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Recurrence Relations for Moments of Generalized Order Statistics from Marshall – Olkin Extended Family of Life Distributions and its Characterization

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Abstract: In this paper, recurrence relations for single and product moments of generalized order statistics from Marshall – Olkin extended family of life distributions are obtained. Specializations to order statistics and records have been made. Further, using a recurrence relation for single moments we obtain characterization of Marshall – Olkin extended family of life distributions.

Keywords: Generalized order statistics - Order statistics – Records –Single and product moments – Recurrence relations - Marshall – Olkin extended distributions - Characterization.

1 Introduction

Marshall and Olkin [16] introduced a new method of adding a parameter into a family of distributions. According to them if $\overline{F}(x)$ denote the survival or reliability function of a continuous random variable X then the timely honored device of adding a new parameter results in another survival function $\overline{G}(x)$ defined by

$$\overline{G}(x) = \frac{\alpha\overline{F}(x)}{1-\overline{\alpha}\overline{F}(x)}; \quad -\infty \prec x \prec \infty, \alpha \succ 0 \text{ and } \overline{\alpha} = 1-\alpha. \tag{1}$$

The probability density function (pdf) and hazard rate function (HF) corresponding to $\overline{G}(x)$ are:

$$g(x) = \frac{\alpha f(x)}{\left[1-\overline{\alpha}\overline{F}(x)\right]^2}; \quad -\infty \prec x \prec \infty, \alpha \succ 0 \text{ and } \overline{\alpha} = 1-\alpha, \tag{2}$$

and

$$r(x) = \frac{h(x)}{1-\overline{\alpha}\overline{F}(x)}; \quad -\infty \prec x \prec \infty, \alpha \succ 0 \text{ and } \overline{\alpha} = 1-\alpha, \tag{3}$$

where $h(x)$, is the HF corresponding $f(x)$.
Here, we denote $\overline{F}(x)$ as

$$\overline{F}(x) = e^{-\lambda(x)}; \quad x \succeq 0, \tag{4}$$

where $\lambda(x)$ is a non-negative, continuous, monotone increasing, differentiable function of x such that $\lambda(x) \to 0$ as $x \to 0^+$ and $\lambda(x) \to \infty$ as $x \to \infty$ (cf Mahmoud and Ghazal [15]). Substituting from (4) in (1), we get

$$\overline{G}(x) = \frac{\alpha e^{-\lambda(x)}}{1-\overline{\alpha}e^{-\lambda(x)}}; \quad x \geq 0, \alpha \succ 0 \text{ and } \overline{\alpha} = 1-\alpha. \tag{5}$$

The pdf and HF corresponding to $\overline{G}(x)$ are:

$$g(x) = \frac{\alpha\lambda'(x)e^{-\lambda(x)}}{\left[1-\overline{\alpha}e^{-\lambda(x)}\right]^2}; \quad x \geq 0, \alpha \succ 0 \text{ and } \overline{\alpha} = 1-\alpha, \tag{6}$$

and

$$r(x) = \frac{\lambda'(x)}{1-\overline{\alpha}\overline{F}(x)}; \quad x \geq 0, \alpha \succ 0 \text{ and } \overline{\alpha} = 1-\alpha, \tag{7}$$

The family of distributions in (6), call Marshall – Olkin extended $(M-OE)$ family of life distributions.

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Now in view of (5) and (6), we get

$$\overline{G}(x) = \frac{g(x)}{\lambda'(x)}\left[1-\overline{\alpha}e^{-\lambda(x)}\right]. \qquad (8)$$

Ten distributions can be obtained from (6), using the following table.

