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Solution of Third Order Fractional Boundary Value Problems by Using Greens Function Method

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Abstract: In this paper, we investigate a solution for certain third order non-homogeneous fractional boundary value problem associated with a fractional boundary condition. To such given problem we construct a Green's function of a fractional order that corresponds to the given problem and provide direct solutions in details. Moreover, we introduce the fractional Leibniz integralrule and discuss several Green's function properties. As an application to this theory, we provide some fractional boundary value problems and obtain their exact solution.

Keywords: Green's function; Derivative; Leibniz integral rule; Boundary valued problem.

1 Introduction

The idea of fractional calculus and fractional boundary value problems have attracted the interest of many researchers due to their excellence in explaining the dynamics of several realworld systems in diffusion, engineering, biology, physics, economics, chemistry, commerce, chaos theory and many others to mention but a few (see, e.g., [1,2,3,4,5,6,7,8,9,10]). The idea of the concept of the fractional derivative was first introduced by LHospital in 1695. Since then, various definitions of the fractional derivative have been given by many authors such as Grnwald Letnikov [11], Atangana-Baleanu [12], Hadamard [13] and Caputo-Fabrizio [14] as well. The most popular ones are the definitions of Riemann-Liouville and Caputo [15] (see, also [16]). In 2014, a new definition of the conformable derivative has been proposed by Khalil et. al [17] named as the conformable fractional derivative. The conformable fractional derivative is a natural definition which gives simple, easy and exact solutions (see, e.g., [17,18,19,20,21,22]). In literature, various methods for solving boundary value problems were given by many scientists from both practical and theoretical points of view. For an example, we recall here the iterative method which has been used for solving high-order non-linear fractional boundary value problem [23], the continuous analytic method for solving the two-point boundary value problem [24] and the Green's function method as well. The Green's function is defined as the response of an impulse of certain inhomogeneous linear differential operator defined over domain with specified boundary or initial conditions(see, e.g., [25], [26] and [27]).

In our paper we establish the following type of fractional non-homogeneous boundary value problem of the third order with a fractional boundary condition

$$D^{\alpha}D^{\alpha}D^{\alpha}x(t) = f(t)\ (\leq t \leq 1),$$

$$x(0) = D^{\alpha}x(\mu) = D^{\alpha}D^{\alpha}x(1) = 0,$$

where $0 < \alpha \leq 1$ and μ is a boundary point satisfying $0 < \mu < 1$. The proposed problem maybe can be solved by other methods like fractional laplace transform, fractional Fourier series with separation of variables, but we can't assure the existence of solution and that will be exact. Then, we define the fractional Leibniz integral rule which will be useful

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in the proofs of existence solution theorem of the given problem and Green's function properties theorems. Therefore, we construct the Green's function corresponding to the established fractional boundary value problem in one theorem. Moreover, as an application to this theory we give a solution to the fractional boundary value problem by using our results and obtain certain exact solution. Finally, we include some graphs performed by using Mathematica 12 to illustrate our given results.

2 Fundamentals

Definition 1. *Let $f : [0;\infty) \to \mathbb{R}$ be a real valued function. Then, the conformable fractional derivative of order α of a function f is given by*

$$D^{\alpha}(f)(t) = \lim_{\varepsilon \to 0} \frac{f(t+\varepsilon t^{1-\alpha}) - f(t)}{\varepsilon} \quad for\ all\ t > 0,\ \alpha \in (0,1).$$

It is clear that when f is $\alpha-$ differentiable function on the open interval $(0,a)$, $a > 0$, and that $\lim_{t\to 0^+} D^{\alpha} f(t)$ exists, then $D^{\alpha} f(0) = \lim_{t\to 0^+} D^{\alpha} f(t)$.

One can easily establish that D^{α} satisfies all desired properties included in the following theorem.

Theorem 1. *Let f and g be $\alpha-$differentiable at a point $t > 0$, $\alpha \in (0,1]$. Then, the following identities hold true.*
(1) *For all $a,b \in \mathbb{R}$, $D^{\alpha}(af+bg) = aD^{\alpha}(f)+bD^{\alpha}(g)$.*
(2) *For all $p \in \mathbb{R}$, $D^{\alpha}(t^p) = pt^{p-\alpha}$.*
(3) *For all constant functions $f(t) = \lambda$, $D^{\alpha}(\lambda) = 0$.*
(4) *$D^{\alpha}(fg) = gD^{\alpha}(f)+fD^{\alpha}(g)$.*
(5) *$D^{\alpha}(\frac{f}{g}) = \frac{gD^{\alpha}(f)-fD^{\alpha}(g)}{g^2}$.*
(6) *$D^{\alpha}(f)(t) = t^{1-\alpha}\frac{df}{dt}(t)$, provided that f is differentiable.*

Proof. See [17].

