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Chaos in a fractional-order neutral differential system

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Abstract: In this paper, chaos in fractional-order neutral delay differential equation (NDDE) is discussed. Chaos in the system is illustrated by presenting its waveform graphs, states diagrams and largest Lyapunov exponent. The largest Lyapunov exponent (LLE) and the LLE of the system with different parameters are derived. In addition, we compare the fractional-order with integer NDD systems, and find that the convergence speed of the synchronization fraction system is faster. We also get the conclusion that the fractional NND system has better anti-interference ability.

Keywords: Fractional-order neutral differential system, Chaos, Largest Lyapunov exponent.

1. Introduction

Fractional-order integrals and derivatives have been known since the development of the regular calculus. Fractional calculus had been studied since the 17th century. In recent years, the behavior of many physical systems has been properly described using the fractional-order system theory. The scientists from physics, demography and finance have focused on the fractional-order differential systems in their research fields [1–3]. It is interesting to investigate the nonlinear fractional-order systems. Complex oscillations are demonstrated in many physical systems due to the effect of delayed feedback. These oscillations have been proved in semiconductor lasers, microwave devices and electronic circuits [4–7]. Time-delay phenomena are also appeared in physical systems, such as AIDS epidemic, aircraft stabilization, chemical engineering systems, control of epidemics, distributed networks, manual control, microwave oscillator and systems with lossless transmission lines [8]. Hence, the time-delay systems have been received considerable attention. The neutral delay differential system which contains derivatives with a delayed argument is more challenging than other systems, and the study of chaotic delay dynamical systems lags behind those ordinary dynamical systems [9]. Existence and uniqueness theorem for delay differential equations have been investigated in [10, 11]. Fractional-order neutral differential system has been focused in the research work of [12–14]. But the chaos in these systems has not been fully investigated in the previous work. A detailed numerical bifurcation analysis of neutral delay differential system was discussed in [15]. The authors in [15] investigated bifurcations of periodic solutions by computing their Floquet multipliers with the methods proposed in [16, 17], and by analogy with the corresponding methods for ordinary differential equations (ODE). Balanov et al. [18] investigated the solutions more complex than periodic ones and the results were presented as plots containing solution regimes that may be considered as bifurcation diagrams for the neutral delay differential equation (NDDE). The map of regimes was demonstrated as a function of two control parameters. They got the range of two control parameters for every state of NDDE, which included a unique equilibrium state, a single period-one limit cycle, period-two, multistability and chaos. Blakely et al. presented the experimental evidence to verify the predictions of chaotic dynamics in a transmission line terminated with a nonlinear element and showed that an extension to the existing theory which added the effects of a finite bandwidth to the negative resistor would generate their experimental results more accurately [19]. All of the studies mentioned above are regarding the integer neutral delay differential system. Balanov

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et al. explored numerically the solution space of a neutral differential delay equation such as Eq. (1) that arises naturally in the Cosserat description of torsional waves on a driven drill-string [18].

$$\frac{dx}{dt}-\frac{dx(t-\tau)}{dt}=\frac{F(x(t)-x(t-\tau)+\Omega)-x(t)-x(t-\tau)+\Omega}{J}. \tag{1}$$

Such torsional vibrations are a major concern to industry, which can occur during the operation of drilling assemblies used in the exploration for oil and gas. A drilling assembly can penetrate several kilometers below the surface using a drill-bit fitted to its lower end. The dynamics of the drill-string is extremely complex and often prone to instabilities that are not fully understood. One particular scenario about the torsional instability of the drill-string arises from self-sustained relaxation oscillations, which induced by non-linear reaction torques due to friction and cutting at the drill-bit. A build-up of torsional vibrations may result in reduced rate-of-penetration, premature component fatigue or even costly fracture of the drill-string. An essential step towards developing practical strategies to combat such torsional vibrations requires a reliable model for the entire drilling assembly and some insight into the structure of its solution space [18]. Model (1) is transformed from the classical solutions of driven drill-string in dynamical boundary conditions. The purely torsional excitations of a drill-string with unstressed length L,described by the rotary angle $\psi(s,t)$ satisfy the wave equation $\ddot{\psi}(s,t)=c^2\psi''(s,t)$,when $s=0$,$\psi(0,t)=\psi$,F is a reaction torque which exist in friction and cutting processes.$x(t)$is the derivative of the function representing upwards moving torsional disturbances. By introducing fractional-order differential into system (1), we obtain a fractional-order neutral delay differential system as follows:

