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Oscillatory features of nonlinear advanced differential equations

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Abstract: In this paper, we rely on some different techniques to study the oscillatory behavior of advanced second-order nonlinear differential equations with multiple deviating arguments. Three oscillation theorems were provided for all solutions of the studied equation by applying the comparison technique with equations of the same order and also using the famous Riccati technique. The results obtained improve and generalize some previous work, as they take into account the effect of deviating arguments on oscillatory behavior, which was previously neglected. Our results can also be applied to more than one type of differential equation, such as linear and ordinary. Finally, the originality and scientific contribution of our results were discussed, and examples were provided that confirmed their applicability and coverage of a broader scope compared to the previous ones.

Keywords: Emden-Fowler equation; neutral differential equation; the canonical case; oscillation.

1 Introduction

Many real-world problems where the evolution rate depends on both the present and the future can benefit from the use of advanced differential equations (ADEs). Therefore, an advance that is currently available and helpful in the decision-making process could be added to the equation to account for the influence of possible future actions. Examples of typical disciplines where such phenomena are believed to occur are population dynamics, economic issues, and mechanical control engineering (see [38,39] for details).

Oscillation theory, which examines the oscillatory and nonoscillatory behavior of DE solution, is one of the most significant subfields of the qualitative theory of DEs. Most researchers in this field are aware that Sturm [41], who developed his renowned technique for inferring the oscillatory properties of solutions of a specific DE from those known to another equation, deserves the credit for the emergence of the theory of oscillation of DEs of light in 1836. After that, Kneser [26] finished his research in this area in 1893 and came up with the kind of answers that bear his name to this day. Fite [20] published the first findings involving the oscillation of DEs with divergent arguments in 1921. In order to further the subject of knowledge, a great quantity of study has been conducted since then. For a summary of the most significant contributions, we recommend reading the monographs by Agarwal et al. [2,3,4], Dosly and Rehak [13], and Gyori and Ladas [22].

In this work, we consider the following second-order nonlinear ADE with multiple deviating arguments:

$$\left(r(t)\left(y'(t)\right)^{\alpha}\right)' + \sum_{i=1}^{n} q_i(t)\, F\left(y\left(h_i(t)\right)\right) = 0, \qquad (1)$$

where $t \in [t_0, \infty)$. We assume the following conditions on the functional coefficient of this equation:

(H_1) $n \in \mathbb{Z}^+$, and $\alpha \in \mathbb{Q}$ is a quotient of odd numbers;
(H_2) $r,\ q_i \in C\left([t_0,\infty), \mathbb{R}^+\right)$ for $i = 1,2,\ldots,n$;
(H_3) $h_i \in C\left([t_0,\infty), \mathbb{R}\right)$ and $h_i(t) \geq t$ for all $i = 1,2,\ldots,n$;
(H_4) $F \in C(\mathbb{R},\mathbb{R})$ and there is a $K > 0$ such that $F(u)/u^{\alpha} \geq K$ for $u \neq 0$.

Under a solution of (1), we mean a function $x \in C([t_a,\infty),\mathbb{R})$ with $t_a \geq t_0$, with the property $r(x')^{\alpha} \in C^1([t_a,\infty),\mathbb{R})$ and satisfies (1) on $[t_a,\infty)$. We consider only those solutions of (1) which satisfy the

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condition $\sup\{|x(t)| : t \geq t_*\} > 0$, for all $t_* \geq t_0$. A solution x of (1) is said to be oscillatory if it is neither eventually positive nor eventually negative; otherwise, it is said to be nonoscillatory. The equation itself is termed oscillatory if all its solutions oscillate. We shall say that (1) is in canonical form if

$$\int_{t_0}^{\infty} r^{-1/\alpha}(s)\mathrm{d}s = \infty. \tag{2}$$

Next, we review some previous results in the literature that dealt with the issue of oscillation of solutions of functional differential equations (delay, advance, and neutral).

Oscillation theory is one of the most important branches of qualitative theory and has undergone many developments. Among the preliminary works that dealt with the oscillations of second-order functional equations [4,19,23,42].

