-degenerate distribution F is said to be stable if given two arbitrary positive constants c_1 and c_2, there exist constants $c > 0$ and γ such that for any r.v.'s X, X_1, X_2 - with X_1 and X_2 independent - from F,

$$c_1X_1 + C_2X_2 \stackrel{d}{=} cX + \gamma. \tag{34}$$

Proof of Equivalence of Definitions 1 and 2 of Stability. Suppose a non-degenerate distribution F is stable according to Definition 1 of stability with c.e. $\alpha, 0 < \alpha \leq 2$. To prove that it is also stable according to Definition 2, suppose first that $\alpha \neq 1$. By Proposition 3(a) then, there is a constant b such that the distribution $F(x-b), -\infty < x < \infty$, is strictly stable, so that given independent r.v.'s X_1, X_2, X from F and real numbers $s, t > 0$, we have by Proposition 2(c) that $s^{\frac{1}{\alpha}}(X_1 + b) + t^{\frac{1}{\alpha}}(X_2 + b) \stackrel{d}{=} (s+t)^{\frac{1}{\alpha}}(X + b)$, or equivalently, that

$$s^{\frac{1}{\alpha}}X_1 + t^{\frac{1}{\alpha}}X_2 \stackrel{d}{=} (s+t)^{\frac{1}{\alpha}}X + b[(s+t)^{\frac{1}{\alpha}} - s^{\frac{1}{\alpha}} - t^{\frac{1}{\alpha}}]. \tag{35}$$

To show that equation (35) implies that equation (34) of Definition 2 is also satisfied for given constants $c_1, c_2 > 0$ and some constants $c > 0$ and $\gamma, -\infty < \gamma < \infty$, just set $c_1 = s^{\frac{1}{\alpha}}$ and $c_2 = t^{\frac{1}{\alpha}}$ in equation (35). It follows then forthwith that equation (35) transforms into equation (34) for given c_1, c_2 with $c = (c_1^{\alpha} + c_2^{\alpha})^{\frac{1}{\alpha}}$ and $\gamma = b[(c_1^{\alpha} + c_2^{\alpha})^{\frac{1}{\alpha}} - c_1 - c_2]$. Similarly when $\alpha = 1$, it follows directly from Proposition 3(b), equation (28) that, for given $c_1, c_2 > 0$ some real γ',

$$c_1X_1 + c_2X_2 \stackrel{d}{=} (c_1 + c_2)X + \gamma'[(c_1 + c_2)ln(c_1 + c_2) - c_1lnc_1 - c_2lnc_2] \stackrel{d}{=} cX + \gamma(say). \tag{36}$$

From equations (35) and (36), it follows that F satisfies equation (34) of Definition 2 of stability. To prove the converse, suppose that Definition 2 of stability holds for the distribution F. Then, given an i.i.d. random sample $\{X_1, X_2, ..., X_n\}$ from F, we have from equation (34) that

$$X_1 + X_2 \stackrel{d}{=} c_{(2)}X_{(2)} + \gamma_{(2)}, \text{ and further that}$$

$$X_1 + X_2 + X_3 = [c_{(2)}X_{(2)} + X_{(3)}] + \gamma_{(2)} \stackrel{d}{=} c_{(3)}X_{(3)} + (\gamma_{(2)} + \gamma_{(3)}), \tag{37}$$

for some constants $c_{(2)}, \gamma_{(2)}, c_{(3)}, \gamma_{(3)}$ and r.v.'s $X_{(2)}, X_{(3)}$, independent of the random sample $\{X_1, X_2, ..., X_n\}$. To establish the required equation (25), we use the induction argument: Assume that, for some n, the equation $S_{n-1} = c_{(n-1)}X_{(n-1)} + \sum_{j=2}^{n-1} \gamma_{(j)}$ holds for some constants $c_{(n-1)}$ and γ_j's, as in equation (37) for $n = 3, 4$ above. Then, using equation (34) of Definition 2 again, we obtain

$$S_n = S_{n-1} + X_n = [c_{(n-1)}X_{(n-1)} + X_n] + \sum_{j=1}^{n-1} \gamma_{(j)} \stackrel{d}{=} c_{(n)}X_{(n)} + \sum_{j=1}^{n} \gamma_{(j)} = c_{(n)}X_{(n)} + \gamma_{(n)}, \tag{38}$$

for some constants $c_{(n)}, \gamma_{(n)} - \sum_{j=1}^{n} \gamma_{(j)}$, and r.v. $X_{(n)}$ distributed according to F. The equations (37), (38) and the induction argument establish the requirement (25) of Definition 1 for F. Thus Definition 2 implies Definition 1 of stability. The proof of equivalence of Definitions 1 and 2 of stability is complete. □

