Mathematical Sciences Letters
An International Journal

On some impulsive differential equations

A. S. Abdel-Rady [1], **A. M. A. El-Sayed** [2], **S. Z. Rida** [1] **and I. Ameen** [1]

[1] *Department of Mathematics, Faculty of Science, South Valley University, Qena, Egypt.*

[2] *Department of Mathematics, Faculty of Science, Alexandria University, Alexandria, Egypt.*
Email: ism_math87@yahoo.com.

Abstract: The existence and uniqueness of solution for the first order impulsive differential equation is established. We show that these results can be applied to second order impulsive differential equation. Examples are given to illustrate our main results.

Keywords: Impulsive differential equations, existence of solution, Banach fixed point theorem, internal nonlocal Cauchy problem

1 Introduction

Impulsive differential equations, that is, differential equations involving impulse effect, appear as a natural description of observed evolution phenomena of several real world problems. There are many good monographs on the impulsive differential equations [1, 5-9]. Many processes studied in applied sciences are represented by differential equations. However, the situation is quite different in many physical phenomena that have a sudden change in their states such as mechanical systems with impact, biological systems such as heart beats, blood flows, population dynamics [10,18], theoretical physics, radiophysics, pharmacokinetics, mathematical economy, chemical technology, electric technology, metallurgy, ecology, industrial robotics, biotechnology processes, chemistry [11], engineering [3], control theory [13,17], medicine [2,14] and so on. Adequate mathematical models of such processes are systems of differential equations with impulses.

The theory of impulsive differential equations is a new and important branch of differential equations. The first paper in this theory is related to A. D. Mishkis and V. D. Mil'man in 1960 and 1963 [19]. The last decades have seen major developments in this theory. In spite of its importance, the development of the theory has been quite slow due to special features possessed by impulsive differential equations in general, such as pulse phenomena, confluence, and loss of autonomy (see for instance, [12]). First and second order ordinary differential equations with impulses have been treated in several works (see [1, 8, 11, 15, 16, 20 and 23]). An impulsive differential equation is described by three components: a continuous-time differential equation, which governs the state of the system between impulses; an impulse equation, which models an impulsive jump defined by a jump function at the instant an impulse occurs; and a jump criterion, which defines a set of jump events in which the impulse equation is active (see [4]). Mathematically this equation takes the form

$$x'(t) = f(t, x(t)), \quad t \neq \tau_k, \quad t \in J,$$

$$\Delta x(\tau_k) = I_k(x(\tau_k)), \quad k = 1,2,\ldots, m,$$

In this paper, we consider the parameterized problem of impulsive differential equation which takes the form

$$x'(t) = f(t, x(t)), \quad t \neq \tau_k, \quad t \in J, \tag{1.1}$$

$$x(\tau_k^-) = \alpha\, x(\tau_k^+) + \beta,\ t = \tau_k,\ \ k = 1,2,\dots, m \text{ and } \alpha, \beta \in R;\ \ \alpha \neq 0, \tag{1.2}$$

$$x(0) = x_0, \tag{1.3}$$

and

$$x'(t) = f(t, x(t)), \quad t \neq \tau_k, \quad t \in J, \tag{1.4}$$

$$x(\tau_k^-) = \alpha\, x(\tau_k^+) + \beta = x_0,\ t = \tau_k,\ \ k = 1,2,\dots, m \text{ and } \alpha, \beta \in R;\ \ \alpha \neq 0, \tag{1.5}$$

where J is any real interval, $f : J \times R \to R$ is a given function, $I_k : J \times R \to R$, $\Delta x(\tau_k) = x(\tau_k^+) - x(\tau_k^-);\ k = 1,2,\dots,m$ and $x_o \in R$. The numbers τ_k are called instants (or moments) of impulse, I_k represents the jump of the state at each τ_k, $x(\tau_k^+) = \lim_{h \to o^+} x(\tau_k + h)$ and $x(\tau_k^-) = \lim_{h \to o^-} x(\tau_k + h)$ represent the right and left limits, respectively, of the state at $t = \tau_k$. More specifically, let $T > 0$, $J = [0, T]$, $0 = \tau_0 < \tau_1 < \dots < \tau_{m+1} = T$.