Using a comparison approach, Kusano [27] studied the oscillation of linear ADE

$$(r(t)x'(t))' + q(t)x(h(t)) = 0. \tag{3}$$

Džurina [15] linked the oscillation of (3) to the oscillation of the solutions of the ordinary equation

$$(r(t)x'(t))' + \left(\frac{R(h(t))}{R(t)}\right)^{\alpha_1} q(t)x(t) = 0,$$

where $\alpha_1 > 0$,

$$R(t) := \int_{t_0}^{\infty} r^{-1}(s)\mathrm{d}s,$$

and

$$R(t)\int_{t}^{\infty} q(s)\,\mathrm{d}s \geq \alpha_1.$$

Using some iterative techniques, Baculíková [7] and Jadlovská [24] established criteria for the linear ADE

$$x''(t) + q(t)x(h(t)) = 0.$$

For non-canonical case, when,

$$\int_{t_0}^{\infty} r^{-1/\alpha}(s)\mathrm{d}s < \infty,$$

Bohner et al. [11] presented a sufficient condition for (3) based on rewriting it in canonical form. On the other hand, Džurina [14] investigated the oscillatory properties of the ADE

$$\left(\frac{1}{a(t)}x'(t)\right)' + p(t)x(t) + q(t)x(h(t)) = 0,$$

in the canonical case

$$\int_{t_0}^{\infty} a(s)\mathrm{d}s = \infty.$$

In the last decade, there have also been many interesting results discussing the oscillation of advanced differential equations, see [5,12,25,28]. On the other hand, the study of oscillation of solutions of delay differential equations has greatly developed. We find that works [6,8,17,40] presented improved oscillation criteria for delay differential equations in several methods. This development was extended to neutral equations through papers [10,16,21,33,36,43]. The studies also extended to equations of higher orders, such as [1,9,18,31,34] (for even order) and [30,32,35,37] (for odd order).

2 Auxiliary Lemmas

In this section we state the preliminary lemmas besides an auxiliary lemma that we rely on to obtain our main results. But, first lets define the following notations:

$$\eta(t) := \int_{t_0}^{t} r^{-1/\alpha}(v)\mathrm{d}v$$

and

$$h(t) = \min\{h_i(t) : i = 1,2,\ldots,n\}. \tag{4}$$

Lemma 1. *[29, Lemma 1.2] Suppose that L and M are real constants, $L > 0$. Then,*

$$Ms - Ls^{(\alpha+1)/\alpha} \leq \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}}\frac{M^{\alpha+1}}{L^\alpha}. \tag{5}$$

To study the oscillation of any differential equation we need first to classify the positive solutions and based on this classification we began to choose the proper methodology to exclude any positive solution of the studied equation which in turn ensures the oscillation of this equation. Therefore we derived the following Lemma:

Lemma 2. *Assume that y be an eventually positive solution of (1), then the is only one possible case for classifying y as:*

$$y'(t) > 0 \text{ and } \left(r(t)\left(y'(t)\right)^\alpha\right)' < 0 \tag{6}$$

eventually.

Proof. Assume that y be an eventually positive solution of (1) for $t_1 \in [t_0,\infty)$ where $y(h(t)) > 0$ for all $t_1 \leq t < \infty$. Equation (1) denotes that

$$\left(r(t)\left(y'(t)\right)^\alpha\right)' = -\sum_{i=1}^{n} q_i(t)F(y(h_i(t))) < 0.$$

I.e.,

$$\left(r(t)\left(y'(t)\right)^\alpha\right)' < 0.$$

Now, we assert the remaining part by letting on the contrary that $y'(t) < 0$ for $t_2 \geq t_1$. Since $r(t)(y'(t))^\alpha$ is decreasing, then

$$r(t)\left(y'(t)\right)^\alpha \leq r(t_2)\left(y'(t_2)\right)^\alpha := -c^2$$

for $t \in [t_2, \infty)$ and c is any positive constant. Integrating the above inequality from t_2 to t , we get

$$y(t) \leq y(t_2) - c^{2/\alpha} \int_{t_2}^{t} \frac{\mathrm{d}v}{r^{1/\alpha}(v)}.$$

But, taking the limit for both sides as $t \to \infty$ and using (2) yields a contradiction. The proof is complete.

