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The Hn- Edge Domination in Graphs

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Abstract: The purpose of this paper is initiate new parameter of domination is called the Hn-edge domination number. This number is determined for the regular graph, the graph has spanning cycle subgraph, and some certain graphs. Moreover, the complement of these graph are calculated. Finally, the effect of removal vertex or removal or add an edge are discussed.

Keywords: The hn- edge dominating set, hn- edge domination number, complement of a graph.

1 Introduction

In recent years graph theory has become the language that can address all sciences such as the science of medicine, engineering, chemistry, computer, etc. One of the most important concepts of this science is the concept of domination, which attracts authors because of its wide application in most fields. This concept has been studied in two ways, one of which depends on the vertex set and the other depends on the edge set. In the vertex set, the first appearance of this concept was in [1], and the first to deal with this concept is Ore in his book[2].In mathematics, this concept deals with various fields such as graph[3,4]and [5,6], fuzzy graph [7] and[8], topological graph [9], topological indices [10,11,12,13,14,15,16,17,18,19,20,21], and others. On the other hand, this concept is calculated by means of the edge set, which is the subject of this paper.Let $X \subseteq E$ then the set X is an edge dominating set of G if every edge in the graph G is in the set X or adjacent to some edges in the set X. The minimum cardinality of all edge dominating set is called the edge domination number and denoted by $\bar{\gamma}(G)$. The concept of edge domination was established by Mitchell & Hedetniemi (1977) [22]. After that, the authors began to study this concept and in many different ways to find solutions to different problems by formulating definitions that represent these problems.Through this research, a new parameter of edge domination is introduced which is called the edge $Hn-$domination number. For regular graph, regular graph, the graph has spanning cycle subgraph, and some certain graphs this number is determined. Also, for the complement of the graphs mentioned above .

Definition 1.*Assume that $G(V,E)$ is a graph has no an isolated vertex, a subset $X \subseteq E$ is called $Hn-$edge dominating set $(Hn-EDS)$, if for all $e_1, e_2 \in E-X$, there exist $x_1, x_2 \in X$ such that if e_1 is adjacent to x_1 and e_2 adjacent to x_2 and e_1 adjacent to e_2 , then x_1 and x_2 are adjacent.*

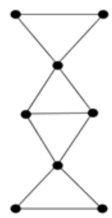


Fig. 1: A graph G

Definition 2.*For a graph $G(V,E)$, if X is $Hn-EDS$, then X is called the minimal $Hn-EDS$ if X has no proper $Hn-EDS$.*

Definition 3.*The smallest cardinality of a minimal $Hn-EDS$ is called $hn-$edge domination number and denoted by $\gamma_{hne}(G)$.*

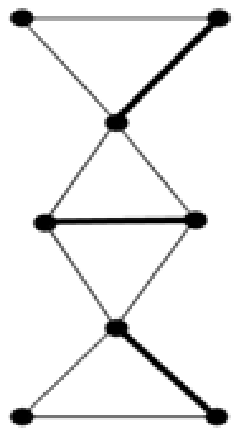


Fig. 2: $Hn-EDS$.

Proposition 1 *Let Gbe a path of order n, then* $\gamma_{hne}(P_n) = \lceil n/2 \rceil - 1$.

Proof. Let $e_1, e_2, \ldots, e_n$ be the edges of P_n consider $D_e = e_{(2+2i)}, i = 0, 1, \ldots, \lceil n/2 \rceil - 1$. One can be concluded that the set D_e is $Hn-EDS$, then $|D_e| = \lceil n/2 \rceil \geqslant \gamma_{hne}$. Assume that there is an edge dominating set M of order $\lceil \frac{n}{2} \rceil - 2$, then there is at least two edges in $E-D_e$ such that these edges are incident and dominated by two distinct edges that are not incident. Then M is not $Hn-EDS$. Thus, D_e is the minimum $Hn-EDS$. Thus, $\gamma_{hne}\ (G) = \lceil \frac{n}{2} \rceil - 1$.

Proposition 2 *If a graph* $G(V,E)$ *of order n and has a spanning cycle subgraph, then* $\gamma_{hne}\ (G) = \lceil \frac{n}{2} \rceil, n \geqslant 3$.