2 Preliminaries

In this section, we need some basic definitions and properties of impulsive differential equation denotes the Banach space of all continuous functions $C[0,T]$ which are used throughout this paper. Here, with the norm $[0,T]$ defined on

$$\| x \|_C = \sup \{ | x(t) | : t \in J \},$$

set $PC(J, R) = \{ x : J \to R \text{ is continuous everywhere excapt for } t = \tau_k \text{ at which } x(\tau_k^-) \text{ and } x(\tau_k^+), k = 1,2,\dots, m \text{ exist and } x(\tau_k^-) = x(\tau_k) \}$

with the norm

$$\| x \|_{PC(J,R)} = \sup \{ | x(t) | : t \in J, t \neq \tau_k \},$$

Definition 2.1. Differential equation of the form

$$\frac{dx}{dt} = f(t, x(t)), \quad t \neq \tau_k, \tag{2.1}$$

with conditions $\Delta x(\tau_k) = x(\tau_k^+) - x(\tau_k^-) = I_k(x(\tau_k))$ where $I_k : J \times R \to R$ are continuous functions, $k = 0, \pm 1, \pm 2, \dots,$ is called impulsive differential equation at fixed impulse and solution $x(t)$ of equation (2.1) with initial condition $x(t_0^+) = x_0$ is the following form

$$x(t) = \begin{cases} x(0) + \int_{t_0}^{t} f(s, x(s))\, ds + \sum_{t_0 \le \tau_k \le t} I_k(x(\tau_k)) \ \text{ if } t \in \Omega^+, \\ \\ x(0) + \int_{t_0}^{t} f(s, x(s))\, ds - \sum_{t \le \tau_k \le t_0} I_k(x(\tau_k)) \ \text{ if } t \in \Omega^-. \end{cases} \tag{2.2}$$

where Ω^{+} and Ω^{-} are maximal intervals on which the solution can be continued to the right or to the left of the point $t = t_0$, respectively.

Definition 2.2. ([5, 21]) $x(t)$ is said to be the solution of problem (1.1) - (1.3) and (1.4)-(1.5) if it satisfies the following conditions:

(1) $\lim_{t\to o^{+}} x(t) = x_0 = x(0^{+})$,

(2) for $(0,+\infty)$, $t \neq \tau_k$, $x(t)$ is differentiable and $x'(t) = f(t, x(t))$,

(3) $x(t)$ is left continuous in $(0,+\infty)$ and if $t = \tau_k$, then $x(\tau_k^{-}) = \alpha\, x(\tau_k^{+})$, $\alpha \neq 1$.

3 Main results

3.1 Existence of Solution

First, we consider the problem (1.1)-(1.3).

Definition 3.1. By a solution of problem (1.1)-(1.3), we mean a function $x \in PC(J, R)$ that satisfies the problem (1.1)-(1.3) itself .

Theorem 3.1. Let $f : J \times R \to R$ *be continuous function and satisfies the lipschitz condition*

$$\left| f(t, x(t)) - f(t, \bar{x}(t)) \right| \leq K \left| x - \bar{x} \right|, \ \forall (t, x), (t, \bar{x}) \in J \times R, \tag{3.1}$$

with lipschitz constant $K > 0$. *If*

$$KT \ \frac{1 - |\alpha|^{k}}{1 - |\alpha|} < 1, \tag{3.2}$$

then, the problem (1.1)-(1.3) has a unique solution. This solution can be expressed by the formula

$$x(t) = \begin{cases} x_0 + \int_0^t f(s, x(s))\, ds & \text{if } \ t \in [0, \tau_1), \\ & \qquad\qquad \text{where } \ k = 1,2,..., m \\ x(\tau_k^{+}) + \int_{\tau_k}^t f(s, x(s))\, ds & \text{if } \ t \in (\tau_k, \tau_{k+1}]. \end{cases} \tag{3.3}$$