Lemma 3. *Assume that y be an eventually positive solution of (1). Let $\mu_0 = 0$ and there is a natural number m with $\mu_j > 0, j = 1, 2, ..., m$ such that*

$$\alpha\eta(t)\left(\int_t^\infty \left(\frac{\eta(h(v))}{\eta(v)}\right)^{\mu_{j-1}} \sum_{i=1}^{n} q_i(v)\,\mathrm{d}v\right)^{1/\alpha} \geq \mu_j, \tag{7}$$

then

$$\left(\frac{y^\alpha(t)}{\eta^{\mu_j}(t)}\right)' \geq 0 \tag{8}$$

eventually.

Proof. Assume that y be an eventually positive solution of (1) for $t_1 \in [t_0, \infty)$. From (1), (4), and (6) we obtain that

$$\begin{aligned} r(t)\left(y'(t)\right)^\alpha &\geq \int_t^\infty \sum_{i=1}^{n} q_i(v)\, y^\alpha(h_i(v))\,\mathrm{d}v \\ &\geq \int_t^\infty y^\alpha(h(v)) \sum_{i=1}^{n} q_i(v)\,\mathrm{d}v \\ &\geq \int_t^\infty y^\alpha(v) \sum_{i=1}^{n} q_i(v)\,\mathrm{d}v \\ &\geq y^\alpha(t) \int_t^\infty \sum_{i=1}^{n} q_i(v)\,\mathrm{d}v, \end{aligned} \tag{9}$$

for t large enough. Therefore,

$$\begin{aligned} \left(\frac{y^\alpha(t)}{\eta^{\mu_1}(t)}\right)' &= \frac{1}{\eta^{\mu_1}(t)} \alpha (y(t))^{\alpha-1} y'(t) \\ &\quad - \mu_1 \frac{1}{\eta^{\mu_1+1}(t)} \frac{1}{r^{1/\alpha}(t)} (y(t))^\alpha \\ &= \frac{1}{r^{1/\alpha}(t)\,\eta^{\mu_1+1}(t)} \\ &\quad \times \left[\alpha\eta(t) \frac{r^{1/\alpha}(t)}{(y(t))^{1-\alpha}} y'(t) - \mu_1 (y(t))^\alpha\right]. \end{aligned}$$

But (9) implies that

$$\left(\frac{y^\alpha(t)}{\eta^{\mu_1}(t)}\right)' \geq \frac{(y(t))^\alpha}{r^{1/\alpha}(t)\,\eta^{\mu_1+1}(t)} \left[\alpha\eta(t)\left(\int_t^\infty \sum_{i=1}^{n} q_i(v)\,\mathrm{d}v\right)^{1/\alpha} - \mu_1\right],$$

which is in the light of (7) is positive, i.e.,

$$\left(\frac{y^\alpha(t)}{\eta^{\mu_1}(t)}\right)' \geq 0.$$

This completes the proof of (8) for $j = 1$. Now, we will rely on the mathematical induction to complete the rest of the proof. So, let (8) holds for $j = k \in \mathbb{N}$, which means that

$$\left(\frac{y^\alpha(t)}{\eta^{\mu_k}(t)}\right)' \geq 0.$$

eventually. And so,

$$y^\alpha(t)\left(\frac{\eta(h(t))}{\eta(t)}\right)^{\mu_k} \leq y^\alpha(h(t)).$$

Equation (1) and (4) denotes that

$$\begin{aligned} r(t)\left(y'(t)\right)^\alpha &\geq \int_t^\infty y^\alpha(h(v)) \sum_{i=1}^{n} q_i(v)\,\mathrm{d}v \\ &\geq \int_t^\infty \left(\frac{\eta(h(v))}{\eta(v)}\right)^{\mu_k} y^\alpha(v) \sum_{i=1}^{n} q_i(v)\,\mathrm{d}v \\ &\geq y^\alpha(t) \int_t^\infty \left(\frac{\eta(h(v))}{\eta(v)}\right)^{\mu_k} \sum_{i=1}^{n} q_i(v)\,\mathrm{d}v. \end{aligned}$$