Table (1) examples of pdf (6)

distribution	pdf	$\lambda(x)$
$M-OE$ linear failure rate	$\frac{\alpha(a+bx)e^{-\left(ax+\frac{b}{2}x^2\right)}}{\left[1-\overline{\alpha}e^{-\left(ax+\frac{b}{2}x^2\right)}\right]^2}$	$\left(ax+\frac{b}{2}x^2\right)$; x,a and $b\succeq 0$
$M-OE$ Weibull	$\frac{\alpha abx^{b-1}e^{-ax^b}}{\left[1-\overline{\alpha}e^{-ax^b}\right]^2}$	$ax^b; a,b\succ 0$ and $x\succeq 0$
$M-OE$ Rayleigh	$\frac{2\alpha axe^{-ax^2}}{\left[1-\overline{\alpha}e^{-ax^2}\right]^2}$	$ax^2; a\succ 0$ and $x\succeq 0$
$M-OE$ exponential	$\frac{\alpha ae^{-ax}}{[1-\overline{\alpha}e^{-ax}]^2}$	$ax; a\succ 0$ and $x\succeq 0$
$M-OE$ modified Weibull	$\frac{\alpha abx^{b-1}(b+\gamma x)}{\left[1-\overline{\alpha}e^{-ax^be^{\gamma x}}\right]^2}$ $\times e^{-ax^be^{\gamma x}+\gamma x}$	$ax^be^{\gamma x}; x\succeq 0$ and $a,b\succ 0$
$M-OE$ Gompertz	$\frac{\alpha ae^{-\frac{a}{c}(e^{cx}-1)+cx}}{\left[1-\overline{\alpha}e^{-\frac{a}{c}(e^{cx}-1)}\right]^2}$	$\frac{a}{c}(e^{cx}-1)$; $c\geq 0$ and $a,x\succ 0$
$M-OE$ Burr type XII	$\frac{\alpha abx^{b-1}(1+x^b)^{-(a+1)}}{\left[1-\frac{\overline{\alpha}}{(1+x^b)^a}\right]^2}$	$a\ln(1+x^b)$; $a,b,x>0$
$M-OE$ Lomax	$\frac{\alpha ab(1+bx)^{-(a+1)}}{[1-\overline{\alpha}(1+bx)^{-a}]^2}$	$a\ln(1+bx)$; $a,b,x>0$
$M-OE$ Pareto	$\frac{\alpha ab(1+x)^{-(a+1)}}{[1-\overline{\alpha}(1+x)^{-a}]^2}$	$a\ln(1+x)$; $a,x>0$
$M-OE$ Gamma	$\frac{\alpha a^2xe^{-ax}}{\left[1-\overline{\alpha}(1+ax)e^{-ax}\right]^2}$	$-\ln(1+ax)$ $+ax; a,x>0$

The concept of generalized order statistics (gos) was introduced by Kamps [10]. A variety of order models of random variables is contained in this concept.

Let, for simplicity F, throughout denote an absolutely continuous distribution function with density function f.

The random variables $X(1.n,\widetilde{m},k),\ldots,X(n.n,\widetilde{m},k)$ are called generalized order statistics based on F, if their joint pdf of the form

$$k\left(\prod_{j=1}^{n-1}\gamma_j\right)\left(\prod_{i=1}^{n-1}\left[\overline{F}(x_i)\right]^{m_i}f(x_i)\right)\left[\overline{F}(x_n)\right]^{k-1}f(x_n),$$

for $F^{-1}(1)>x_1\geq x_2\geq\ldots\geq x_n>F^{-1}(0)$,
with parameters $n\in N, n\geq 2, k>0$,
$\widetilde{m}=(m_1,m_2,\ldots,m_{n-1})\in R^{n-1}, M_r=\sum_{i=r}^{n-1}m_i$,
such that
$\gamma_r=k+n-r+M_r>0$, for all $r\in\{1,2,\ldots,n-1\}$.

For $\gamma_i\neq\gamma_j, i\neq j$ for all $i,j\in(1,2,\ldots,n)$ the pdf of $X(r.n,\widetilde{m},k)$ is given by Cramer and Kamps [8] in the following way

$$f_{X(r.n,\widetilde{m},k)}(x)=C_{r-1}f(x)\sum_{i=1}^{r}a_i(r)\left[\overline{F}(x)\right]^{\gamma_i-1}. \qquad (9)$$