Proof. Integrating equation (1.1) over $t \in [0, \tau_1)$ with the initial condition (1.3), we obtain

$$x(t) = x_0 + \int_0^t f(s, x(s))\, ds \ . \tag{3.4}$$

Since, $x(\tau_1^{-}) = \alpha\, x(\tau_1^{+}) + \beta \Rightarrow x(\tau_1^{+}) = \frac{1}{\alpha}(x(\tau_1^{-}) - \beta)$, but $\tau_1^{-} \in [0, \tau_1)$,

$$x(\tau_1^{-}) = \lim_{t \to \tau_1^{-}} x(t) = x_0 + \int_0^{\tau_1} f(s, x(s))\, ds \ ,$$

$$x(\tau_1^{+}) = \frac{1}{\alpha}\Big[(x_0 - \beta) + \int_0^{\tau_1} f(s, x(s))\, ds\Big], \tag{3.5}$$

integrating equation (1.1) over $t \in (\tau_1, \tau_2)$, we obtain

$$x(t) = x(\tau_1^+) + \int_{\tau_1}^{t} f(s, x(s))\, ds \ ,$$

then, form equation (3.5)

$$x(t) = \frac{1}{\alpha}\Big[(x_0 - \beta) + \int_0^{\tau_1} f(s, x(s))\, ds\Big] + \int_{\tau_1}^{t} f(s, x(s))\, ds \ ,$$

Since, $x(\tau_2^-) = \alpha\, x(\tau_2^+) + \beta \Rightarrow x(\tau_2^+) = \frac{1}{\alpha}(x(\tau_2^-) - \beta)$, but $\tau_2^- \in (\tau_1, \tau_2)$,

$$x(\tau_2^-) = \lim_{t \to \tau_2^-} x(t) = \frac{1}{\alpha}\Big[(x_0 - \beta) + \int_0^{\tau_1} f(s, x(s))\, ds\Big] + \int_{\tau_1}^{\tau_2} f(s, x(s))\, ds \ ,$$

$$x(\tau_2^+) = \frac{1}{\alpha}\left[\frac{1}{\alpha}\Big[(x_0 - \beta) + \int_0^{\tau_1} f(s, x(s))\, ds\Big] + \int_{\tau_1}^{\tau_2} f(s, x(s))\, ds - \beta\right], \qquad (3.6)$$

integrating equation (1.1) over $t \in (\tau_2, \tau_3)$, we obtain

$$x(t) = x(\tau_2^+) + \int_{\tau_2}^{t} f(s, x(s))\, ds \ ,$$

then, form equation (3.6)

$$x(t) = \frac{1}{\alpha}\left[\frac{1}{\alpha}\Big[(x_0 - \beta) + \int_0^{\tau_1} f(s, x(s))\, ds\Big] + \int_{\tau_1}^{\tau_2} f(s, x(s))\, ds - \beta\right] + \int_{\tau_2}^{t} f(s, x(s))\, ds \, .$$

By repeating the same procedure, we can easily deduce that

$$x(t) = x(\tau_k^+) + \int_{\tau_k}^{t} f(s, x(s))\, ds \ \ if \ \ t \in (\tau_k, \tau_{k+1}], k = 1,2,..., m. \qquad (3.7)$$

Since,

$$x(\tau_k^+) = \frac{1}{\alpha}\left[x(\tau_{k-1}^+) + \int_{\tau_{k-1}}^{\tau_k} f(s, x(s))\, ds - \beta\right].$$

Combining (3.4)-(3.7), we obtain (3.3).

Now, we study the existence of solution of the problem (1.1)-(1.3). Let the operator $F_\alpha : PC(J, R)$ $\to PC(J, R)$ be defined by

$$F_\alpha(x)(t) = \begin{cases} F_1(x)(t) = x_0 + \int_0^t f(s, x(s))\, ds & if \ \ t \in [0, \tau_1), \\ & where \ \ k = 1,2,..., m \\ F_{2,\alpha}(x)(t) = x(\tau_k^+) + \int_{\tau_k}^t f(s, x(s))\, ds & if \ \ t \in (\tau_k, \tau_{k+1}]. \end{cases}$$

By the Banach contraction fixed point theorem, it is clear that if $K\, T \frac{1 - |\alpha|^k}{1 - |\alpha|}$, then F_α has a unique solution $x \in PC(J, R)$.

Next, we consider the problem (1.4)-(1.5).

Definition 3.2. By a solution of problem (1.4)-(1.5), we mean a function $x \in PC(J,R)$ that satisfies the problem (1.4)-(1.5) itself.