On the other hand,

$$\begin{aligned} \left(\frac{y^\alpha(t)}{\eta^{\mu_{k+1}}(t)}\right)' &= \frac{1}{\eta^{2\mu_{k+1}}(t)} \left[\alpha\eta^{\mu_{k+1}}(t) \frac{y'(t)}{(y(t))^{1-\alpha}} \right. \\ &\quad \left. - \mu_{k+1} \frac{\eta^{\mu_{k+1}-1}(t)}{r^{1/\alpha}(t)} (y(t))^\alpha\right] \\ &= \frac{r^{-1/\alpha}(t)}{\eta^{\mu_{k+1}+1}(t)} \left[\alpha\eta(t)\, r^{1/\alpha}(t) (y(t))^{\alpha-1} y'(t) \right. \\ &\quad \left. - \mu_{k+1} (y(t))^\alpha\right] \\ &\geq \frac{(y(t))^\alpha}{r^{1/\alpha}(t)\,\eta^{\mu_{k+1}+1}(t)} \left[A(t) - \mu_{k+1}\right] \\ &\geq 0, \end{aligned}$$

where

$$A(t) = \alpha\eta(t)\left(\int_t^\infty \left(\frac{\eta(h(v))}{\eta(v)}\right)^{\mu_k} \sum_{i=1}^{n} q_i(v)\,\mathrm{d}v\right)^{1/\alpha}.$$

This completes the proof.

3 Oscillation Theorems

In the subsequent theorems, we assume that all of the improper integrals involved are convergent.

Theorem 1. *Let $\mu_0 = 0$ and there is a natural number m with $\mu_j > 0$ such that (7) holds. If the following differential equation*

$$\left(r(t)\left(x'(t)\right)^\alpha\right)' + \left(\frac{\eta(h(t))}{\eta(t)}\right)^{\mu_j} q(t) F(x(t)) = 0 \tag{10}$$

is oscillatory for , $j = 1, 2, ..., m$. Then, (1) is oscillatory.

Proof. Assume that y be an eventually positive solution of (1) for $t_1 \in [t_0,\infty)$. From the increasing monotonicity of $y^{\alpha}(t)/\eta^{\mu_j}(t)$, we get

$$y^{\alpha}(h(t)) \geq \left(\frac{\eta(h(t))}{\eta(t)}\right)^{\mu_j} y^{\alpha}(t).$$

Substituting into (1) yields

$$\left(r(t)\left(y'(t)\right)^{\alpha}\right)' + \left(\frac{\eta(h(t))}{\eta(t)}\right)^{\mu_j} y^{\alpha}(t)\sum_{i=1}^{n} q_i(t) \leq 0.$$

for $j = 1,2,\ldots,m$. By integrating the above inequality from t to ∞, we obtain

$$y'(t) \geq \left(\frac{1}{r(t)}\int_t^{\infty}\left(\frac{\eta(h(v))}{\eta(v)}\right)^{\mu_j} y^{\alpha}(v)\sum_{i=1}^{n} q_i(v)\,\mathrm{d}v\right)^{1/\alpha}.$$

Once more, integrate the above inequality from t_0 to t, we have

$$y(t) \geq y(t_0) + \int_{t_0}^{t}\left(\frac{1}{r(u)}\int_t^{\infty}\left(\frac{\eta(h(v))}{\eta(v)}\right)^{\mu_j} y^{\alpha}(v)\sum_{i=1}^{n} q_i(v)\,\mathrm{d}v\right)^{1/\alpha}\mathrm{d}u.$$

