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Some New Perturbed Generalizations of Ostrowski-Grüss Type Inequalities for Bounded Differentiable Mappings and Applications

Shunfeng Wang[1], *Qiaoling Xue* [2] *and Wenjun Liu*[2,*]

[1] Binjiang College, Nanjing University of Information Science and Technology, Nanjing 210044, China
[2] College of Mathematics and Statistics, Nanjing University of Information Science and Technology, Nanjing 210044, China

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Abstract: In this note, we establish some new perturbed generalizations of Ostrowski-Grüss type inequalities with a parameter for bounded differentiable mappings. Our results in special cases give new bounds for Ostrowski-Grüss type or Ostrowski type inequalities. Some applications to probability density functions are also given.

Keywords: Ostrowski-Grüss type inequalities, Differentiable mappings, Ostrowski inequality

1 Introduction

The following Ostrowski-Grüss type integral inequality was proved by Dragomir and Wang in [2].

Theorem 1. *Let $I \subset \mathbf{R}$ be an open interval, $a, b \in I, a < b$. If $f : I \to \mathbf{R}$ is a differentiable function such that there exist constants $\gamma, \Gamma \in \mathbf{R}$, with $\gamma \leq f'(x) \leq \Gamma$, $x \in [a,b]$. Then for all $x \in [a,b]$, we have*

$$\left| f(x) - \frac{f(b)-f(a)}{b-a}\left(x - \frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)dt \right| \leq \frac{1}{4}(b-a)(\Gamma - \gamma). \tag{1}$$

In [1], Cheng not only gave a sharp version of the above inequality but also generalized it as follows.

Theorem 2. *Let the assumptions of Theorem 1 hold. Then*

$$\left| \frac{1}{2} f(x) - \frac{(x-b)f(b)-(x-a)f(a)}{2(b-a)} - \frac{1}{b-a}\int_a^b f(t)dt \right| \leq \frac{(x-a)^2+(b-x)^2}{8(b-a)}(\Gamma - \gamma), \tag{2}$$

for all $x \in [a,b]$.

In [9], two perturbations of an Ostrowski type inequality were established. Recently in [10], the present authors obtained the following perturbed generalization of Ostrowski-Grüss type inequality for bounded differentiable mapping with a parameter, which not only generalize Theorem 2, but also give some other interesting inequalities as special cases.

Theorem 3. *Let the assumptions of Theorem 1 hold. Then*

$$\left| \left(1 - \frac{\lambda}{2}\right) f(x) - \lambda \frac{(x-b)f(b)-(x-a)f(a)}{2(b-a)} - \frac{\Gamma+\gamma}{2}(1-\lambda)\left(x - \frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)dt \right| \leq \left(1 - \lambda + \frac{\lambda^2}{2}\right) \frac{(x-a)^2+(b-x)^2}{4(b-a)}(\Gamma - \gamma), \tag{3}$$

for all $x \in [a,b]$ and $\lambda \in [0,2]$.

More recently, Theorem 3 was proved for general time scales by Tuna and Daghan in [8]. Sarikaya [6] established a similar inequality of Ostrowski-type involving functions of two independent variables. In [7], Set, Sarikaya and Ahmad improved and further generalized some Čebyšev type inequalities involving functions whose derivatives belong to L_p spaces via

* Corresponding author e-mail: wjliu@nuist.edu.cn

certain integral identities. In [3], Liu derived the following sharp generalized Ostrowski-Grüss inequality by using a variant of Grüss inequality (See also [4,5] for other related works).

Theorem 4. *Let the assumptions of Theorem 1 hold. Then*

$$\left|\left(1-\frac{\lambda}{2}\right)f(x)-\lambda\frac{(x-b)f(b)-(x-a)f(a)}{2(b-a)}\right.$$
$$\left.-S(1-\lambda)\left(x-\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)dt\right|$$
$$\leq\frac{\Gamma-\gamma}{2}I(\lambda,x), \tag{4}$$

for all $x\in[a,b]$ *and* $\lambda\in[0,2]$, *where* $S=(f(b)-f(a))/(b-a)$ *and*

$$I(\lambda,x)=\begin{cases}\left[\frac{a+b}{2}-\left(1-\frac{\lambda}{2}\right)a-\frac{\lambda}{2}x\right]^2, \\ \quad x\in\left[a,\ \frac{a+(1-\lambda)b}{2-\lambda}\right], \\ \frac{1}{4}[1+(\lambda-1)^2][(x-a)^2+(b-x)^2], \\ \quad x\in\left(\frac{a+(1-\lambda)b}{2-\lambda},\frac{(1-\lambda)a+b}{2-\lambda}\right), \\ \left[\frac{\lambda}{2}x+\left(1-\frac{\lambda}{2}\right)b-\frac{a+b}{2}\right]^2, \\ \quad x\in\left[\frac{(1-\lambda)a+b}{2-\lambda},\ b\right],\end{cases} \tag{5}$$