The joint pdf of $X(r.n,\widetilde{m},k)$ and $X(s.n,\widetilde{m},k), 1\leq r<s\leq n$ is given as

$$f_{X(r.n,\widetilde{m},k),X(s.n,\widetilde{m},k)}(x,y)$$
$$=C_{s-1}\left(\sum_{i=r+1}^{s}a_i^{(r)}(s)\left[\frac{\overline{F}(y)}{\overline{F}(x)}\right]^{\gamma_i}\right)$$
$$\left(\sum_{i=1}^{r}a_i(r)\left[\overline{F}(x)\right]^{\gamma_i}\right)\frac{f(x)f(y)}{\overline{F}(x)\overline{F}(y)}, \qquad (10)$$

where $x<y$ and

$$a_i(r)=\prod_{\substack{j=1\\ j\neq i}}^{r}\frac{1}{\gamma_j-\gamma_i}, 1\leq i\leq r\leq n,$$

$$a_i^{(r)}(s)=\prod_{\substack{j=r+1\\ j\neq i}}^{s}\frac{1}{\gamma_j-\gamma_i}, r+1\leq i\leq s\leq n.$$

It may be noted that for
$m_1=m_2=\ldots=m_{n-1}=m\neq -1$,

$$a_i(r)=\frac{(-1)^{r-i}\binom{r-1}{r-i}}{(m+1)^{r-1}(r-1)!},. \qquad (11)$$

and

$$a_i^{(r)}(s)=\frac{(-1)^{s-i}\binom{s-r-1}{s-i}}{(m+1)^{s-r-1}(s-r-1)!},. \qquad (12)$$

Therefore pdf of $X(r.n,\widetilde{m},k)$ given in (9) reduces to

$$f_{X(r.n,m,k)}(x)=\frac{C_{r-1}}{(r-1)!}\left[\overline{F}(x)\right]^{\gamma_r-1}f(x)g_m^{r-1}\left[F(x)\right], \qquad (13)$$

and joint pdf of $X(r.n,\widetilde{m},k)$ and $X(s.n,\widetilde{m},k)$, $1 \leq r < s \leq n$ given in (10) reduces to

$$f_{X(r.n,m,k),X(s.n,m,k)}(x,y) = \frac{C_{s-1}}{(r-1)!(s-r-1)!}\left[\overline{F}(x)\right]^{m} f(x)\, g_m^{r-1}[F(x)] \{h_m[F(y)] - h_m[F(x)]\}^{s-r-1}\left[\overline{F}(y)\right]^{\gamma_s-1} f(y), \quad x<y, \tag{14}$$

where

$C_{r-1} = \prod_{i=1}^{r}\gamma_i$, $\gamma_i = k+(n-i)(m+1)$,

$$h_m(x) = \begin{cases} \frac{-1}{m+1}x^{m+1}, m \neq -1 \\ -\ln(x), m=-1 \end{cases}$$

and

$g_m(x) = h_m(x) - h_m(1), x \in [0,1)$.

We shall also take $X(0.n,m,k)=0$. If $m=0, k=1$, then $X(r.n,m,k)$ reduces to the $(n-r+1)^{th}$ order statistics, $X_{n-r+1:n}$ from the sample $X_1, X_2, \ldots, X_n$ and when $m=-1$, then $X(r.n,m,k)$ reduces to the k^{th} record values (Pawlas and Szynal [17]).

Many authors utilized the gos in their work, such as Kamps and Gather [11], Keseling [12], Cramer and Kamps [8], Ahsanullah [4], Pawlas and Szynal [17], Ahmed [2], Ahmed and Fawzy [3], Khan et al. [13], AL-Hussaini et al. [5] and Kumar [14]. Abdul-Moniem [1] obtained recurrence relations for moments of lower gos from exponentiated Lomax distribution and its characterization.

In this paper, we have established explicit expressions and some recurrence relations for single and product moments of gos from $M-OE$ family of life distributions. Further its various deductions and particular cases are discussed. Characterization of $M-OE$ family of life distributions has been obtained on using a recurrence relation for single moments.