Furthermore, lets define the sequence $\{\omega_l(t)\}_{l\in\mathbb{N}_0}$ by

$$\omega_0 = y(t)$$

$$\omega_{l+1}(t) = y(t_0) + \int_{t_0}^{t}\left(\frac{1}{r(u)}\int_t^{\infty}\left(\frac{\eta(h(v))}{\eta(v)}\right)^{\mu_j} q(v)\,\omega_l^{\alpha}(v)\,\mathrm{d}v\right)^{1/\alpha}\mathrm{d}u.$$

The mathematical induction indicates that the sequence $\{\omega_l(t)\}$ is decreasing and

$$y^{\alpha}(t_0) \leq \omega_l(t) \leq y^{\alpha}(t),$$

for $l \in \mathbb{N}_0$. As a result, there is a function $\omega(t)$ satisfies that for $t \in [t_0,\infty)$

$$\lim_{l\longrightarrow\infty}\omega_l(t) = \omega(t)$$

and

$$y^{\alpha}(t_0) \leq \omega_l(t) \leq y^{\alpha}(t).$$

Following Lebesgue's dominated convergence theorem, we obtain that

$$\omega_l(t) = y(t_0) + \int_{t_0}^{t}\left(\frac{1}{r(u)}\int_t^{\infty}\left(\frac{\eta(h(v))}{\eta(v)}\right)^{\mu_j} \omega_l^{\alpha}(v)\sum_{i=1}^{n} q_i(v)\,\mathrm{d}v\right)^{1/\alpha}\mathrm{d}u.$$

Getting the second derivative of the above inequality implies that the function $\omega(t)$ considered as a positive solution of (10). Which contradicts with our oscillation assumption of this equation, and this completes the proof.

Theorem 2. *Suppose that $\mu_0 = 0$ and there exist $m \in N$ and $\mu_j > 0$, for $j = 1,2,\ldots,m$, such that (7) holds. If there is a $\rho \in C([t_0,\infty),\mathbb{R}^+)$ such that*

$$\limsup_{t\to\infty}\int_{t_1}^{t}\left(B(v) - \frac{r(v)(\rho'(v))^{\alpha+1}}{(\alpha+1)^{\alpha+1}\rho^{\alpha}(v)}\right)\mathrm{d}v = \infty, \qquad (11)$$

for $t_1 \geq t_0$, then equation (1) is oscillatory, where

$$B(v) := K\rho(v)\sum_{i=1}^{n} q_i(v)\frac{\eta^{\mu_j}(h_i(v))}{\eta^{\mu_j}(v)}$$

Proof. Suppose the contrary that (1) has an eventually positive solution y. Then, there is a $t_1 \geq t_0$ such that $y(t) > 0$ and $y(h_i(t)) > 0$ for $t \geq t_1$ and $i = 1,2,\ldots,n$.

Using (H_4), equation (1) becomes

$$\left(r(t)\left[y'(t)\right]^{\alpha}\right)' = -\sum_{i=1}^{n} q_i(t)F(y(h_i(t))) \leq -K\sum_{i=1}^{n} q_i(t)y^{\alpha}(h_i(t)). \qquad (12)$$

Now, we define the function

$$Z := \rho\cdot r\cdot\left[\frac{y'}{y}\right]^{\alpha}.$$

Hence, $Z(t) > 0$, for $t \geq t_1$, and

$$\begin{aligned} Z' &= \frac{\rho'}{\rho}Z + \rho\left[\frac{\left(r\cdot(y')^{\alpha}\right)'}{y^{\alpha}} - \frac{r\cdot(y')^{\alpha}}{y^{2\alpha}}\alpha y^{\alpha-1}y'\right] \\ &= \frac{\rho'}{\rho}Z + \rho\frac{\left(r\cdot(y')^{\alpha}\right)'}{y^{\alpha}} - \alpha\rho\frac{r\cdot(y')^{\alpha+1}}{y^{\alpha+1}} \\ &= \frac{\rho'}{\rho}Z + \rho\frac{\left(r\cdot(y')^{\alpha}\right)'}{y^{\alpha}} - \alpha\frac{Z^{1+1/\alpha}}{\rho^{1/\alpha}\cdot r^{1/\alpha}}. \end{aligned}$$