Theorem 3.2. *Let the assumptions of theorem (3.1) are satisfied .If $KT < 1$, then the problem (1.4)-(1.5) has a unique solution. This solution can be expressed by the formula*

$$x(t) = \begin{cases} x_0 + \int_t^{\tau_1} f(s,x(s))\,ds & \text{if } t \in [0,\tau_1), \\ & \text{where } k = 1,2,\ldots,m \\ x(\tau_k^+) + \int_{\tau_k}^{t} f(s,x(s))\,ds & \text{if } t \in (\tau_k,\tau_{k+1}]. \end{cases} \tag{3.8}$$

Proof. Integrating equation (1.4) over $t \in [0,\tau_1)$, we obtain

$$x(\tau_1) - x(t) = \int_t^{\tau_1} f(s,x(s))\,ds ,$$

from Eq.(1.5), we have

$$x(t) = x_0 - \int_t^{\tau_1} f(s,x(s))\,ds , \tag{3.9}$$

integrating equation (1.4) over $t \in (\tau_1,\tau_2)$, we obtain

$$x(t) = x(\tau_1^+) + \int_{\tau_1}^{t} f(s,x(s))\,ds ,$$

form equation (1.5) , we obtain

$$x(\tau_1^+) = \frac{1}{\alpha}(x_0 - \beta),$$

thus,

$$x(t) = \frac{1}{\alpha}(x_0 - \beta) + \int_{\tau_1}^{t} f(s,x(s))\,ds ,$$

integrating equation (1.4) over $t \in (\tau_2,\tau_3)$, we obtain

$$x(t) = x(\tau_2^+) + \int_{\tau_2}^{t} f(s,x(s))\,ds ,$$

from equation (1.5), we have

$$x(\tau_2^+) = \frac{1}{\alpha}(x_0 - \beta),$$

thus,

$$x(t) = \frac{1}{\alpha}(x_0 - \beta) + \int_{\tau_2}^{t} f(s,x(s))\,ds .$$

By repeating the same procedure, we can easily deduce that

$$x(t) = x(\tau_k^+) + \int_{\tau_k}^{t} f(s,x(s))\,ds \quad \text{if } t \in (\tau_k,\tau_{k+1}],\ k = 1,2,\ldots,m. \tag{3.10}$$

Since,

$$x(\tau_k^+) = \frac{1}{\alpha}(x_0 - \beta).$$

Combining (3.9)-(3.10), we obtain (3.8).

Now, we study the existence of solution of the problem (1.4)-(1.5). Let the operator $F_\alpha : PC(J,R) \to PC(J,R)$ be defined by

$$F_\alpha(x)(t) = \begin{cases} F_1(x)(t) = x_0 - \int_t^{\tau_1} f(s,x(s))\,ds & \text{if } t \in [0,\tau_1), \\ & \text{where } k = 1,2,\dots,m \\ F_{2,\alpha}(x)(t) = x(\tau_k^+) + \int_{\tau_k}^{t} f(s,x(s))\,ds & \text{if } t \in (\tau_k, \tau_{k+1}]. \end{cases}$$

By the Banach contraction fixed point theorem, it is clear that if $KT < 1$, then F_α has a unique solution $x \in PC(J,R)$.

3.2 Continuation theorem

Now, we have the following theorems:

Theorem 3.3. *If* $\alpha \to 1$ and $\beta \to 0$, *then the problems (1.1)-(1.3) and the following IVP*

$$x'(t) = f(t, x(t)), \quad t \in J,$$

$$x(0) = x_0,$$

are coincide with the same solution.

Theorem 3.4. *If* $\alpha \to 1$ and $\beta \to 0$, *then the problems (1.4)-(1.5) and the following internal nonlocal Cauchy problem [22]*

$$x'(t) = f(t, x(t)), \quad t \in J,$$

$$x(\tau) = x_0, \quad \tau \in (0, T),$$

are coincide with the same solution.