To illustrate our idea, we present the following example including conformable derivative of certain class of functions.

Example 1. The following hold true.
(1) For all $p \in \mathbb{R}$, we have $D^{\alpha}(t^p) = pt^{p-\alpha}$.
(2) $D^{\alpha}(1) = 0$.
(3) For $c \in \mathbb{R}$, we have $D^{\alpha}(e^{ct}) = ct^{1-\alpha}e^{ct}$.
(4) For $b \in \mathbb{R}$, we have $D^{\alpha}(\sin\ bt) = bt^{1-\alpha}\ \cos\ bt$.
(5) For $b \in \mathbb{R}$, we have $D^{\alpha}(\cos\ bt) = -bt^{1-\alpha}\ \sin\ bt$.
(6) $D^{\alpha}(\frac{1}{\alpha}t^{\alpha}) = 1$.
(7) $D^{\alpha}(\sin\frac{1}{\alpha}t^{\alpha}) = \cos\frac{1}{\alpha}t^{\alpha}$.
(8) $D^{\alpha}(\cos\frac{1}{\alpha}t^{\alpha}) = -\sin\frac{1}{\alpha}t^{\alpha}$.
(9) $D^{\alpha}(e^{\frac{1}{\alpha}t^{\alpha}}) = e^{\frac{1}{\alpha}t^{\alpha}}$.

Definition 2. *Let f be a real valued function defined on $[0;\infty)$ and $\alpha \in (0,1)$. Then, the conformable integral of order α of a function f is defined by*

$$I^a_{\alpha}(f)(t) = I^a_1(t^{\alpha-1}f(t)) = \int_a^t \frac{f(x)}{x^{1-\alpha}}dx,$$

when the integral at the right-hand side is a Riemann integral.

Following is an interesting result that we need in the sequel.

Theorem 2. *Let $f : [0;\infty) \to \mathbb{R}$ be a function continuous in the domain of I_{α} and $0 < \alpha \le 1$. Then, we have*

$$D^{\alpha}I^a_{\alpha}(f)(t) = f(t),\ for\ t \ge a\ and\ 0 < \alpha \le 1.$$

Proof. See [17].

3 Fractional Green's function and its properties

We start our main results by establishing the fractional Leibniz theorem as follows.

Theorem 3.*(Fractional Leibniz integral rule)*
Let the α-differentiation of a function f under the integral sign be given by

$$\int_{a(x)}^{b(x)} f(x,t)d_\alpha t, \text{ where } 0<\alpha\leq 1 \text{ and } -\infty<a(x),b(x)<+\infty.$$

Then, the α-derivative of this integral can be expressed as

$$\frac{d^\alpha}{dx}\left(\int_{a(x)}^{b(x)} f(x,t)d_\alpha t\right)=f(x,b(x)).\frac{d^\alpha}{dx}b(x)-f(x,a(x)).\frac{d^\alpha}{dx}a(x)+\int_{a(x)}^{b(x)}\frac{\partial^\alpha}{\partial x}f(x,t)d_\alpha t,$$

where the partial derivative indicates that only the variation of $f(x,t)$ with x is considered inside the integral sign. Notice that when b and a be given as constants, not as functions of x, then we obtain the following special case

$$\frac{d^\alpha}{dx}\left(\int_a^b f(x,t)d_\alpha t\right)=\int_a^b\frac{\partial^\alpha}{\partial x}f(x,t)d_\alpha t.$$

*Proof.*Let us start proving the case where the upper and lower bounds a and b of the integral are constants. Let x and $x+h$ be within the interval $[x_0,x_1]$, $h>0$. Then, by using the Fubini's theorem we write

$$\begin{aligned}\int_x^{x+h}\int_a^b\frac{\partial^\alpha}{\partial x}f(x,t)d_\alpha t\ d_\alpha x &= \int_a^b\int_x^{x+h}\frac{\partial^\alpha}{\partial x}f(x,t)d_\alpha x\ d_\alpha t\\ &=\int_a^b\left(f(x+h,t)-f(x,t)\right)d_\alpha t\\ &=\int_a^b f(x+h,t)d_\alpha t-\int_a^b f(x,t)d_\alpha t.\end{aligned}$$