2 Characterization based on recurrence relation for single moments of *gos*

Theorem 2.1. let X be a random variable has pdf (6). Then for integer j such that $j>0$, the following recurrence relation is satisfied.

$$E\left[X^j(r.n,\widetilde{m},k)\right] - E\left[X^j(r-1.n,\widetilde{m},k)\right] = \frac{j}{\gamma_r}E\left[\frac{X^{j-1}(r.n,\widetilde{m},k)}{\lambda'(X(r.n,\widetilde{m},k))}\right] - \frac{j\overline{\alpha}}{\gamma_r}\sum_{l=0}^{\infty}\frac{(-1)^l}{l!} E\left[\frac{X^{j-1}(r.n,\widetilde{m},k)\,\lambda^l(X(r.n,\widetilde{m},k))}{\lambda'(X(r.n,\widetilde{m},k))}\right]. \tag{15}$$

Proof. We have from Lemma 2.3 (Athar and Islam [6]) that

$E[\xi\{X(r.n,\widetilde{m},k)\}] - E[\xi\{X(r-1.n,\widetilde{m},k)\}] = C_{r-2}\int_{\theta}^{\beta}\xi'(x)\sum_{i=1}^{r}a_i(r)\left[\overline{F}(x)\right]^{\gamma_r}dx$

If we let $\xi(x) = x^j$, then

$$E\left[X^j(r.n,\widetilde{m},k)\right] - E\left[X^j(r-1.n,\widetilde{m},k)\right] = jC_{r-2}\int_{\theta}^{\beta}x^{j-1}\sum_{i=1}^{r}a_i(r)\left[\overline{F}(x)\right]^{\gamma_r}dx. \tag{16}$$

On using (8) in (16), we get

$$E\left[X^j(r.n,\widetilde{m},k)\right] - E\left[X^j(r-1.n,\widetilde{m},k)\right] = \frac{j}{\gamma_r}C_{r-1}\int_0^{\infty}x^{j-1}\sum_{i=1}^{r}a_i(r)\left[\overline{F}(x)\right]^{\gamma_r-1}\frac{f(x)}{\lambda'(x)}\left[1-\overline{\alpha}e^{-\lambda(x)}\right]dx$$
$$= \frac{j}{\gamma_r}C_{r-1}\int_0^{\infty}\frac{x^{j-1}}{\lambda'(x)}\sum_{i=1}^{r}a_i(r)\left[\overline{F}(x)\right]^{\gamma_r-1}f(x)dx - \frac{j\overline{\alpha}}{\gamma_r}C_{r-1}\sum_{l=1}^{\infty}\frac{(-1)^l}{l!}\int_0^{\infty}\frac{x^{j-1}\lambda^l(x)}{\lambda'(x)}\sum_{i=1}^{r}a_i(r)\left[\overline{F}(x)\right]^{\gamma_r-1}f(x)dx.$$

Which after simplification leads to (15).

Corollary 2.2. For $m_1 = m_2 = \ldots = m_{n-1} = m \neq -1$, the recurrence relations for single moment of gos for $M-OE$ family of life distributions is given as

$$E\left[X^j(r.n,m,k)\right] - E\left[X^j(r-1.n,m,k)\right] = \frac{j}{\gamma_r}E\left[\frac{X^{j-1}(r.n,m,k)}{\lambda'(X(r.n,m,k))}\right] - \frac{j\overline{\alpha}}{\gamma_r}\sum_{l=0}^{\infty}\frac{(-1)^l}{l!} E\left[\frac{X^{j-1}(r.n,m,k)\,\lambda^l(X(r.n,m,k))}{\lambda'(X(r.n,m,k))}\right]. \tag{17}$$

Proof. This can easy be deduced from (15) in view of the relation (11).

Note that: We can obtain the recurrence relations for moments of gos for Marshall-Olkin extended Weibull distribution by taking $\lambda(x) = x^{\theta}$ in (17), established by Athar et. al. [7].