$$\begin{aligned}{}_aD_t^{\alpha}y(t)={}_aD_t^{\beta}y(t-\tau)+F(y(t)-y(t-\tau)\\+\Omega)-y(t)-y(t-\tau)+\Omega,\end{aligned} \tag{2}$$

where$F(z)=-\frac{Az}{\sqrt{(z^2+\varepsilon^2)}}(1+bexp(-\frac{\sqrt{(z^2+\varepsilon^2)}}{\Delta}))$,which is the expression of non-dimensional reaction torque based on recent drilling data. A,b,ε are certain constants.α,β as the fractional-order, are non-integer constants. The chaotic behavior of system (2) will be explored in this paper. The research on chaotic characteristics is focused on phase plot, bifurcation diagram and the largest Lyapunov exponent (LLE) which are the most important characteristic quantities to perform chaos in both fractional-order system and integral-order system. One positive Lyapunov exponent indicates the existence of chaos. Wolf [20] and Jacobian [21] algorithms are the most popular algorithms to calculate the largest Lyapunov exponent of integral-order system. However, Jacobian algorithm is not applicable for calculating LLE of a fractional-order system since Jacobian matrix of fractional-order system is hard to be obtained. Furthermore, Wolf algorithm is relatively difficult to implement. The method of the small sets [22] presented by Rosenstein et al. can calculate LLE of a Fractional-order system. In this paper, the method of the small sets will be used which mainly focuses on analyzing the time sequence of fractional systems. The importance of this method is the phase space reconstruction and the small data sets in time sequence. The C-C method [23,**?**] which uses the correlation integral is chosen to determine the parameters of phase space reconstruction, which are the delay time d and the embedding dimension m. In this way, we can get the characteristics of chaotic attractors in the chaotic time series. In addition, the information hidden in the sequence may be revealed as well. Then we can calculate the LLE by the small data sets. Chaotic phenomena will be exhibited if the LLE is positive. The remainder of this paper is organized as followings. In Section 2, Fractional derivative and Grunwald-Letnikov (GL) definition are introduced. In Section 3, we present the method of calculating of the largest Lyapunov exponent. The mathematical model is presented in section 4 with the chaotic phenomena and the LLE. In section 5, the difference between fractional and integer NDDE are discussed. Section 6 is the conclusion.

2. Fractional derivative and its approximation

Initial Capitals There are many definitions for fractional differential operator [25,**?**,**?**]. One of the definitions is Grunwald-Letnikov (GL) definition [25,**?**,**?**], which can be described by

$${}_aD_t^{\alpha}f(t)=\lim_{h\to 0}h^{-\alpha}\sum_{j=0}^{[t-a/h]}\binom{\alpha}{j}f(t-hj), \tag{3}$$

where: $\binom{\alpha}{j}=\frac{\alpha(\alpha-1)\cdots(\alpha-j+1)}{j!}$. This formula can be reduced to

$${}_0D_t^{\alpha}y(t_m)\approx h^{-\alpha}\sum_{j=0}^{m}\omega_j^{(\alpha)}y_{m-j}, \tag{4}$$

h is the time step. For a general nonlinear fractional-order differential system, the explicit analytical solution is usually not existed. A numerical algorithm is required for the solution. For example, the nonlinear system in [30]:

$${}_0D_t^{\alpha}y(t_m)+N(y(t))=g(t),\alpha>0 \tag{5}$$

where, N represents a nonlinear operator, $g(t)$ is a function with respect to t. From Eq. (4), it can be transformed to a discrete form as follows:

$$h^{-\alpha}\sum_{j=0}^{m}\omega_j^{(\alpha)}y_{m-j}+N(y_m)=g(t_m), \tag{6}$$

$$m=1,2,3,...,[\frac{t-a}{h}]$$

By Eq. (6), we can have

$$y_m = h^{-\alpha}(g(t_m) - N(y_m)) - \sum_{j=0}^{m} \omega_j^{(\alpha)} y_{m-j}, \qquad (7)$$