3.3 Extension

The present results of first's order impulsive differential equation can be extended for the second order impulsive differential equation as follow:

$$x''(t) = f(t, x'(t)), \quad t \neq \tau, \quad t \in J,$$

$$x(\tau^-) = \alpha_1 x(\tau^+) + \beta_1, \quad x'(\tau^-) = \alpha_2 x'(\tau^+) + \beta_2,$$

$$x(0) = \gamma_1, \quad x'(0) = \gamma_2.$$

Put $x'(t) = p(t)$, thus we can write

$$p'(t) = f(t, p(t)), \quad t \neq \tau, \quad t \in J,$$

$$p(\tau^-) = \alpha_2 p(\tau^+) + \beta_2,$$

with initial condition

$$p(0) = \gamma_2,$$

we can find the solution $p(t)$ of this problem by the same method of problem (1.1)-(1.3), so we have

$$x'(t) = p(t), \quad t \neq \tau, \quad t \in J,$$

$$x(\tau^-) = \alpha_1 x(\tau^+) + \beta_1,$$

with initial condition

$$x(0) = \gamma_1.$$

Now, we can find $x(t)$.

4 Examples

In this section, we consider some second order impulsive differential equations and the following examples will be helpful to illustrate the main results of this paper.

Example **4.1.** *Consider the following problem of impulsive differential equation*

$$\csc(t)\, x''(t) = 1, \quad t \neq \frac{\pi}{2}, \quad t \in [0,6],$$

$$x(\frac{\pi}{2}^-) = \alpha_1 x(\frac{\pi}{2}^+) + \beta_1, \; x'(\frac{\pi}{2}^-) = \alpha_2 x'(\frac{\pi}{2}^+) + \beta_2, \; t = \frac{\pi}{2},$$

$$x(0) = 0, \; x'(0) = -1.$$

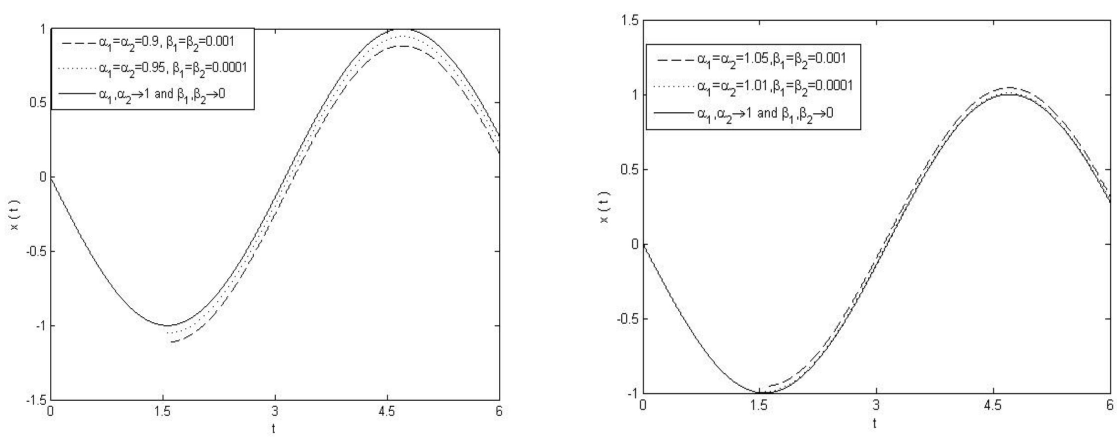

Figure 1: show the continuation of solutions of Ex.(4.1).

Example **4.2.** *Consider the following problem of impulsive differential equation*