From (12), we obtain

$$Z' \leq \frac{\rho'}{\rho}Z - K\rho\sum_{i=1}^{n} q_i\frac{(y^{\alpha}\circ h_i)}{y^{\alpha}} - \alpha\frac{Z^{1+1/\alpha}}{\rho^{1/\alpha}\cdot r^{1/\alpha}}. \qquad (13)$$

Using Lemma 3, we arrive at

$$\frac{y\circ h_i}{y} \geq \frac{\eta^{\mu_j}\circ h_i}{\eta^{\mu_j}},$$

which with (13) gives

$$Z' \leq -K\rho\sum_{i=1}^{n} q_i\frac{\eta^{\mu_j}\circ h_i}{\eta^{\mu_j}} + \frac{\rho'}{\rho}Z - \alpha\frac{Z^{1+1/\alpha}}{\rho^{1/\alpha}\cdot r^{1/\alpha}}. \qquad (14)$$

Using Lemma 1, with

$$M = \frac{\rho'}{\rho},\; L = \frac{\alpha}{\rho^{1/\alpha}\cdot r^{1/\alpha}},$$

and $s = Z$, we get

$$\frac{\rho'}{\rho}Z - \alpha\frac{Z^{1+1/\alpha}}{\rho^{1/\alpha}\cdot r^{1/\alpha}} \le \frac{1}{(\alpha+1)^{\alpha+1}}\frac{r(\rho')^{\alpha+1}}{\rho^{\alpha}}.$$

Therefore, (14) reduces to

$$Z' \le -K\rho\sum_{i=1}^{n} q_i\frac{\eta^{\mu_j}\circ h_i}{\eta^{\mu_j}} + \frac{1}{(\alpha+1)^{\alpha+1}}\frac{r(\rho')^{\alpha+1}}{\rho^{\alpha}}.$$

By integrating this inequality from t_1 to t, we obtain

$$\int_{t_1}^{t}\left(B(v) - \frac{r(v)(\rho'(v))^{\alpha+1}}{(\alpha+1)^{\alpha+1}\rho^{\alpha}(v)}\right)dv \le Z(t_1),$$

which contradicts (11).

Theorem 3.*Let* $\mu_0 = 0$ *and there is a with* $\mu_j > 0$ *such that (7) holds and*

$$\mu_m > \frac{\alpha}{(1+\alpha)^{(1+\alpha)/\alpha}}, \tag{15}$$

$j = 1,2,\ldots,m$. *Then, every solution of (1) oscillates.*

*Proof.*Assume that the natural number m is the greater number satisfies (15). Otherwise, one is the essential number for the validity of (15). Inequality (7) and (15) implies that

$$\alpha\eta(t)\left(\int_t^{\infty}\left(\frac{\eta(h(v))}{\eta(v)}\right)^{\mu_{j-1}}\sum_{i=1}^{n} q_i(v)\,dv\right)^{1/\alpha} \ge \mu_m,$$

for $\frac{\alpha}{(1+\alpha)^{(1+\alpha)/\alpha}} < \mu_m < \infty$. By replacing $\left(\frac{\eta(h(t))}{\eta(t)}\right)^{\alpha\mu_j}\sum_{i=1}^{n} q_i(t)$ by $\sum_{i=1}^{n} q_i(t)$ in (10) in Theorem 1, the proof becomes clear, and so we omit it.

*Example 1.*Consider the following functional differential equation with multiple deviating arguments:

$$\left(\frac{\alpha^{\alpha}}{(1+\alpha)^{1+\alpha}}t^{-1}\left(y'(t)\right)^{\alpha}\right)' + \frac{q_0}{t^{\alpha+2}}\sum_{i=1}^{n} y^{\alpha}(\varepsilon_i t) = 0, \tag{16}$$

for all $t \in (0,\infty)$, $q_0 \in (0,\infty)$, $\varepsilon_i \in [1,\infty)$, and $i = 1,2,\ldots n, n \in \mathbb{N}$. It is clear that the assumptions $(\mathrm{H}_1 - \mathrm{H}_4)$ holds eventually with