$$m = 1, 2, 3, ..., [\frac{t-a}{h}]$$

For Eq. (7), an iterative formula to solve y_m can be described as follows:

$$y_m^{(l)} = h^{-\alpha}(g(t_m) - N(y_m)) - \sum_{j=0}^{m} \omega_j^{(\alpha)} y_{m-j}^{(l)}, \qquad (8)$$

$$m = 1, 2, 3, ..., [\frac{t-a}{h}]$$

. where l is the iteration number.If $|h^{\alpha}N'| < 1$, the algorithm is convergent, and $y_m = y_m^{(l)}$. For system (2), one of the discrete forms is

$$h^{-\alpha}\sum_{j=0}^{m} \omega_j^{(\alpha)} y_{m-j} = (y_{m\tau} - y_{m\tau-1})/h + F(y_m \qquad (9)$$

$$+y_{m\tau} + \Omega) + y_m + y_{m\tau} + \Omega = 0.$$

We use the Newton-Raphson method, which is a convergent iterative algorithm $f(x) = 0$ for an algebraic equation , solution is obtained by calculating iterative formula, solution x is obtained by calculating iterative formula $x^{(l)} = x^{(l-1)} - \frac{f(x^{(l-1)})}{f'(x^{(l-1)})}$. Thus Eq.(9) can be rewritten as

$$h^{-\alpha}\sum_{j=0}^{m} \omega_j^{(\alpha)} y_{m-j} - (y_{m\tau} - y_{m\tau-1})/h - F(y_m \qquad (10)$$

$$+y_{m\tau} + \Omega) - y_m - y_{m\tau} + \Omega.$$

According to Newton-Raphson method, we get that:

$$y_{(m)}^{(l)} = y_{(m)}^{(l-1)}$$

$$-\frac{(y_m^{(l-1)} - \frac{(h^{\alpha} * (y'_{m\tau} + F(z) - y_m\tau + \Omega)) - y_{(m)}^{(l)} \sum_{j=0}^{m} \omega_j^{(\alpha)} y_{m-j}}{1+h^{\alpha}})}{1 - h^{\alpha} * F'(z)} \qquad (11)$$

where,$z = y_{(m)}^{l-1} + y_{m\tau} + \Omega$, $F'(z)$ is the derivative of $F(z)$.

3. The method of calculating largest Lyapunov exponent

The largest Lyapunov exponent is an important characteristic not only in integral-order system but also in fractional-order system, because any system containing at least one positive Lyapunov exponent is defined to be chaotic [31, 32]. Wolf algorithm, Jacobian algorithm and the small data sets have been mentioned in section 1. In this paper, we apply the small data sets and the C-C method to calculate the LLE of fractional-order system. That is, the delay time d and embedding dimension m can be determined by the C-C method. Then we can use the small sets method to estimate the LLE.

The small sets method will be introduced briefly in this section. The parameters d and m can be derived from the C-C method which reconstructs the attractor dynamics from the single time series of the dynamic system. The scalar time series x_l, $l = 1, 2, 3 \cdots N$, in a m-dimensional space are defined by the vectors $Y_l = (x_l, x_{l+\tau}, x_{l+(m-l)\tau},) \in R^m$, where $M = N - (m-1)\tau$. After reconstructing the dynamics, the C-C method locates the nearest neighbor of each point on the trajectory. The nearest neighbor $Y_{\hat{k}}$ can be found by searching for the point that minimizes the distance to the particular reference point Y_k which can be expressed as: $d_k(0) = x_{l+\tau} \min_{Y_{\hat{k}}} = \|Y_k - Y_{\hat{k}}\|, |k - \hat{k}| > p$, ,where p is mean period, and $\| \cdots \|$ denotes the Euclidean norm. Then we can obtain the following largest Lyapunov exponent based on the method proposed by Sato et al. [29].