The definite integrals at the right-hand side are indeed well-defined as the partial derivative $\frac{\partial^\alpha}{\partial x}f(x,t)$ is continuous on $[a,b]\times[x_0,x_1]$ and, hence uniformly continuous. Therefore, the integrals with $d_\alpha t$ or $d_\alpha x$ are indeed continuous and integrable. Therefore,

$$\begin{aligned}\frac{\int_a^b f(x+h,t)d_\alpha t-\int_a^b f(x,t)d_\alpha t}{h}&=\frac{1}{h}\int_x^{x+h}\int_a^b\frac{\partial^\alpha}{\partial x}f(x,t)d_\alpha t\ d_\alpha x\\ &=\frac{F(x+h)-F(x)}{h},\end{aligned}$$

where,

$$F(u)=\int_{x_0}^{u}\int_a^b\frac{\partial^\alpha}{\partial x}f(x,t)d_\alpha t\ d_\alpha x.$$

It is clear that F is α-differentiable with the α-derivative $\int_a^b\frac{\partial^\alpha}{\partial x}f(x,t)d_\alpha t$ and hence we may consider the limit as h approaches zero. The limit on the left hand side is

$$\frac{d^\alpha}{dx}\int_a^b f(x,t)d_\alpha t,$$

whereas for the right hand side becomes

$$\frac{d^\alpha}{dx}F(x)=\int_a^b\frac{\partial^\alpha}{\partial x}f(x,t)d_\alpha t.$$

Thus, have proved the desired result

$$\frac{d^\alpha}{dx}\int_a^b f(x,t)d_\alpha t=\int_a^b\frac{\partial^\alpha}{\partial x}f(x,t)d_\alpha t.$$

Similarly, we prove the following result for variable limits.

Now, we turn back to our main problem which is a fractional boundary value problem in the following type

$$D^{\alpha}D^{\alpha}D^{\alpha}x(t)=f(t),\quad 0\leq t\leq 1, \tag{1}$$

$$x(0)=D^{\alpha}x(\mu)=D^{\alpha}D^{\alpha}x(1)=0, \tag{2}$$

where $0<\alpha\leq 1$ and μ is a boundary point satisfying $0<\mu<1$.

Our approach to the existence of solutions associates fractional Green's function. By establishing the following theorems, we can employ a fixed point theorem to obtain solutions for (1) subject to the boundary conditions (2).

Theorem 4. *Let $0<\alpha\leq 1$, $\mu\in(0,1)$ and*

$$D^{\alpha}D^{\alpha}D^{\alpha}x(t)=0,$$

then the corresponding fractional Green's function of the above fractional homogeneous problem with the conditions (2) is given as

$$G(t,s)=\begin{cases} s\in[0,\mu]: & \begin{cases} u(t,s) & :0\leq t\leq s\leq 1\\ x(0,s) & :0\leq s\leq t\leq 1\end{cases}\\ \\ s\in[\mu,1]: & \begin{cases} u(t,\mu) & :0\leq t\leq s\leq 1\\ u(t,\mu)+x(t,s) & :0\leq s\leq t\leq 1\end{cases}\end{cases} \tag{3}$$

where

$$u(t,s)=\frac{2t^{\alpha}s^{\alpha}-t^{2\alpha}}{2\alpha^2} \tag{4}$$

and

$$x(t,s)=\frac{(t^{\alpha}-s^{\alpha})^2}{2\alpha^2}, \tag{5}$$

is the Cauchy function. Then, for any continuous function h we have

$$x(t)=\int_0^1 G(t,s)h(s)d_{\alpha}s=\int_0^1 G(t,s)h(s)s^{\alpha-1}ds$$

is the solution of the following fractional boundary value problem

$$D^{\alpha}D^{\alpha}D^{\alpha}x(t)=h(t),$$

with the boundary condition (2), where G is the fractional Green's function constructed above.