Proof. Let G be a graph has a spanning cycle subgraph C_n and let $e_1, e_2, \ldots, e_n$ be a set of all edges clockwise in C_n, then two cases are appeared as follows.
Case1. If n is even, then consider $D_e = e_{(2+2i)}, i = 0, 1, \ldots, \frac{n}{2-1}$. Every edge in G make in two disjoint vertices, so it is common with at least one edge in D_e. Thus, the set D_e is $Hn-EDS$ of G with $|D_e| = \frac{n}{2} \geqslant \gamma_{hne}$. Suppose that the set D_{e1} is $Hn-EDS$ with $|D_{e1}| = \frac{2}{2-1}$, then there are at least two edges in $E-D_e$ say $e_i, e_{(i+1)}$. These edges are incident the edge e_i is dominated by the edge $e_{(i-1)}$ and the edge $e_{(i+1)}$ is dominated by $e_{(i+2)}$ such that $e_{(i-1)}, e_{(i+2)} \in E-D_e$ are not incident. Then D_{e1} is not $Hn-EDS$ (for example, see Figure 3.(b)). Thus, $D_e$is the minimum dominating set of G , and $\gamma_{hne}(G) = \frac{n}{2}$.
Case2. If n is odd, then let $D_{e1} = e_{(2+2i)}, i = 0, 1, \ldots, \lceil \frac{n}{2} \rceil$ be an EDS in G. It is obvious that the set D_{e1} is not $Hn-EDS$ (as an example, see Figure 3.(d)), since the two edges e_n, e_1 are common by a vertex but the two edges $e_2, e_{(n-1)}$ are not common by any vertex. Thus, we must add either $e_n$or e_1 , say e_1.
Therefore, $D_e = D_{e1} \cup e_1$ is $Hn-EDS$ in G,
so every two incident edges in G are dominated by the same edge in $E-D_e$ At the same manner in *case*1, the set D_e is the minimum $Hn-EDS$. Thus, $\gamma_{hne}(G) = \lceil \frac{n}{2} \rceil$.

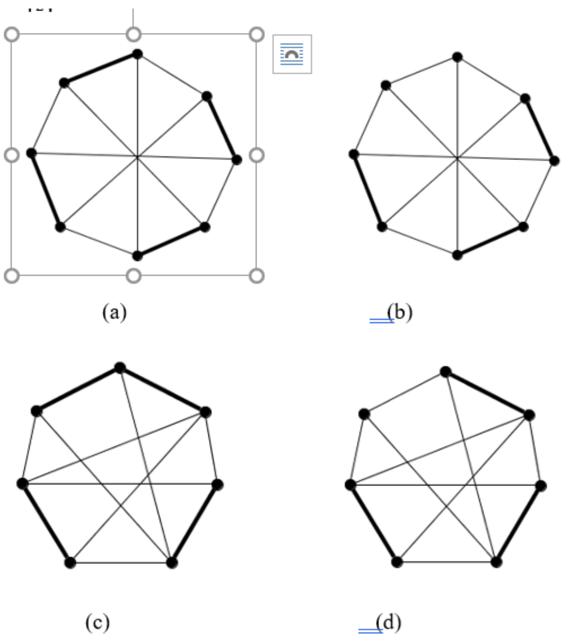


Fig. 3: $Hn-EDS$.

Proposition 3 *Assume that G is a cycle of order n, then* $\gamma_{hne}(C_n) = \lceil \frac{n}{2} \rceil, n \geqslant 3$.

Proof. It is straightforward from proposition2 $\gamma_{hne}(C_n) = \lceil \frac{n}{2} \rceil$. .

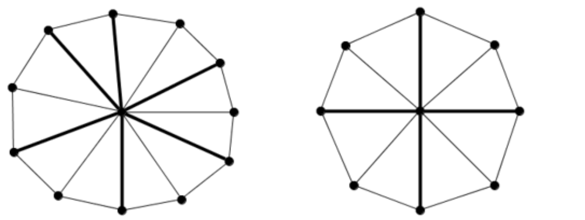
Proposition 4 *Assume that G is a complete of order n,* $\gamma_{hne}(K_n) = \lceil \frac{n}{2} \rceil$, $n \geqslant 3$.

Proof. The graph K_n has a spanning cycle subgraph C_n, then according to proposition2, $\gamma_{hne}(K_n) = \lceil \frac{n}{2} \rceil$.