$$\lambda_1(l) = \frac{1}{l \cdot \triangle t} \frac{1}{(m-l)} \sum_{k=1}^{M-k} ln \frac{d_k(l)}{d_k(0)}. \qquad (12)$$

where $\triangle t$ is the sampling period of the time series, and $d_k(l)$ is the distance between the k^{th} pair of nearest neighbors after l discrete-time steps. Sato et al [33] also presented an improved expression as the following:

$$\lambda_1(k\prime) = \frac{1}{k\prime \cdot \triangle t} \frac{1}{(m-k\prime)} \sum_{k=1}^{M-k\prime} ln \frac{d_k(l+k\prime)}{d_k(l)}, \qquad (13)$$

where $k\prime$ is held constant, and λ_1 is extracted by locating the plateau of $\lambda_1(l, k\prime)$ with respect to l. From the definition of $\lambda_1(l, k\prime)$, we assume that the k^{th} pair of nearest neighbors diverge approximately at a rate defined by the largest Lyapunov exponent:

$$d_k(l) = C_k e^{\lambda_1(l \cdot \triangle t)}. \qquad (14)$$

Here C_k is the initial separation. Applying the logarithm to both sides of Eq. (5), the following equation can be derived:

$$lnd_k(l) \approx lnC_k + \lambda_1(l \cdot \triangle t)(k = 1, 2 \cdots, M). \qquad (15)$$

Eq. (6) represents a set of approximately parallel lines each with a slope roughly proportional to λ_1. Then we can get the largest Lyapunov exponent easily and accurately by using a least squares fit to the "average" line defined by

$$y(l) = \frac{1}{\triangle t}\langle lnd_k(l) \rangle. \qquad (16)$$

here$\langle lnd_k(l) \rangle$ is the average value for all values of k. This process of averaging is the key step to calculate accurate value of λ_1. In order to calculate more accurately, we also calculate the average value of the nearest neighbors of $y(l)$.

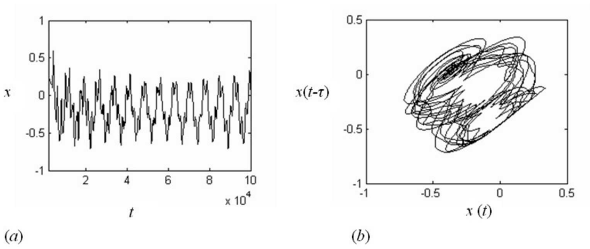

Fig.1. Waveform diagram (a) and the Phase portrait (b) of system (1) with α=1.02, A=0.83, Ω=0.461

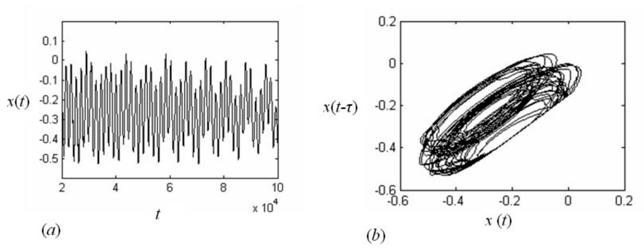

Fig.4. Waveform diagram (a) and the Phase portrait (b) of system (1) with α=1.01, A=0.65, Ω=0.15

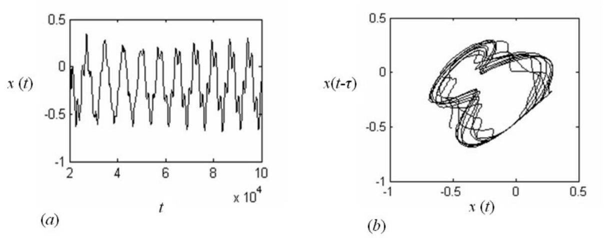

Fig.2. Waveform diagram (a) and the Phase portrait (b) of system (1) with α=1.06, A=0.83, Ω=0.461

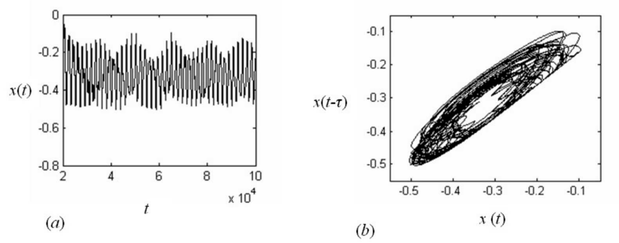

Fig.5. Waveform diagram (a) and the Phase portrait (b) of system (1) with α=1.01, A=0.65, Ω=0.06