Proof. First, let $t\in[0,\mu]$. Then, we write

$$x(t)=\int_0^t x(0,s)h(s)d_{\alpha}s+\int_t^{\mu}u(t,s)h(s)d_{\alpha}s+\int_{\mu}^1 u(t,\mu)h(s)d_{\alpha}s. \tag{6}$$

Clearly $x(0)=0$, by (5). By differentiating (6), using the fractional Leibniz theorem, we get

$$\begin{aligned}D^{\alpha}x(t)&=x(t,0)h(t)t^{\alpha-1}t^{1-\alpha}+\int_t^{\mu}\left(\frac{s^{\alpha}-t^{\alpha}}{\alpha}\right)h(s)s^{\alpha-1}ds\\ &-u(t,t)h(t)t^{\alpha-1}t^{1-\alpha}+\int_{\mu}^1\left(\frac{\mu^{\alpha}-t^{\alpha}}{\alpha}\right)h(s)s^{\alpha-1}ds\\ &=\frac{t^{2\alpha}}{2\alpha^2}h(t)+\int_t^{\mu}\left(\frac{s^{\alpha}-t^{\alpha}}{\alpha}\right)h(s)s^{\alpha-1}ds\\ &-\frac{t^{2\alpha}}{2\alpha^2}h(t)+\int_{\mu}^1\left(\frac{\mu^{\alpha}-t^{\alpha}}{\alpha}\right)h(s)s^{\alpha-1}ds\\ &=\int_t^{\mu}\left(\frac{s^{\alpha}-t^{\alpha}}{\alpha}\right)h(s)s^{\alpha-1}ds+\int_{\mu}^1\left(\frac{\mu^{\alpha}-t^{\alpha}}{\alpha}\right)h(s)s^{\alpha-1}ds.\end{aligned}$$

Clearly, the boundary condition $D^{\alpha}x(\mu) = 0$ has been met. Once again, differentiating reveals

$$D^{\alpha}D^{\alpha}x(t) = \int_t^{\mu} \left(\frac{-\alpha t^{\alpha-1}t^{1-\alpha}}{\alpha}\right) h(s)s^{\alpha-1}ds + \int_{\mu}^{1} \left(\frac{-\alpha t^{\alpha-1}t^{1-\alpha}}{\alpha}\right) h(s)s^{\alpha-1}ds$$
$$= \int_1^t h(s)s^{\alpha-1}ds = \int_1^t h(s)d_{\alpha}s.$$

It follows that

$$D^{\alpha}D^{\alpha}x(1) = \int_1^1 h(s)d_{\alpha}s = 0.$$

By Theorem 2 we obtain

$$D^{\alpha}D^{\alpha}D^{\alpha}x(t) = D^{\alpha}\int_1^t h(s)d_{\alpha}s = D^{\alpha}I_{\alpha}h(t) = h(t).$$

This proves the claim for this case. Now, let $t \in [\mu, 1]$. Then, we have

$$x(t) = \int_0^{\mu} x(0,s)h(s)d_{\alpha}s + \int_{\mu}^{t} x(t,s)h(s)d_{\alpha}s + u(t,\mu)\int_{\mu}^{1} h(s)d_{\alpha}s. \tag{7}$$

Again $x(0) = 0$, by (5). Differentiating (7), using the fractional Leibniz theorem, implies

$$D^{\alpha}x(t) = \int_{\mu}^{t} \left(\frac{t^{\alpha}-s^{\alpha}}{\alpha}\right) h(s)d_{\alpha}s + \left(\frac{\mu^{\alpha}-t^{\alpha}}{\alpha}\right)\int_{\mu}^{1} h(s)d_{\alpha}s.$$

The boundary condition $D^{\alpha}x(\mu) = 0$ has clearly been met. Therefore, differentiating again reveals

$$D^{\alpha}D^{\alpha}x(t) = \int_1^t h(s)s^{\alpha-1}ds = \int_1^t h(s)d_{\alpha}s.$$

As in the previous case,

$$D^{\alpha}D^{\alpha}x(1) = \int_1^1 h(s)d_{\alpha}s = 0.$$

By Theorem 2 we obtain

$$D^{\alpha}D^{\alpha}D^{\alpha}x(t) = D^{\alpha}\int_1^t h(s)d_{\alpha}s = D^{\alpha}I_{\alpha}h(t) = h(t).$$

This proves the claim in this case as well.

The fractional Green's function that we are constructing above has some properties that we will be presented in the following theorems.

Theorem 5. *Let G be the function expressed by (3). Then, we have*

$$0 < G(t,s) \leq G(\mu,s), \tag{8}$$

where $t \in (0,1]$, $s \in (0,1]$ and the condition $u(1,\mu) > 0$ applies for u in (4), i.e., the inequality

$$\mu > \left(\frac{1}{2}\right)^{\frac{1}{\alpha}}$$

hold.