Proposition 5. *Let a graph G be a wheel of order n, then* $\gamma_{hne}(W_n) = \lceil \frac{n}{2} \rceil$.

Proof. The wheel graph has a spanning subgraph isomorphic to the star graph S_n. Let $e_1, e_2, \ldots, e_n$ be the edges in S_n , so two cases are appeared as follows.
Case1. If n is even, consider $D_e = e_{(1+2i)}, i = 0, 1, \ldots, \frac{n}{2} - 1$. Since all edges in D_e are incident, then D_e is $Hn-EDS$, so $\gamma_{hne}(W_n) \leq |D_e| = \frac{n}{2}$. If we assume that there is dominating set D_{e1} is $hn-edge$ dominating set with $|D_{e1}| = \frac{n}{2} - 1$. Then there are at least two incident edges dominated by not incident edges or not dominated by any edge in D_{e1}. Thus, D_e is a $Hn-MEDS$ (as an example see Figure 4.(a)), so $\gamma_{hne}(W_n) = \frac{n}{2}$.
Case2. If n is odd, consider

$D_e = e_{(1+2i)}, i = 0, 1, \ldots, \lceil \frac{n}{2} \rceil - 1$ such that all edges in D_e are incident, so D_e is $Hn-EDS$ with $\gamma_{hne} \leq |D_e| = \lceil \frac{n}{2} \rceil$. If we assume that there is a set D_{e2} is $Hn-EDS$ with $|D_{e2}| = \lceil \frac{n}{2} \rceil - 1$. Then there is at least one incident edges in $E - D_{e2}$ are not dominated by any edge in D_{e2} and this is a contracts. Thus, the set D_e is $Hn-MEDS$ (as an example see Figure 4.(b)), so $\gamma_{hne}(W_n) = \lceil \frac{n}{2} \rceil$.



(a) Hn-MEDS of W_8 (b) Hn-MEDS of W_{11}

Fig. 4: $Hn-MEDS$ of even and odd in wheel graph.

Proposition 6*Assume that the graph G is a complete bipartite of order $n+m$, then $\gamma_{hne}(K_{(m,n)}) = minm, n$.*

*Proof.*Let $K_{(m,n)}(V,E)$ be bipartite graph with vertex set $V = V_1 \cup V_2$ and $V_1 \cap V_2 = \phi$, and $|V_1| = m$ and $|V_2| = n$,such that $m \leq n$.
If D_e is the set of all edges that incident to one vertex from V_2 , then every edge in D_e is incident to one vertex in the set V_1 , so it dominate on all edges that incident to the same vertex in V_1. Since V_1 is incident to all edges in $K_{(m,n)}$, then D_e is $Hn-EDS$ of $K_{(m,n)}$ with $|D_e| = m$.
If we assume that D_{e1} is $Hn-EDS$ such that $|D_{e1}| = m-1$, then there is at least one edge in $E - D_{e1}$ is not dominated by any edge in D_{e1} and this is a contracts, so D_e is $Hn-MEDS$. Thus, $\gamma_{hne}(K_{(m,n)}) = m$.

Proposition 7*A regular graph $G(V,E)$ has spanning cycle subgraph with C_n such that $|V| = n$.*

*Proof.*Suppose that a graph G is $k-$regular graph, then all regular graphs are Hamiltonian, so a graph G is hamiltonian. By definition of hamiltonion graph, G has spanning cycle subgraph with all the vertices of a graph C_n $(asanexample, seeFigure.5)$. Then we get the result

2 The edge Hn-domination of the complement of some graphs

Proposition 8*Let G be a path of order n, then $\gamma_{hne}((\bar{P}_n)) = n-3$.*