Remark 2.1 Putting $m=0, k=1$ in Theorem 2.1., we obtain recurrence relations for single moments of order statistics as

$$E\left[X_{r:n}^j\right] - E\left[X_{r-1:n}^j\right] = \frac{j}{n-r+1}E\left[\frac{X_{r:n}^{j-1}\left(1-\overline{\alpha}e^{-\lambda(X_{r:n})}\right)}{\lambda'(X_{r:n})}\right]. \tag{18}$$

Remark 2.2 Setting $m=-1,k=1$ in Theorem 2.1., we obtain the recurrence relations of upper record values as

$$E\left[X^{j}(r.n,-1,1)\right]-E\left[X^{j}(r-1.n,-1,1)\right]$$
$$=jE\left[\frac{X^{j-1}(r.n,-1,1)\left(1-\overline{\alpha}e^{-\lambda(X(r.n,-1,1))}\right)}{\lambda'(X(r.n,-1,1))}\right]. \quad (19)$$

3 Characterization based on recurrence relation for product moments of *gos*

Theorem 3.1 let X be a random variable has pdf (6). Then for integer i,j such that $i,j>0$, the following recurrence relation is satisfied.

$$E\left[X^{i}(r.n,\widetilde{m},k)X^{j}(s.n,\widetilde{m},k)\right]-$$
$$E\left[X^{i}(r.n,\widetilde{m},k)X^{j}(s-1.n,\widetilde{m},k)\right]$$
$$=\frac{j}{\gamma_s}E\left[\frac{X^{i}(r.n,\widetilde{m},k)X^{j-1}(s.n,\widetilde{m},k)}{\lambda'(X(s.n,\widetilde{m},k))}\right]-\frac{j\overline{\alpha}}{\gamma_s}\sum_{l=0}^{\infty}\frac{(-1)^l}{l!}$$
$$E\left[\frac{X^{i}(r.n,\widetilde{m},k)X^{j-1}(s.n,\widetilde{m},k)\lambda^{l}(X(s.n,\widetilde{m},k))}{\lambda'(X(s.n,\widetilde{m},k))}\right]. \quad (20)$$

Proof. We have from Lemma 3.2 (Athar and Islam [6]) that

$$E\left[\xi\left\{X(r.n,\widetilde{m},k),X(s.n,\widetilde{m},k)\right\}\right]-$$
$$E\left[\xi\left\{X(r.n,\widetilde{m},k),X(s-1.n,\widetilde{m},k)\right\}\right]$$
$$=C_{s-2}\int_{\theta}^{\beta}\int_{x}^{\beta}\frac{\partial}{\partial y}\xi(x,y)\sum_{l=r+1}^{s}a_l^{(r)}(s)\left[\frac{\overline{F}(y)}{\overline{F}(x)}\right]^{\gamma_l}$$
$$\sum_{l=1}^{r}a_l(r)\left[\overline{F}(x)\right]^{\gamma_l}\frac{f(x)}{\overline{F}(x)}dydx$$

If we let $\xi(x,y)=x^iy^j$, then

$$E\left[X^{i}(r.n,\widetilde{m},k)X^{j}(s.n,\widetilde{m},k)\right]-$$
$$E\left[X^{i}(r.n,\widetilde{m},k),X^{j}(s-1.n,\widetilde{m},k)\right]$$
$$=\frac{jC_{s-1}}{\gamma_s}\int_{\theta}^{\beta}\int_{x}^{\beta}x^iy^{j-1}\sum_{l=r+1}^{s}a_l^{(r)}(s)\left[\frac{\overline{F}(y)}{\overline{F}(x)}\right]^{\gamma_l}$$
$$\sum_{l=1}^{r}a_l(r)\left[\overline{F}(x)\right]^{\gamma_l}\frac{f(x)}{\overline{F}(x)}dydx$$

In view of (8), note that

$$\frac{\overline{F}(y)}{f(y)}=\frac{\left[1-\overline{\alpha}e^{-\lambda(y)}\right]}{\lambda'(y)}$$