Proof. Referring to (3), (4) and (5), we see

$$u(s,s) = x(0,s) = \frac{s^{2\alpha}}{2\alpha^2},$$

and

$$x(0,\mu) = u(t,\mu) + x(t,\mu) = \frac{\mu^{2\alpha}}{2\alpha^2},$$

assuring that the function G is well-defined. Moreover, $G(t,s)=0$ for $s=t=0$.
In view of (4), we have

$$\frac{d}{dt}u(t,s)=\frac{t^{\alpha-1}\left(s^{\alpha}-t^{\alpha}\right)}{\alpha}\geq 0 \quad for\ \ s\geq t.$$

Furthermore, in view of (5) we have $x(s,s)=0$ and

$$\frac{d}{dt}x(t,s)=\frac{t^{\alpha-1}\left(t^{\alpha}-s^{\alpha}\right)}{\alpha}\geq 0 \quad for\ \ t\geq s$$

and $u(0,s)=0$ for all s. Thus, G is a non-decreasing function on $[0,\mu]$ and a non-increasing function on $[\mu,1]$. Hence, (8) holds if $G(1,\mu)>0$ satisfies, provided $u(1,\mu)>0$. Equivalently, we write

$$u(1,\mu)=\frac{2\mu^{\alpha}-1}{2\alpha^{2}}>0 \ \Rightarrow\ 2\mu^{\alpha}-1>0 \ \Rightarrow\ \mu^{\alpha}>\frac{1}{2}.$$

Hence

$$\mu>\left(\frac{1}{2}\right)^{\frac{1}{\alpha}}.$$

This finishes the proof of theorem.

Theorem 6.*Let $t,s\in[0,1]$. Then, we have*

$$g(t)G(\mu,s)\leq G(t,s)\leq G(\mu,s) \tag{9}$$

where

$$g(t)=\min\left\{\frac{t^{\alpha}}{\alpha\mu^{2\alpha}}\left[2\alpha\mu^{\alpha}-\alpha t^{\alpha}\right],\frac{1-t}{1-\mu}\right\}. \tag{10}$$

*Proof.*In view of the preceding result, we infer $G(t,s)\leq G(\mu,s)$, $t,\ s\in[0,1]$. Now, we, for the lower bound, continue by following the conditions imposed on the branches of the fractional Green's function (3) and make use of (4) and (5).
1) $0\leq t\leq s\leq\mu$: Here $G(t,s)=u(t,s),\ G(\mu,s)=x(0,s)=\frac{s^{2\alpha}}{2\alpha^{2}}$. For t,s we derive the inequality

$$\frac{u(t,s)}{x(0,s)}\geq\frac{u(t,\mu}{x(0,\mu)}\geq\frac{t^{\alpha}}{\alpha\mu^{2\alpha}}\left[2\alpha\mu^{\alpha}-\alpha t^{\alpha}\right],$$

which implies

$$G(t,s)\geq\frac{t^{\alpha}}{\alpha\mu^{2\alpha}}\left[2\alpha\mu^{\alpha}-\alpha t^{\alpha}\right]G(\mu,s).$$

2) $0\leq t\leq\mu\leq s\leq 1$: This reveals that $G(\mu,s)=u(\mu,\mu)$ and $G(t,s)=u(t,\mu)$. Hence, we obtain

$$G(t,s)\geq\frac{t^{\alpha}}{\alpha\mu^{2\alpha}}\left[2\alpha\mu^{\alpha}-\alpha t^{\alpha}\right]G(\mu,s).$$

3) $0\leq s\leq t\leq\mu$ or $0\leq s\leq\mu\leq t\leq 1$: As $G(t,s)=G(\mu,s)=\frac{s^{2\alpha}}{2\alpha^{2}}$, we derive

$$G(t,s)=G(\mu,s).$$

4) $\mu\leq t\leq s\leq 1$: As in 2), $G(t,s)=u(t,\mu)$ and $G(\mu,s)=u(\mu,\mu)$. Let

$$\begin{aligned} w(t)&=u(t,\mu)-\left(\frac{1-t}{1-\mu}\right)u(\mu,\mu)\\ &=G(t,s)-\left(\frac{1-t}{1-\mu}\right)G(\mu,s). \end{aligned}$$

Then, $w(\mu)=0$, $w^{'}(\mu)>0$, and $w(1)=G(1,s)>0$ by the definition of $w(t)$.
Since w concaves down, we declare that $w(t)\geq 0$ on $[\mu,1]$. Therefore,

$$G(t,s)\geq\left(\frac{1-t}{1-\mu}\right)G(\mu,s).$$

5) $\mu \leq s \leq t \leq 1$: It is noted that $G(\mu,s) = u(\mu,\mu)$, whereas

$$G(t,s) = u(t,\mu) + x(t,s) \geq u(t,\mu).$$

Consequently, by employing w as above we derive

$$G(t,s) \geq \left(\frac{1-t}{1-\mu}\right) G(\mu,s).$$

This finishes the proof of the theorem.