Remark 2. Under the seemingly stronger Definition 2 of stability (although equivalent to Definition 1 above), the stability of $F_{(1)}$ follows even more readily from the "convolution" property of Theorem 1. To see this, set $\tau = \sqrt{c_1}$ and $\lambda = \sqrt{c_2}$, so that the densities $f_{(\tau)}$ and $f_{(\lambda)}$ on $(0,\infty)$ are (see the notations of Theorem 1 and Theorem 2) those of r.v.'s $X_{(\tau)} = \left(\frac{\tau^2}{Z_1^2}\right) = (\tau^2 X_1) = c_1 X_1$ and $X_{(\tau)} = \left(\frac{\lambda^2}{Z_2^2}\right) = (\lambda^2 X_2) = c_2 X_2$, where $X_1 = \left(\frac{1}{Z_1^2}\right)$ and $X_2 = \left(\frac{1}{Z_2^2}\right)$ with Z_1 and Z_2 independent $N(0,1)$ r.v.'s. By Theorem 1, since $f_{(\tau)} * f_{(\lambda)} = f_{(\tau+\lambda)}$, it follows that

$$c_1X_1 + c_2X_2 = \left(\frac{\tau^2}{Z_1^2}\right) + \left(\frac{\lambda^2}{Z_2^2}\right) \stackrel{d}{=} \left(\frac{(\tau+\lambda)^2}{Z^2}\right) = cX, \tag{39}$$

where $c = \left(\sqrt{c_1} + \sqrt{c_2}\right)^2$ and $X \stackrel{d}{=} \left(\frac{1}{Z^2}\right)$ with Z a $N(0,1)$ r.v.. Since the constant γ of equation (34) is zero in (39), the distribution $F_{(1)}$ is strictly stable. □

Stability and Infinite Divisibility. As mentioned above, a thorough discussion of stability of distributions requires other probabilistic concepts and tools, such as domain of attraction, infinite divisibility, slow and regular variation of functions, Fourier transforms, convolution semi-groups, and so forth. Nevertheless, we state below (without proof) a proposition, involving the first two, that throws further light on the concept of stable distributions.

Definition 1*.A distribution G belongs to the *Domain of Attraction* of a (non-degenerate) distribution F if and only if there exist constants $a_n > 0$ and b_n such that, based on an i.i.d. sample from G, $a_n^{-1}(S_n - b_n) \xrightarrow{d} F$, as $n \to \infty$.

Definition 2*. A (non-degenerate) distribution F is said to be Infinitely Divisible if and only if for each positive integer n, there exists a distribution F_n such that, based on an i.i.d. sample of size n from F_n, we always have $S_n \stackrel{d}{=} F, n = 1,2,3....$

Proposition 4. (a) A distribution G belongs to the "domain of attraction" of some (non-degenerate) distribution F if and only if, for some index $\alpha, 0 < \alpha \le 2$, and $p,q \ge 0$ with $p+q = 1$,

$$\frac{x^2[1-G(x)]}{\mu(x)} \to p\left(\frac{2-\alpha}{\alpha}\right) and \frac{x^2G(-x)}{\mu(x)} \to q\left(\frac{2-\alpha}{\alpha}\right), \tag{40}$$

as $x \to \infty$, where $\mu(x) = \int_{-x}^{x} t^2 dG(t), x > 0$, is the truncated moment function of G; (b) A (non-degenerate) distribution F possesses a 'domain of attraction' if and only if it is stable. A stable F belongs to its own domain of attraction; (c) The class of stable distributions $\{F = F_\alpha : 0 < \alpha \le 2\}$ coincides with the class of all infinitely divisible distributions that are limits of normed sums $\left[\frac{(S_n - b_n)}{b_n}\right]$ (defined in Deffiniton 1*), as $n \to \infty$, and for which the limiting member $F = F_\alpha$ corresponding to (40), besides (40), also satisfies the tail conditions