Therefore,

$$E\left[X^{i}(r.n,\widetilde{m},k)X^{j}(s.n,\widetilde{m},k)\right]-$$
$$E\left[X^{i}(r.n,\widetilde{m},k),X^{j}(s-1.n,\widetilde{m},k)\right]$$
$$=\frac{jC_{s-1}}{\gamma_s}\int_{\theta}^{\beta}\int_{x}^{\beta}x^iy^{j-1}\frac{\left[1-\overline{\alpha}e^{-\lambda(y)}\right]}{\lambda'(y)}\sum_{l=r+1}^{s}a_l^{(r)}(s)$$
$$\left[\frac{\overline{F}(y)}{\overline{F}(x)}\right]^{\gamma_l}\sum_{l=1}^{r}a_l(r)\left[\overline{F}(x)\right]^{\gamma_l}\frac{f(x)}{\overline{F}(x)}\frac{f(y)}{\overline{F}(y)}dydx$$
$$=\frac{jC_{s-1}}{\gamma_s}\int_{\theta}^{\beta}\int_{x}^{\beta}\frac{x^iy^{j-1}}{\lambda'(y)}\sum_{l=r+1}^{s}a_l^{(r)}(s)\left[\frac{\overline{F}(y)}{\overline{F}(x)}\right]^{\gamma_l}$$
$$\sum_{l=1}^{r}a_l(r)\left[\overline{F}(x)\right]^{\gamma_l}\frac{f(x)}{\overline{F}(x)}\frac{f(y)}{\overline{F}(y)}dydx$$
$$-\frac{j\overline{\alpha}C_{s-1}}{\gamma_s}\int_{\theta}^{\beta}\int_{x}^{\beta}\frac{x^iy^{j-1}e^{-\lambda(y)}}{\lambda'(y)}\sum_{l=r+1}^{s}a_l^{(r)}(s)\left[\frac{\overline{F}(y)}{\overline{F}(x)}\right]^{\gamma_l}$$
$$\sum_{l=1}^{r}a_l(r)\left[\overline{F}(x)\right]^{\gamma_l}\frac{f(x)}{\overline{F}(x)}\frac{f(y)}{\overline{F}(y)}dydx$$

Which after simplification leads to (20).

Corollary 3.2 For $m_1=m_2=...=m_{n-1}=m\neq-1$, the recurrence relations for product moments of gos for $M-OE$ family of life distributions is given as

$$E\left[X^{i}(r.n,m,k)X^{j}(s.n,m,k)\right]-$$
$$E\left[X^{i}(r.n,m,k)X^{j}(s-1.n,m,k)\right]$$
$$=\frac{j}{\gamma_s}E\left[\frac{X^{i}(r.n,m,k)X^{j-1}(s.n,m,k)}{\lambda'(X(s.n,\widetilde{m},k))}\right]-\frac{j\overline{\alpha}}{\gamma_s}\sum_{l=0}^{\infty}\frac{(-1)^l}{l!}$$
$$E\left[\frac{X^{i}(r.n,m,k)X^{j-1}(s.n,m,k)\lambda^{l}(X(s.n,m,k))}{\lambda'(X(s.n,m,k))}\right]. \quad (21)$$

Proof. This can easy be deduced from (20) in view of the relation (12).

Note that: We can obtain the recurrence relations for product moments of *gos* for Marshall-Olkin extended Weibull distribution by taking $\lambda(x)=x^{\theta}$ in (21), established by Athar et. al. [7].

Remark 3.1 Putting $m=0,k=1$ in (21), we obtain recurrence relations for product moments of order statistics as

$$E\left[X_{r,s:n}^{i,j}\right]-E\left[X_{r,s-1:n}^{i,j}\right]$$
$$=\frac{j}{n-s-1}\left\{E\left[\frac{X_{r,s:n}^{i,j-1}}{\lambda'(X_{r,s:n})}\right]-\right.$$
$$\left.\overline{\alpha}\sum_{l=0}^{\infty}\frac{(-1)^l}{l!}E\left[\frac{X_{r,s:n}^{i,j-1}\lambda^{l}(X_{r,s:n})}{\lambda'(X_{r,s:n})}\right]\right\}. \quad (22)$$