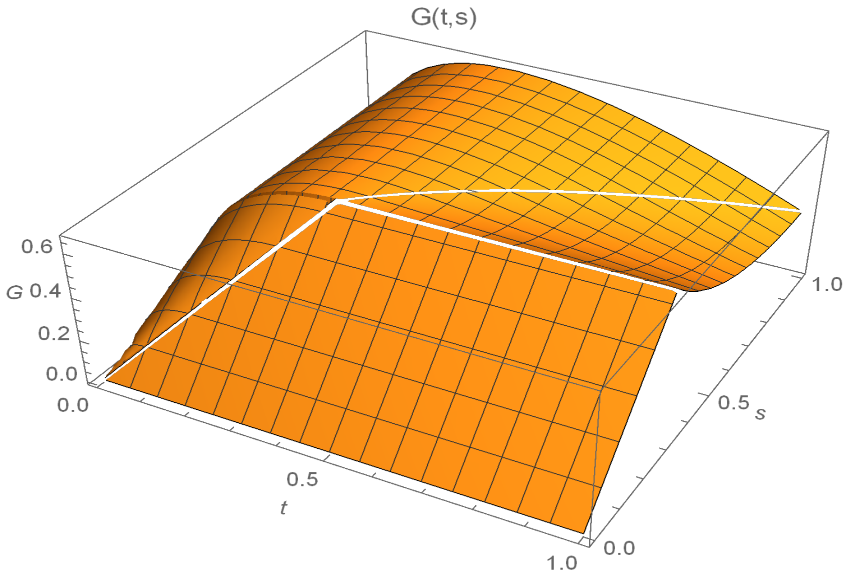

Fig. 1: The fractional Green's function $G(t,s)$ defined in (3)