for $\lambda\in[0,1]$, *and*

$$I(\lambda,x)=\begin{cases}\left[\frac{a+b}{2}-\frac{\lambda}{2}a-\left(1-\frac{\lambda}{2}\right)x\right]^2, \\ \quad x\in\left[a,\ \frac{a+(\lambda-1)b}{\lambda}\right], \\ \frac{1}{4}[1+(\lambda-1)^2][(x-a)^2+(b-x)^2], \\ \quad x\in\left(\frac{a+(\lambda-1)b}{\lambda},\frac{(\lambda-1)a+b}{\lambda}\right), \\ \left[\left(1-\frac{\lambda}{2}\right)x+\frac{\lambda}{2}b-\frac{a+b}{2}\right]^2, \\ \quad x\in\left[\frac{(\lambda-1)a+b}{\lambda},\ b\right],\end{cases} \tag{6}$$

for $\lambda\in[1,2]$.

In this note, motivated by above research, we shall establish another perturbed generalization of Ostrowski-Grüss type inequalities with a parameter for bounded differentiable mappings. Our results in special cases give new bounds for Ostrowski-Grüss type or Ostrowski type inequalities. Some applications to probability density functions are also given.

2 Main Results

Theorem 5. *Let* $I\subset\mathbf{R}$ *be an open interval,* $a,b\in I, a<b$. *If* $f:I\to\mathbf{R}$ *is a differentiable function such that there exist constants* $\gamma,\Gamma\in\mathbf{R}$, *with* $\gamma\leq f'(x)\leq\Gamma$, $x\in[a,b]$, *then, we have*

$$\left|\left(1-\frac{\lambda}{2}\right)f(x)-\lambda\frac{(x-b)f(b)-(x-a)f(a)}{2(b-a)}\right.$$
$$\left.-\gamma(1-\lambda)\left(x-\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)dt\right|$$
$$\leq\frac{1}{2}[1+|\lambda-1|]\left[\frac{b-a}{2}+\left|x-\frac{a+b}{2}\right|\right](S-\gamma), \tag{7}$$

and

$$\left|\left(1-\frac{\lambda}{2}\right)f(x)-\lambda\frac{(x-b)f(b)-(x-a)f(a)}{2(b-a)}\right.$$
$$\left.-\Gamma(1-\lambda)\left(x-\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)dt\right|$$
$$\leq\frac{1}{2}[1+|\lambda-1|]\left[\frac{b-a}{2}+\left|x-\frac{a+b}{2}\right|\right](\Gamma-S), \tag{8}$$

for all $x\in[a,b]$ *and* $\lambda\in[0,2]$, *where* $S=(f(b)-f(a))/(b-a)$.

Proof. Let us define the following mapping as in [10]:

$$K(x,t)=\begin{cases}t-\left(a+\lambda\frac{x-a}{2}\right), & t\in[a,x], \\ t-\left(b-\lambda\frac{b-x}{2}\right), & t\in(x,b].\end{cases} \tag{9}$$

Integrating by parts, we have

$$\frac{1}{b-a}\int_a^b K(x,t)f'(t)dt=\left(1-\frac{\lambda}{2}\right)f(x)$$
$$-\lambda\frac{(x-b)f(b)-(x-a)f(a)}{2(b-a)}-\frac{1}{b-a}\int_a^b f(t)dt. \tag{10}$$

We also have

$$\frac{1}{b-a}\int_a^b K(x,t)dt=(1-\lambda)\left(x-\frac{a+b}{2}\right). \tag{11}$$

Let $C\in\mathbf{R}$ be a constant. From (10) and (11), it follows that

$$\frac{1}{b-a}\int_a^b K(x,t)[f'(t)-C]dt$$
$$=\left(1-\frac{\lambda}{2}\right)f(x)-\lambda\frac{(x-b)f(b)-(x-a)f(a)}{2(b-a)}$$
$$-C(1-\lambda)\left(x-\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)dt. \tag{12}$$

If we choose $C=\gamma$ in (12), then we get

$$\frac{1}{b-a}\int_a^b K(x,t)[f'(t)-\gamma]dt$$
$$=\left(1-\frac{\lambda}{2}\right)f(x)-\lambda\frac{(x-b)f(b)-(x-a)f(a)}{2(b-a)}$$
$$-\gamma(1-\lambda)\left(x-\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)dt. \tag{13}$$

On the other hand, we have

$$\left|\frac{1}{b-a}\int_a^b K(x,t)[f'(t)-\gamma]dt\right| \leq \max_{t\in[a,b]}|K(x,t)|\frac{1}{b-a}\int_a^b |f'(t)-\gamma|dt. \quad (14)$$