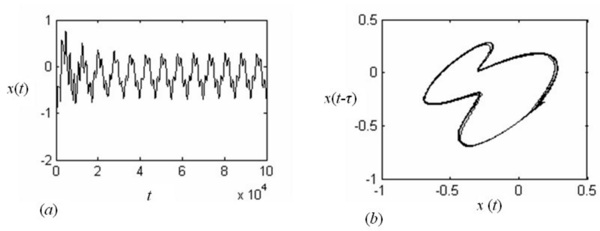

Fig.3. Waveform diagram (a) and the Phase portrait (b) of system (1) with α=1.1, A=0.83, Ω=0.461

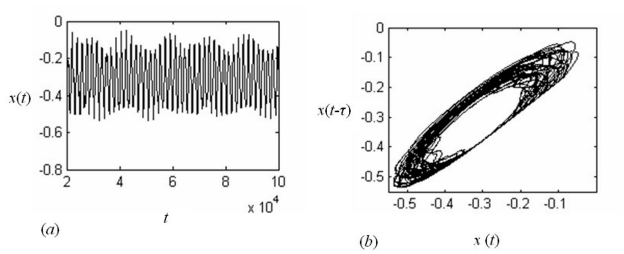

Fig.6. Waveform diagram (a) and the Phase portrait (b) of system (1) with α=1.01, A=0.65, Ω=0.08

4. Chaotic phenomena and the LLE of the system (2)

In the integer-order NDDE mentioned in [16], chaos was found in different range of the control parameter. In this paper, we also find similar phenomenon in fractional-order NDDE. For the system (2) (represented by Eq. (2)), we take $\alpha = 1.02, 1.06, 1.1$, $\beta = 1, A = 0.83$, $\tau = 2, \Omega = 0.461, \Delta = 0.1, b = 0.2, \varepsilon = 0.001, h = 0.001$. Fig. 1-3 illustrates the waveform diagrams and phase portraits respective to . From above simulation results, we can see that when $\alpha = 1.02$ and 1.06, the system (2) is chaotic. When $\alpha = 1.1$, the system presents the period-1, and the corresponding LLE are 0.0138, 0.0440, 0, respectively. The numerical simulations are consistent with the result of theoretical calculation. When $\alpha = 1.01, \beta = 1, A = 0.65, \tau = 2, \Omega = 0.15, 0.08, 0.06, \Delta = 0.1, b = 0.2, \varepsilon = 0.001, h = 0.001$, the attractor is changed. Fig. 4-6 depicts the state of attractor in detail. When $\Omega = 0.06, 0.08, 0.15$, the LLE are 0.3572, 0.2211 0.8759, so we can get that when $\Omega = 0.06, 0.08, 0.15$, the system is chaotic. Except above chaotic attractor, there is another chaotic phenomenon, when $\alpha = 1.01, \beta = 1, A = 1.5, \tau = 2, \Omega = 0.01, \Delta = 0.1, b = 0.2, \varepsilon = 0.001, h = 0.001$, which is illustrated in the following. The LLE of the system (2) is 0.1625. We calculate the LLE of the system (2) using the methods in the previous section. And we can get that the results are consistency with the waveform diagram and phase portrait.

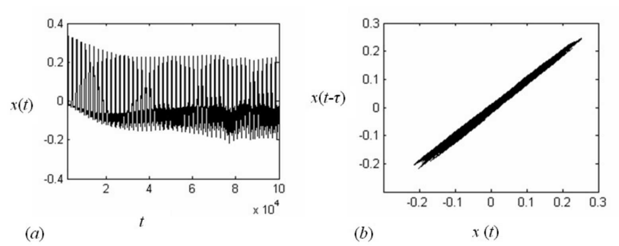

Fig.7. Waveform diagram (a) and the Phase portrait (b) of system (1) with α=1.01, A=1.5, Ω=0.01