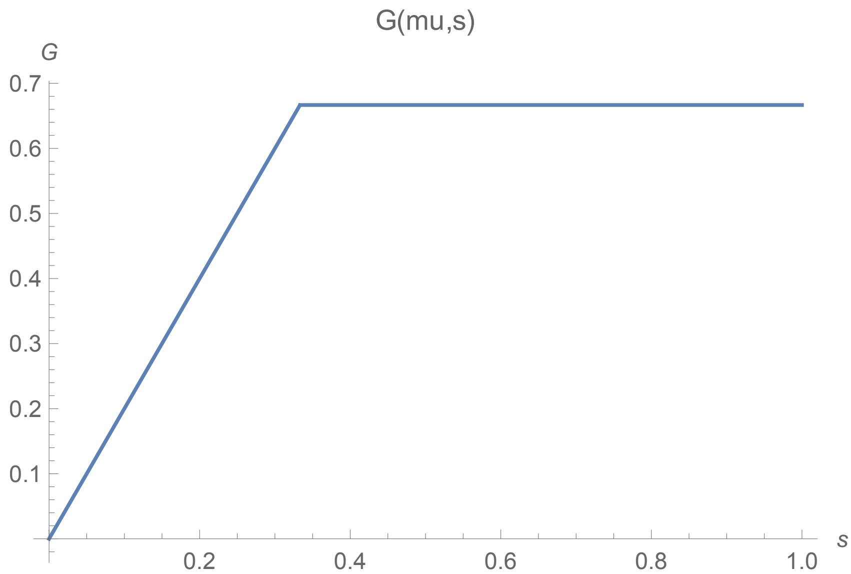

Fig. 2: The fractional Green's function $G(t,s)$ at $t = \mu$

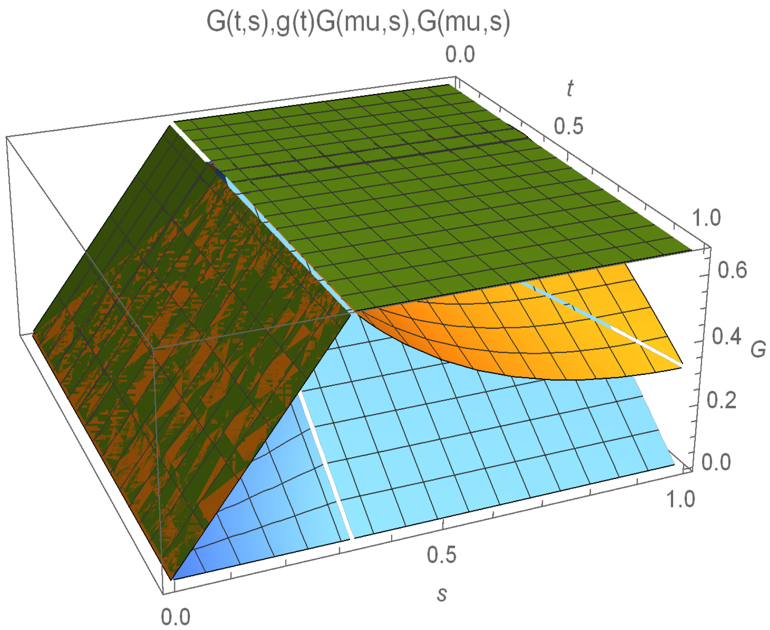

Fig. 3: The fractional Green's function $G(t,s)$ alongside its upper bound $G(\mu,s)$ and its lower bound $g(t)G(\mu,s)$.

Discussion : 1) **Figure 1** represents the Fractional Green's function $G(t,s)$, corresponding to the fractional boundary value problem (1) which was defined in Theorem 4 (Equation (3)).
2) **Figure 2** represents the fractional Green's function $G(t,s)$ at $t=\mu$.
3) **Figure 3** represents the fractional Green's function $G(t,s)$ alongside its upper bound $G(\mu,s)$ and its lower bound $g(t)G(\mu,s)$.

We introduce **Figure 1** and **Figure 2** just to distinguish between $G(t,s)$, $G(\mu,s)$ and $g(t)G(\mu,s)$. **Figure 3** provides the result in Theorem 6 (Inequality (9)), and we see that the fractional Green's function $G(t,s)$ corresponds to the fractional boundary value problem (1) subject to the boundary conditions (2) which was defined in Theorem 4 (Equation (3)) have an upper bound $G(\mu,s)$ and a lower bound $g(t)G(\mu,s)$.

As an application to this theory we consider the following example.

*Example 2.*Consider the following fractional boundary value problem.

$$D^{\alpha}D^{\alpha}D^{\alpha}x(t)=1, \quad 0\le t\le 1, 0<\alpha<1$$

subject to the fractional boundary condition,

$$x(0)=D^{\alpha}x(\mu)=D^{\alpha}D^{\alpha}x(1)=0,$$

where $0<\alpha\le 1$ and the boundary point μ satisfies $0<\mu<1$.

Solution. By Theorem 4, the solution of the considered fractional boundary value problem in our application is given as

$$x(t)=\int_0^1 G(t,s)d_{\alpha}s=\int_0^1 G(t,s)s^{\alpha-1}ds.$$

(1) If $s \in [0,\mu)$, then we have

$$
\begin{aligned}
x(t) &= \int_0^t x(0,s)s^{\alpha-1}ds + \int_t^\mu u(t,s)s^{\alpha-1}ds + \int_\mu^1 u(t,\mu)s^{\alpha-1}ds \\
&= \int_0^t \frac{s^{2\alpha}}{2\alpha^2}s^{\alpha-1}ds + \int_t^\mu \frac{2t^\alpha s^\alpha - t^{2\alpha}}{2\alpha^2}s^{\alpha-1}ds + \int_\mu^1 \frac{2t^\alpha\mu^\alpha - t^{2\alpha}}{2\alpha^2}s^{\alpha-1}ds \\
&= \int_0^t \frac{s^{3\alpha-1}}{2\alpha^2}ds + \int_t^\mu \frac{2t^\alpha s^{2\alpha-1} - t^{2\alpha}s^{\alpha-1}}{2\alpha^2}ds + \int_\mu^1 \frac{2t^\alpha\mu^\alpha - t^{2\alpha}}{2\alpha^2}s^{\alpha-1}ds \\
&= \frac{t^{3\alpha}}{6\alpha^3} + \frac{t^\alpha\mu^{2\alpha} - t^{2\alpha}\mu^\alpha}{2\alpha^3} + \frac{2t^\alpha\mu^\alpha - t^{2\alpha} - 2t^\alpha\mu^{2\alpha} + t^{2\alpha}\mu^\alpha}{2\alpha^3} \\
&= \frac{t^{3\alpha}}{6\alpha^3} + \frac{2t^\alpha\mu^\alpha - t^\alpha\mu^{2\alpha} - t^{2\alpha}}{2\alpha^3}.
\end{aligned}
$$