Remark 3.2 Setting $m=-1,k=1$ in (21), we obtain the recurrence relations for product moments of k^{th} record values as

$$E\left[\left(X_r^{(k)}\right)^i\left(X_s^{(k)}\right)^j\right]-E\left[\left(X_r^{(k)}\right)^i\left(X_{s-1}^{(k)}\right)^j\right]$$
$$=\frac{j}{k}\left\{E\left[\frac{\left(X_r^{(k)}\right)^i\left(X_s^{(k)}\right)^{j-1}}{\lambda'\left(X_s^{(k)}\right)}\right]-\right.$$
$$\left.\overline{\alpha}\sum_{l=0}^{\infty}\frac{(-1)^l}{l!}E\left[\frac{\left(X_r^{(k)}\right)^i\left(X_s^{(k)}\right)^{j-1}\lambda^l\left(X_s^{(k)}\right)}{\lambda'\left(X_s^{(k)}\right)}\right]\right\}. \quad (23)$$

4 Characterization

Theorem 4.1 Let X be a non-negative random variable having an absolutely continuous distribution function $F(x)$ with $F(0)=0$ and $0<F(x)<1$ for all $x>0$, then

$$E\left[X^j(r.n,m,k)\right]-E\left[X^j(r-1.n,m,k)\right]$$
$$=\frac{j}{\gamma_r}E\left[\frac{X^{j-1}(r.n,m,k)}{\lambda'(X(r.n,m,k))}\right]-\frac{j\overline{\alpha}}{\gamma_r}\sum_{l=0}^{\infty}\frac{(-1)^l}{l!}$$
$$E\left[\frac{X^{j-1}(r.n,m,k)\,\lambda^l(X(r.n,m,k))}{\lambda'(X(r.n,m,k))}\right]. \quad (24)$$

if and only if $\frac{\overline{F}(x)}{f(x)}=\frac{\left[1-\overline{\alpha}e^{-\lambda(x)}\right]}{\lambda'(x)}$.

Proof: The necessary part follows immediately from equation (17). On the other hand if the recurrence relation in equation (24) is satisfied, then by using equation (13), we have

$$\frac{C_{r-1}}{(r-1)!}\int_0^{\infty}x^j\left[\overline{F}(x)\right]^{\gamma_r-1}f(x)\,g_m^{r-1}[F(x)]\,dx-$$
$$\cdot\frac{C_{r-2}}{(r-2)!}\int_0^{\infty}x^j\left[\overline{F}(x)\right]^{\gamma_{r-1}-1}f(x)\,g_m^{r-2}[F(x)]\,dx$$
$$=\frac{jC_{r-1}}{\gamma_r(r-1)!}\int_0^{\infty}x^{j-1}\left[\overline{F}(x)\right]^{\gamma_r-1}\frac{\left[1-\overline{\alpha}e^{-\lambda(x)}\right]}{\lambda'(x)}$$
$$f(x)\,g_m^{r-1}[F(x)]\,dx$$

Integrating the first term in left hand side by parts, we get

$$\frac{jC_{r-1}}{\gamma_r(r-1)!}\int_0^{\infty}x^{j-1}\left[\overline{F}(x)\right]^{\gamma_r}g_m^{r-1}[F(x)]\,dx$$
$$.=\frac{jC_{r-1}}{\gamma_r(r-1)!}\int_0^{\infty}x^{j-1}\left[\overline{F}(x)\right]^{\gamma_r-1}\frac{\left[1-\overline{\alpha}e^{-\lambda(x)}\right]}{\lambda'(x)}$$
$$f(x)\,g_m^{r-1}[F(x)]\,dx$$

This is implies that

$$\frac{jC_{r-1}}{\gamma_r(r-1)!}\int_0^{\infty}x^{j-1}\left[\overline{F}(x)\right]^{\gamma_r-1}g_m^{r-1}[F(x)]$$
$$\left\{\overline{F}(x)-\frac{\left[1-\overline{\alpha}e^{-\lambda(x)}\right]}{\lambda'(x)}f(x)\right\}dx$$
$$=0 \quad (25)$$

Now applying a generalization of the Muntz-Szasz theorem (Hwang and Lin [9]) to equation (25), we get $\frac{\overline{F}(x)}{f(x)}=\frac{\left[1-\overline{\alpha}e^{-\lambda(x)}\right]}{\lambda'(x)}$.

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