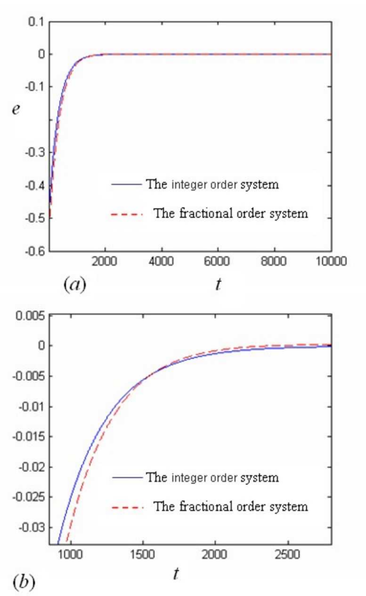

Fig.8. Error of system (18) with k=2, A=0.83, Ω=0.461 (a), (b) is the partial enlarged view

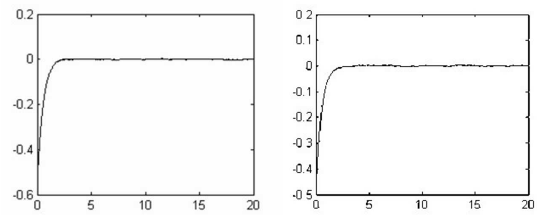

Fig.9. Error of system with nosie with k=2, A=0.83, Ω=0.461,the order are 1.02(a) and the 1(b)

5. Synchronization of fractional system with noise

In this section, we discuss the synchronization of the fractional and integer NDDE. We employ (2) as the response system, and the drive system is as Eq.(17).

$$\frac{dx}{dt} - \frac{dx(t-\tau)}{dt} = \frac{F(x(t)-x(t-\tau)+\Omega)-x(t)-x(t-\tau)+\Omega}{J} - k(x(t)-y(t)). \tag{17}$$

Coupling (2) with (17), a synchronization system is obtained as (18).

$$\begin{cases} {}_aD_t^a y(t) = {}_a D_t^a y(t-\tau) + F(y(t)-y(t-\tau)+\Omega) + Ly \\ {}_aD_t^a x(t) = {}_a D_t^a y(t-\tau) + F(y(t)-y(t-\tau)+\Omega) + Lx \end{cases} \tag{18}$$

When $Ly = -y(t) - y(t-\tau) + \Omega, Lx = -x(t) - y(t-\tau) + \Omega - k(x(t) - y(t))$. Defining the synchronization error as e(t)=y(t)-x(t), we can obtain the error system from system (17) as Eq.(19).

$$\frac{d^\alpha e(t)}{dt^\alpha} = -(1+k)e(t).. \tag{19}$$

Taking the Laplace transformation in both sides of Eq.(19), we obtain

$$s^\alpha L(e(t)) - s^{\alpha-1}e(0) = -(1+k)Le(t). \tag{20}$$

It follows that$L(e(t)) = \frac{s^{\alpha-1}e(0)}{s^\alpha+(1+k)}$,By the final-value theorem of the Laplace transformation,if $k \neq 1$, we have $\lim\limits_{h\to 0} e(t) = \lim\limits_{h\to 0} sL(e(t)) = \lim\limits_{h\to 0} \frac{s^\alpha e(0)}{s^\alpha+(1+k)} = 0$, The above analysis implies that coupled system with fractional-order response system (2) is synchronized as long as $k \neq 1$. The simulation is illustrated in Fig. 8. From the simulation, we can get that fractional order NND system achieves synchronization earlier than the integer order NND system. According to the experimental data, the error of fractional order NND system reach $10^{-}4$ at 2271, but the integer order NND system is at 2845. We add the White noise which the Mean value is 0 in system 17. We get the conclusion that the fractional NND system have better anti-interference ability from the simulation results (Fig.9).

6. Conclusion

In this paper, we have studied the dynamics of the fractional-order neutral differential system by means of the largest Lyapunov exponent. The novel numerical algorithm and the Newton-Raphson method are used to analyze the NNDE. We also calculate the largest Lyapunov exponent by using the small data sets instead of Wolf algorithm. The Lyapunov exponents with different situation are given, chaotic phenomenon are presented respectively, and in addition, we compare fractional order with integer order NDDE, and find that the fractional-order system achieve synchronize more quickly than the integer order system, and the fractional-order system has better anti-interference ability.

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