(2) If $s \in [\mu,1]$, then we have

$$
\begin{aligned}
x(t) &= \int_0^\mu x(0,s)s^{\alpha-1}ds + \int_\mu^t x(t,s)s^{\alpha-1}ds + \int_\mu^1 u(t,\mu)s^{\alpha-1}ds \\
&= \int_0^\mu \frac{s^{3\alpha-1}}{2\alpha^2}ds + \int_\mu^t \frac{(t^\alpha - s^\alpha)^2}{2\alpha^2}s^{\alpha-1}ds + u(t,\mu)\int_\mu^1 s^{\alpha-1}ds \\
&= \frac{\mu^{3\alpha}}{6\alpha^3} + \frac{t^{3\alpha} - \mu^{3\alpha}}{6\alpha^3} + \frac{-t^{2\alpha}\mu^\alpha + t^\alpha\mu^\alpha}{2\alpha^3} + \frac{2t^\alpha\mu^\alpha - t^{2\alpha} - 2t^\alpha\mu^{2\alpha} + t^{2\alpha}\mu^\alpha}{2\alpha^3} \\
&= \frac{t^{3\alpha}}{6\alpha^3} + \frac{2t^\alpha\mu^\alpha - t^\alpha\mu^{2\alpha} - t^{2\alpha}}{2\alpha^3}.
\end{aligned}
$$

Clearly in both cases we have the same solution which equals

$$
x(t) = \frac{t^{3\alpha}}{6\alpha^3} + \frac{2t^\alpha\mu^\alpha - t^\alpha\mu^{2\alpha} - t^{2\alpha}}{2\alpha^3}.
$$

Also, when we check the fractional boundary condition and the solution we get $x(0) = 0$. And

$$
D^\alpha x(t) = \frac{\alpha t^{2\alpha}}{2\alpha^3} + \frac{2\alpha\mu^\alpha - 2\alpha t^\alpha - \alpha\mu^{2\alpha}}{2\alpha^3}, \; then \;\; D^\alpha x(\mu) = 0.
$$

Moreover,

$$
D^\alpha D^\alpha x(t) = \frac{t^\alpha}{\alpha} - \frac{1}{\alpha}, \; so \;\; D^\alpha D^\alpha x(1) = 0
$$

and

$$
D^\alpha D^\alpha D^\alpha x(t) = D^\alpha\left(\frac{t^\alpha}{\alpha} - \frac{1}{\alpha}\right) = 1.
$$

Hence the result is obtained.

4 Conclusion

This paper has considered a new type of fractional non-homogeneous boundary value problem of third order with a fractional boundary condition. To obtain as exact solution of the given fractional boundary value problem, a fractional Green's function, which corresponds to the fractional boundary value problem was defined and used in finding the desired solution. Some exact solution was declared to illustrate that the proposed method is indeed straightforward and appropriate with efficiency for finding solutions the fractional boundary value problem. In addition, several graphs are provided to show accuracy and efficiency of the our given method. In the future work, we aim to develop the fractional Green's function to deal with more complicated types of fractional boundary value problems in the forms :

$$
(i)\; D^\gamma D^\beta D^\alpha x(t) = f(t), \quad 0 \le t \le 1, \quad 0 < \gamma, \beta, \alpha \le 1,
$$

$$x(0) = D^{\alpha}x(\mu) = D^{\beta}D^{\alpha}x(1) = 0,$$

with a boundary point μ satisfying $0 < \mu < 1$.

$$(ii)\ D^{\delta}D^{\gamma}D^{\beta}D^{\alpha}x(t) = f(t), 0 \le t \le 1, 0 < \delta, \gamma, \beta, \alpha \le 1,$$
$$x(0) = D^{\alpha}x(0) = D^{\beta}D^{\alpha}x(1) = D^{\gamma}D^{\beta}D^{\alpha}x(1) = 0.$$

$$(iii)\ D^{\alpha}D^{\alpha}D^{\alpha}D^{\alpha}x(t) = f(t), \quad 0 \le t \le 1, \quad 0 < \alpha \le 1,$$
$$x(0) = D^{\alpha}x(0) = x(1) = D^{\alpha}x(1) = 0.$$

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