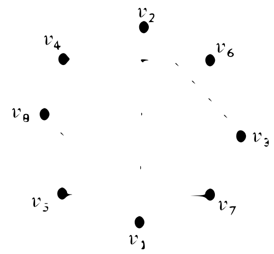


Fig. 5: A spanning cycle of $3-reguler$

*Proof.*Let $v_i, i = 1, 2, .., n$ be the set of the vertices that are adjacent clockwise in P_n and $e_i, i = 1, 2, \ldots, n-2$ be the set of edges in $(\bar{P}_n)$ that incident to pendent vertex v_1 or v_n say v_1 such that $e_1 = v_1v_n, e_2 = v_1v_{(n-1)}, e_3 = v_1v_{(n-2)}$ and so on. Consider $D_e = e_i, i = 1, \ldots, n-3$ is dominating set. Since all edges in D_e are incident, then D_e is $hn-edge$ dominating set with $|D_e| = n-3 \geqslant \gamma_{hne}$.
Now, suppose that D_e is $hn-dominating$ set with $|D_{e1}| = n-4$, then there is at least two incident edges in $E - D_{e1}$. These edges are dominating by two edges in D_{e1} that are not incident. Therefore, D_{e1} is not $hn-dominating$ set. Thus, D_e is minimum $hn-domination$ number of$\overline{P_n}$, so$\gamma_{hne}(\bar{P}_n) = n-3$.

Proposition 9$\gamma_{hne}(\bar{C}_n) = \lceil \frac{n}{2} \rceil, n \geq 4$.

*Proof.*Since $\bar{c}_n$ has a spanning cycle subgraph C_n , then by proposition 2, we get the result.

remark 10*If G of order n and has a vertex v such that $d(v) = n-1$.,then a vertex v is isolated vertex in $\bar{G}$.*

Example: A graphs $(\overline{K_n}), (\overline{W_n}, (\bar{P}_n), (\overline{C_n})$has a vertex of degree $n-1$, so these graphs have no hn-edge dominating set according to the Remark 10.

Proposition 11$\gamma_{hne}(\overline{K_{m,n}}) = \lceil \frac{m}{2} \rceil + \lceil \frac{n}{2} \rceil$, $m, n \geq 3$.

*Proof.*Since $G \cong K_{(m,n)}$,then the graph $\overline{G}$ contains two components one of them is a complete graph of order m and the other is a complete graph of order n. Then $\gamma_{hne}(\overline{K_{m.n}}) = \gamma_{hne}(K_m) + \gamma_{hne}(K_n$ and by proposition 4, $\gamma_{hne}(K_m) = \lceil \frac{n}{2} \rceil$ and $\gamma_{hne}(K_n) = \lceil \frac{n}{2} \rceil$. Thus, $\gamma_{hne}(\overline{K_{m,n}}) = \lceil \frac{m}{2} \rceil + \lceil \frac{n}{2} \rceil$.

3 The changing and unchanging

Theorem 12.*Assume that G be a graph has γ_{hne} ,then $\gamma_{hne}(G-v) \leq \gamma_{hne}(G)$.*

*Proof.*Two cases are appeared as follows.
Caes1. If the graph $G-v$ has an isolated vertex then $G-v$ has no $Hn-EDS$.
Case2. If $G-v$ has no isolated vertex, so three cases are appeared as follows.
If v is incident to only one edge from D_e say $e = vu$, then