We also have

$$\max_{t\in[a,b]}|K(x,t)| = \frac{1}{2}[1+|\lambda-1|]\left[\frac{b-a}{2}+\left|x-\frac{a+b}{2}\right|\right] \quad (15)$$

and (see [9])

$$\frac{1}{b-a}\int_a^b |f'(t)-\gamma|dt = S-\gamma. \quad (16)$$

From (14)-(16), it follows that

$$\left|\frac{1}{b-a}\int_a^b K(x,t)\left[f'(t)-\gamma\right]dt\right| \leq \frac{1}{2}[1+|\lambda-1|]\left[\frac{b-a}{2}+\left|x-\frac{a+b}{2}\right|\right](S-\gamma). \quad (17)$$

From (13) and (17) we see that (7) holds.

If we choose $C=\Gamma$ in (12), then we get (8) similarly.

Corollary 1. *Under the assumptions of Theorem 5 and with $\lambda=1$, we have*

$$\left|\frac{1}{2}f(x)-\frac{(x-b)f(b)-(x-a)f(a)}{2(b-a)}-\frac{1}{b-a}\int_a^b f(t)dt\right| \leq \frac{1}{2}\left[\frac{b-a}{2}+\left|x-\frac{a+b}{2}\right|\right](S-\gamma) \quad (18)$$

and

$$\left|\frac{1}{2}f(x)-\frac{(x-b)f(b)-(x-a)f(a)}{2(b-a)}-\frac{1}{b-a}\int_a^b f(t)dt\right| \leq \frac{1}{2}\left[\frac{b-a}{2}+\left|x-\frac{a+b}{2}\right|\right](\Gamma-S), \quad (19)$$

for all $x\in[a,b]$.

Corollary 2. *Under the assumptions of Theorem 5 and with $\lambda=0$, we have*

$$\left|f(x)-\gamma\left(x-\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)dt\right| \leq \left[\frac{b-a}{2}+\left|x-\frac{a+b}{2}\right|\right](S-\gamma) \quad (20)$$

and

$$\left|f(x)-\Gamma\left(x-\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)dt\right| \leq \left[\frac{b-a}{2}+\left|x-\frac{a+b}{2}\right|\right](\Gamma-S), \quad (21)$$

for all $x\in[a,b]$.

Corollary 3. *Under the assumptions of Theorem 5 and with $\lambda=2$, we have*

$$\left|\gamma\left(x-\frac{a+b}{2}\right)-\frac{(x-b)f(b)-(x-a)f(a)}{b-a}-\frac{1}{b-a}\int_a^b f(t)dt\right| \leq \left[\frac{b-a}{2}+\left|x-\frac{a+b}{2}\right|\right](S-\gamma) \quad (22)$$

and

$$\left|\Gamma\left(x-\frac{a+b}{2}\right)-\frac{(x-b)f(b)-(x-a)f(a)}{b-a}-\frac{1}{b-a}\int_a^b f(t)dt\right| \leq \left[\frac{b-a}{2}+\left|x-\frac{a+b}{2}\right|\right](\Gamma-S), \quad (23)$$

for all $x\in[a,b]$.

Corollary 4. *Under the assumptions of Theorem 5 and with $x=\frac{a+b}{2}$, we have*

$$\left|\left(1-\frac{\lambda}{2}\right)f\left(\frac{a+b}{2}\right)+\frac{\lambda}{2}\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)dt\right| \leq \frac{b-a}{4}[1+|\lambda-1|](S-\gamma) \quad (24)$$

and

$$\left|\left(1-\frac{\lambda}{2}\right)f\left(\frac{a+b}{2}\right)+\frac{\lambda}{2}\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)dt\right| \leq \frac{b-a}{4}[1+|\lambda-1|](\Gamma-S), \quad (25)$$

for all $\lambda\in[0,2]$.

Corollary 5. *Under the assumptions of Theorem 5 and with $x=a$, we have*

$$\left|\left(1-\frac{\lambda}{2}\right)f(a)+\frac{\lambda}{2}f(b)+\frac{\gamma}{2}(1-\lambda)(b-a)-\frac{1}{b-a}\int_a^b f(t)dt\right| \leq \frac{b-a}{2}[1+|\lambda-1|](S-\gamma) \quad (26)$$

and

$$\left|\left(1-\frac{\lambda}{2}\right)f(a)+\frac{\lambda}{2}f(b)+\frac{\Gamma}{2}(1-\lambda)(b-a)-\frac{1}{b-a}\int_a^b f(t)dt\right| \leq \frac{b-a}{2}[1+|\lambda-1|](\Gamma-S), \quad (27)$$

for all $\lambda\in[0,2]$.