$$x^\alpha[1-F(x)] \to cp\left[\frac{(2-\alpha)}{\alpha}\right] and\, x^\alpha F(-x) \to cq\left[\frac{(2-\alpha)}{\alpha}\right], \tag{41}$$

as $x \to \infty$. The conditions (40) and (41) determine the stable distribution F_α uniquely, but only up to arbitrary centering and scale parameters; (d) For any distribution G belonging to a 'Domain of Attraction' with index $\alpha, 0 < \alpha \le 2$, all absolute moments m_β of order $\beta < \alpha$ exist, whereas if $\alpha < 2$, no moment of order $\beta > \alpha$ exists. □

Proof. For proofs, the reader is referred to Feller (1971)[5], to Section IX 8, Theorem 1 on pp 312-315 for Proposition 1* (a) and Section XVII 5, Theorem 1 on pp 576-577 for Proposition 1* (b) and (c), and the Lemma on p. 578 for 1*(d) . For Proposition 1* (a) and (c) note that if distributions F_α or G in there have a finite variance, then by the CLT the F must be normal with $\alpha = 2$, so that in this case the limits on the RHS of (40) and (41) reduce to zero. □

5 Concluding Remarks

In Section 3 above, we have presented a direct proof of the "convolution" property for the family $\{f_{(\tau)} : \tau > 0\}$ of heavy-tailed stable densities defined by equation (4). The family contains the density $f_{(1)}$ given by equation (1), especially important in applications. The mean and variance for members of this family, which differ from each other only in scale, do not exist; so the conditions required for Central Limit Theorem are not satisfied for the members of this family. Theorem 2 above points out that the sample mean $\bar{X}_n$ from distribution $f_{(1)}$, or for that matter from any distribution $f_{(\tau)}$ in the family, has the same distribution as that of n times a single observation from it (Feller 1971, p.52; Romano and Siegel 1986, pp59-60)[5, 6].

Clearly, $\bar{X}_n$ is more variable than a single observation X_1 and increases by an order of n, instead of converging in distribution to a limiting random variable, that is, certainly not to a normal distribution, as $n \to \infty$. It should be noted that the density (1) corresponds to an important class of densities in applications. It is the density of first passage times in a one-dimensional Brownian motion. It is also the limiting density of normalized average $\left[\frac{\bar{X}_n}{n}\right]$ of waiting times $X_1, X_2, ..., X_n$

of successive returns to the origin in a symmetric random walk. The densities (4) differing from each other only in scale are typical of limiting densities, without expectation, of such time averages of recurrence of events in many physical and economic processes (Feller 1968, p. 90, 246)[4].

Besides Lèvy's distribution $F_{(1)}$, the distribution of first passage times in one-dimensional Brownian Motion with c.e. $\alpha = \frac{1}{2}$, there are many other important stable distributions which arise naturally in applications, like Cauchy (c.e. $\alpha = 1$), Normal (c.e. $\alpha = 2$), and Holtsmark's (c.e. $\alpha = \frac{3}{2}$) etc, the last distribution being that of X_λ, the random "x - component of the gravitational force of a stellar system with density λ". For a thorough study of stable distributions - beyond the basic results presented in section 4 above - the reader is referred to Sections VI 1-4, VIII 1-4, IX 1-6, 8, XIII 4-7 and XVII 1-6 of Feller (1971)[5] among others. □

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References

[1] Bagui, S.C., Bhaumilk, D.K., and Mehra, K.L. (2013). A few counter examples useful in teaching central limit theorems, *The American Statistician* **67**(1), 49-56.
[2] Casella, G. and Berger, R. (2002). Statical Inference (2nd ed.), Pacific Grove, CA: Duxbury.
[3] Johnson, N., Kotz, S., and Balakrishnam, N. (1995). *Continuous Univariate Distributions* (2nd ed., Vol.),
[4] Feller W. (1968). *An Introduction to Probability and Its Applications* (Vol. I), New York:Wiley.
[5] Feller W. (1971). *An Introduction to Probability and Its Applications* (Vol. II), New York:Wiley.
[6] Romano, R.J. (1986). *Counter Examples in Probability and Statistics*, Monterery, Ca:Wadsworth and Brooks/Cole.

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