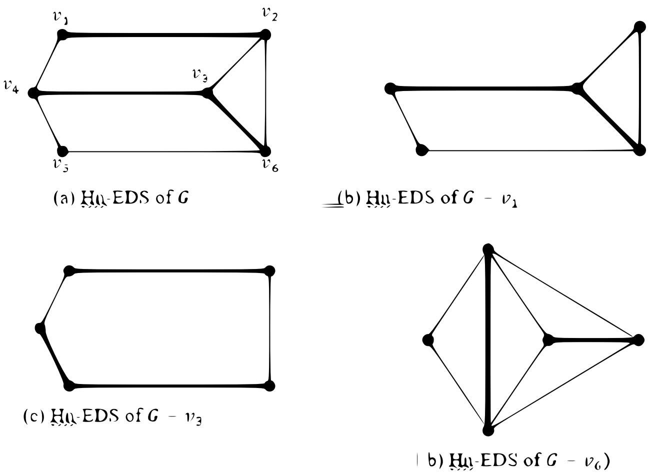


Fig. 6: cases of $\gamma_{hne}(G-e)$ with $\gamma_{hne}(G)$ in deferent graph

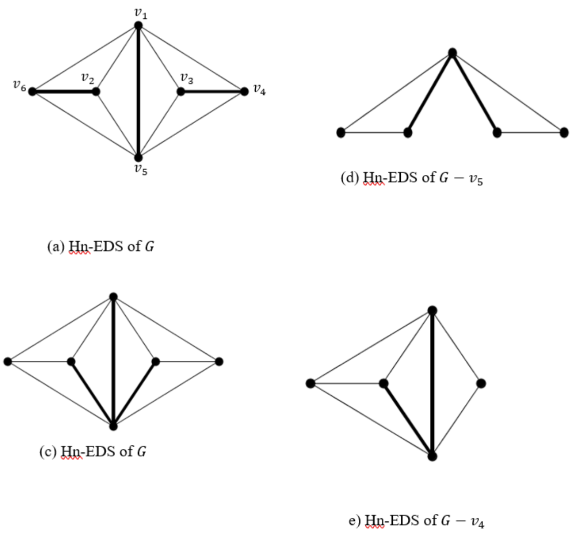


Fig. 7: cases of $\gamma_{hne}(G-e)$ with $\gamma_{hne}(G)$

if all edges that incident to the vertex u in $G-v$ are dominate by an edges in D_{e-e}, then $\gamma_{hne}(G-v) < \gamma_{hne}(G)$. (as an example, see Fig. 6.(b)). Otherwise, $\gamma_{hne}(G-v) = \gamma_{hne}(G)$.
If v is incident to at least two edges in D_e, then there are three cases as follows.
If at least one edges in D_e is not incident to v and all neighborhoods of the edges in D_e that incident to the vertex v are dominated by other edges in $D_e$or only one edge from these neighborhoods
is not dominate by any edge in D_e, then $\gamma_{hne}(G-v) < \gamma_{hne}(G)$. Otherwise, $\gamma_{hne}(G-v) = \gamma_{hne}(G)$. (as an example, see Fig. 4.(c)).
If v is incident to all edges in D_e, then in G if the neighborhoods
of every edge in D_e are incident to distinct one edge in $E-D_e$, then γ_{hne} $(G-v) = \gamma_{hne}(G)$. Otherwise, $\gamma_{hne}(G-v) < \gamma_h ne(G)$. (see Fig. 6.(d)). If v is incident to edges in $E-D_e$ such that these edges are dominating by incident edges in D_e, then $\gamma_{hne}(G-v) < \gamma_{hne}(G)$. (as an example, see Fig.7(e)). Otherwise, $\gamma_{hne}(G-v) = \gamma_{hne}(G)$. Thus, $\gamma_{hne}(G-v) \leq \gamma_{hne}(G)$.

Theorem 13.*Assume that G be a graph has γ_{hne}, then if $e \in D_e$, $G-e$ is disconnected graph with at least two disjoint edges not dominated by other edges then $\gamma_{hne}(G-e) \geqslant \gamma_{hne}(G)$. Otherwise, $\gamma_{hne}(G-e) \leq \gamma_{hne}(G)$.*

*Proof.*If an edge $e = uv(u,v \in G)$ is added, then following two cases are getting as follows.
Case1. If $G-e$ is disconnected graph, then there are two cases as follows. If $G-e$ has isolated vertex, then $G-e$ has no $hn-edge$ dominating set. If $G-e$ has no isolated vertex with $e \in D_e$ and at least one edge incident to v and other incident two u are not dominating by any other edges in D_e, then $\gamma_{hne}(G-e) > \gamma_{hne}(G)$. But if e not in D_e, then $hn-edge$ domination number is not influence by deletion. (as an example, see Fig.8(b)).
Case2. If $G-e$ is connected graph, then two cases are appeared as the following. If $e \in D_e$, then we have three cases as follows.
(a) If all edges in $E-D_e$ that incident to the edge e are dominating by other edges in D_e, then $\gamma_{hne}(G-e) < \gamma_{hne}(G)$. (see Fig. 4.3(c)).
(b) If at least two adjacent edges that incident to e are not dominating by edges in $D_e - e$ and these two edges are not incident to pendent vertex, then $\gamma_{hne}(G-e) > \gamma_{hne}(G)$. (as an example, see Fig. 9(b)).
Otherwise, $\gamma_{hne}(G-e) = \gamma_{hne}(G)$. If $e \in E-D_e$, then $\gamma_h ne(G-e) \leq \gamma_{hne}(G)$.(see Fig. 8(d)).
Thus, we get the result