$$\eta(t) = (1+\alpha)^{\frac{1}{\alpha}}t^{1+1/\alpha}$$

approaches to ∞ as t approaches to ∞. Moreover, we define

$$\varepsilon = \min\{\varepsilon_i : i = 1,2,\ldots,n\}.$$

Now, by applying (7), we obtain

$$\alpha\eta(t)\left(\int_t^{\infty}\left(\frac{\eta(h(v))}{\eta(v)}\right)^{\mu_{j-1}}q(v)\,dv\right)^{1/\alpha}$$
$$= \alpha(1+\alpha)^{\frac{1}{\alpha}}t^{1+1/\alpha}\left(\int_t^{\infty}\left(\varepsilon^{1+1/\alpha}\right)^{\mu_{j-1}}\frac{q_0}{v^{\alpha+2}}dv\right)^{1/\alpha}$$
$$= q_0^{1/\alpha}\left(\varepsilon^{(1/\alpha)+1}\right)^{\mu_{j-1}} \ge \mu_j$$

Case	(1)	(2)	(3)	(4)
α	0.6	0.6	1.0	3.0
q_0	0.3	0.6	0.2	0.5
ε	1.9	1.9	2.0	1.2
γ	0.1713	0.1713	0.2500	0.4725
j	3	1	2	1
μ_j	0.1796	0.4268	0.2639	0.7937

Table 1: Values of μ_j for different values of q_0, ε and α

for $j = 1,2,\ldots,m$, $m \in \mathbb{N}$. The validity of Theorem 3 depending on the values of μ_j that satisfying Condition (15), i.e., the values of μ_j that meets the inequality

$$\mu_j > \frac{\alpha}{(1+\alpha)^{(1+\alpha)/\alpha}}.$$

Table 1 illustrates these starting values of μ_j for different values of q_0, ε and α, where

$$\gamma := \frac{\alpha}{(1+\alpha)^{(1+\alpha)/\alpha}}.$$

*Remark.*The previous results can be generalized by ensuring that, if

$$\mu_j \le \frac{\alpha}{(1+\alpha)^{(1+\alpha)/\alpha}}, \quad \text{for} \quad j = 1,2,\ldots,m-1$$

and

$$\mu_m > \frac{\alpha}{(1+\alpha)^{(1+\alpha)/\alpha}}, \quad \text{for} \quad m \ge 2.$$

Then, every solution of (1) oscillate.

4 Conclusion

In this study, we carried out a thorough investigation into the oscillatory behavior of advanced second-order nonlinear differential equation with multiple deviating arguments (1). Through a combination of novel methods, we obtained three different oscillation theorems (Theorem 1–3) that hold for any solution of the (1). Our approach used the comparison technique with equation (10) of the same order to obtain Theorems 1 and 3, as well as the well-known Riccati technique for deducing criterion (11) of Theorem 2. The conclusions of our study constitute a significant improvement over previous research, mainly because we addressed the previously unconsidered effect of deviating arguments $h_i(t)$ on oscillatory behavior by improving the previous oscillation criterion

$$\eta(t)\left(\int_t^{\infty} q(v)\,dv\right)^{1/\alpha} \ge \mu > \frac{\alpha}{(1+\alpha)^{(1+\alpha)/\alpha}}$$

where μ is constant to criterion (7). Relying our criteria on the presence of $h_i(t)$ greatly affects the oscillation of

(1) and gives improved results as represented in Example 1. Moreover, our findings are generalizable and can be applied to different types of differential equations, such as ordinary ($h_i(t) \equiv t$) and linear ($\alpha = 1$) forms. In addition, we provided further details regarding the novelty and scientific contributions of our work, supported by Example 1 that validate the wider relevance and expanded reach of our conclusions in comparison to earlier research as shown in Table 1.

Our research paves the way for future studies in this field by thoroughly addressing these aspects and providing a comprehensive framework for understanding the oscillatory behavior of advanced second-order nonlinear differential equations with multiple deviating arguments.

Conflict of Interest

The authors declare that there is no conflict of interest regarding the publication of this article.

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