Corollary 6. *Under the assumptions of Theorem 5 and with $x=b$, we have*

$$\left|\left(1-\frac{\lambda}{2}\right)f(b)+\frac{\lambda}{2}f(a)-\frac{\gamma}{2}(1-\lambda)(b-a)-\frac{1}{b-a}\int_a^b f(t)dt\right|\le\frac{b-a}{2}[1+|\lambda-1|](S-\gamma) \quad (28)$$

and

$$\left|\left(1-\frac{\lambda}{2}\right)f(b)+\frac{\lambda}{2}f(a)-\frac{\Gamma}{2}(1-\lambda)(b-a)-\frac{1}{b-a}\int_a^b f(t)dt\right|\le\frac{b-a}{2}[1+|\lambda-1|](\Gamma-S), \quad (29)$$

for all $\lambda\in[0,2]$.

Remark. We point out that, as in [8] and [6], Theorem 5 may be proved for general time scales or be similarly extended to inequalities involving functions of two independent variables. The details are left for the interested reader.

3 Application to probability density functions

Now, let X be a random variable taking values in the finite interval $[a,b]$, with the probability density function $f:[a,b]\to\mathbf{R}^+$ and with the cumulative distribution function

$$F(x)=Pr(X\le x)=\int_a^x f(t)dt.$$

The following result holds:

Theorem 6. *With the above assumptions and that the probability density function f satisfies $\gamma\le f(x)\le\Gamma$, $x\in[a,b]$ for some constants $\gamma,\Gamma\in\mathbf{R}^+$, then, we have*

$$\left|\left(1-\frac{\lambda}{2}\right)Pr(X\le x)-\frac{\lambda}{2}\frac{x-b}{b-a}-\gamma(1-\lambda)\left(x-\frac{a+b}{2}\right)-\frac{b-E(X)}{b-a}\right|\le\frac{1}{2}[1+|\lambda-1|]\left[\frac{b-a}{2}+\left|x-\frac{a+b}{2}\right|\right]\left(\frac{1}{b-a}-\gamma\right), \quad (30)$$

and

$$\left|\left(1-\frac{\lambda}{2}\right)Pr(X\le x)-\frac{\lambda}{2}\frac{x-b}{b-a}-\Gamma(1-\lambda)\left(x-\frac{a+b}{2}\right)-\frac{b-E(X)}{b-a}\right|\le\frac{1}{2}[1+|\lambda-1|]\left[\frac{b-a}{2}+\left|x-\frac{a+b}{2}\right|\right]\left(\Gamma-\frac{1}{b-a}\right), \quad (31)$$

for all $x\in[a,b]$ and $\lambda\in[0,2]$, where $E(X)$ is the expectation of X.

Proof. By choosing $f=F$ in (7) and (8), and taking into account $F(a)=0$, $F(b)=1$ and

$$E(X)=\int_a^b tdF(t)=b-\int_a^b F(t)dt,$$

we obtain (30) and (31).

In particular, we have:

Corollary 7. *With the above assumptions, we have the inequalities*

$$\left|\frac{1}{2}Pr(X\le x)-\frac{1}{2}\frac{x-b}{b-a}-\frac{b-E(X)}{b-a}\right|\le\frac{1}{2}\left[\frac{b-a}{2}+\left|x-\frac{a+b}{2}\right|\right]\left(\frac{1}{b-a}-\gamma\right), \quad (32)$$

and

$$\left|\frac{1}{2}Pr(X\le x)-\frac{1}{2}\frac{x-b}{b-a}-\frac{b-E(X)}{b-a}\right|\le\frac{1}{2}\left[\frac{b-a}{2}+\left|x-\frac{a+b}{2}\right|\right]\left(\Gamma-\frac{1}{b-a}\right), \quad (33)$$

for all $x\in[a,b]$.

Proof. We set $\lambda=1$ in Theorem 6.

Corollary 8. *With the above assumptions, we have the inequalities*

$$\left|\gamma\left(x-\frac{a+b}{2}\right)-\frac{x-b}{b-a}-\frac{b-E(X)}{b-a}\right|\le\left[\frac{b-a}{2}+\left|x-\frac{a+b}{2}\right|\right]\left(\frac{1}{b-a}-\gamma\right), \quad (34)$$

and

$$\left|\Gamma\left(x-\frac{a+b}{2}\right)-\frac{x-b}{b-a}-\frac{b-E(X)}{b-a}\right|\le\left[\frac{b-a}{2}+\left|x-\frac{a+b}{2}\right|\right]\left(\Gamma-\frac{1}{b-a}\right), \quad (35)$$

for all $x\in[a,b]$.

Proof. We set $\lambda=2$ in Theorem 6.

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