Theorem 14.*Assume that G be a graph has γ_{hne} and $e = uv(u,v \in G)$, then in $G+e$ if u and v are incident to edges in D_e say e_1 and e_2, then in $G+e$ if all neighborhoods of e_1 and e_2 are incident to e, then $\gamma_{hne}(G+e) < \gamma_{hne}(G)$. otherwise, $\gamma_{hne}(G+e) \geq \gamma_{hne}(G)$, where $e \in \bar{G}$*

*Proof.*Let D_e belong to $\gamma_{hne}-set$ of G, then if we add an edge $e = uv(u,v \in G)$,then three cases are appeared as the following.
Case1. If u and v are not incident to any edge in D_e, then $\gamma_{hne}(G+e) > \gamma_{hne}(G)$.(as an example, see Fig. 4.5(a)).
Case2. If u or v are incident to an edge in D_e say e_1, then in $G+e$ If e is incident to edge in $E-D_e$ and this edge dominated by edge that disjoint with e_1, then $\gamma_{hne}(G+e) > \gamma_{hne}(G)$.(see Fig. 4.5(b)). Otherwise, $\gamma_{hne}(G+e) = \gamma_{hne}(G)$(as an example, see Fig. 4.5(c)).
Case3. If u and v are incident to edges in D_e say e_1 and e_2, then in $G+e$ if all neighborhoods of e_1 and e_2 are incident to e , then $\gamma_{hne}(G+e) < \gamma_{hne}(G)$.(see Fig.

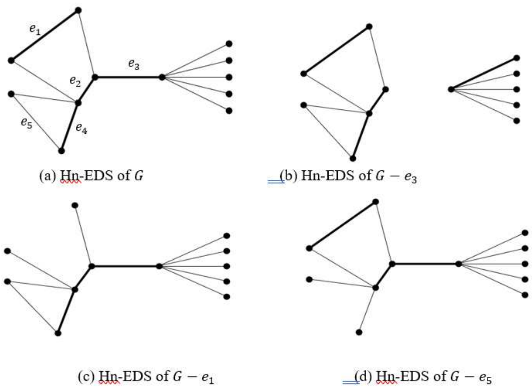


Fig. 8: $\gamma_{hne}(G-e) \leq \gamma_{hne}(G)$.

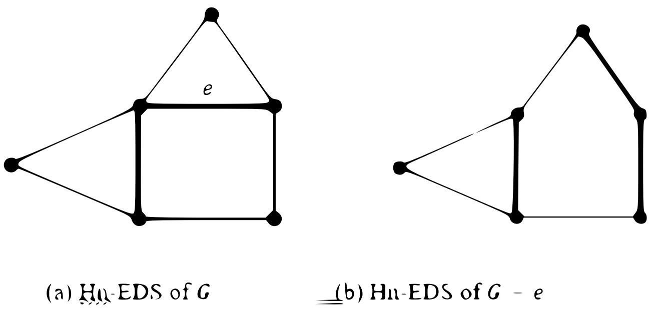


Fig. 9: $\gamma_{hne}(G-e) > \gamma_{hne}(G)$.

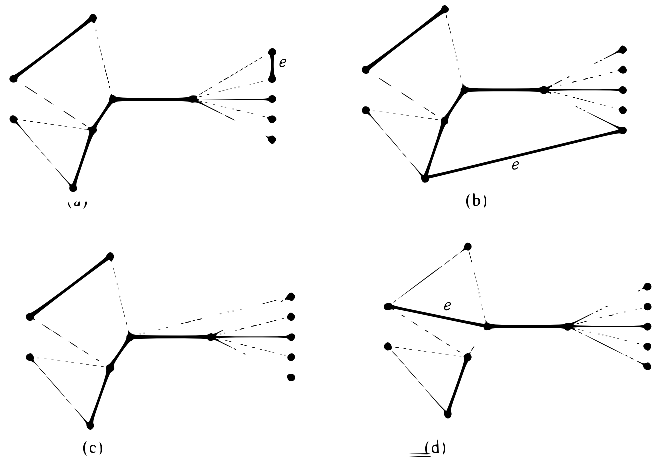


Fig. 10: The add and deletion an edge

4.5(d)). Otherwise, $\gamma_{hne}(G+e) = \gamma_{hne}(G)$. Thus, we get the result.

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