$$x''(t) = \frac{x'(t)}{t+1}, \quad t \neq 1, \quad t \in [0,2],$$

$$x'(1^-) = \alpha\, x'(1^+) + \beta, \; t = 1,$$

$$x(0) = 0, \; x'(0) = 1.$$

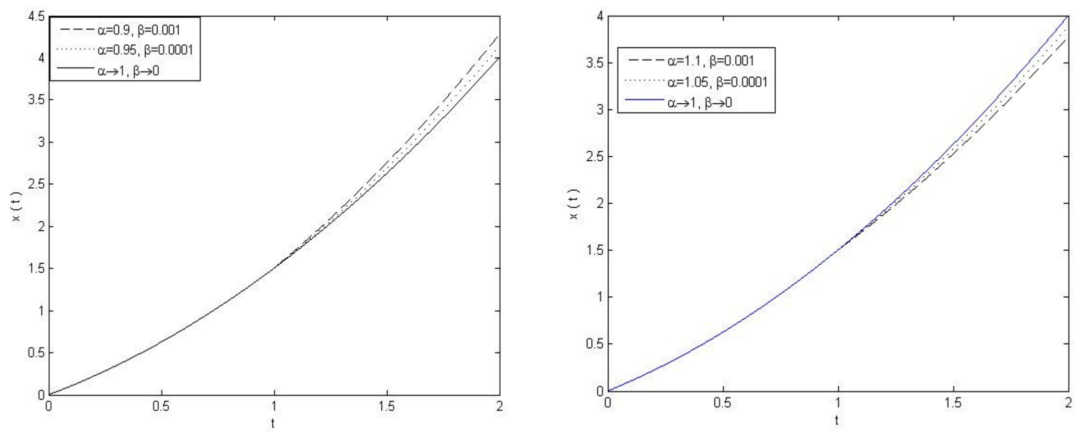

Figure 2: show the continuation of solutions of Ex.(4.2).

Example **4.3.** *Consider the following problem of impulsive differential equation*

$$x''(t) = \frac{x'(t) - 1}{t}, \quad t \neq 1, \quad t \in [0,2],$$

$$x(1^-) = \alpha\, x(1^+) + \beta,\ t = 1,$$

$$x(0) = \frac{-1}{2}, \quad x'(0) = 1.$$

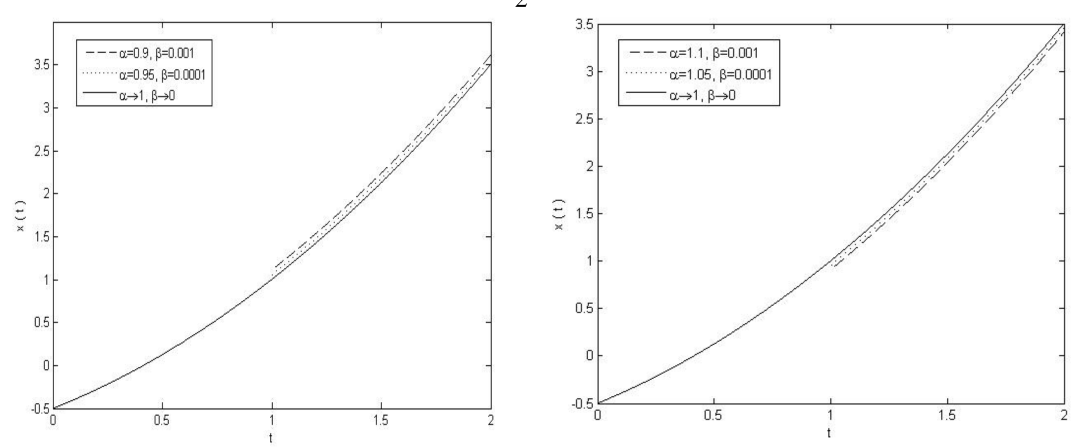

Figure 3: show the continuation of solutions of Ex.(4.3).

5 Conclusion

In this work, we established theorems with some examples for linear first/second order impulsive differential equations with linear impulse effect which is a generalization to that in previous work [22] and [23].

References

[1] A. M. Samoilenko and N. A. Perestyuk, *Impulsive Differential Equations*, vol.**14** of World Scientific Series on Nonlinear Science. Series A: Monographs and Treatises, World Scientific, Singapore, (1995).

[2] A. D'onofrio, On pulse vaccination strategy in the SIR epidemic model with vertical transmission, *Appl. Math. Lett.*, **18**, (2005), 729-732.

[3] A. Dishliev and D.D. Bainov, Dependence upon initial conditions and parameters of solutions of impulsive differential equations with variable structure, *International Journal of Theoretical Physics*, **29,** (1990), 655-676.

[4] A. V. Roup and others, Limit cycle analysis of the verge and foliot clock escapement using impulsive differential equations and poincar maps, *Int. J. Control*, **76**, (2003), 1685-1698.

[5] D. D. Bainov and P. S. Simeonov, *Oscillation Theory of Impulsive Differential Equations*, International Publications, Orlando, Fla, USA, (1998).

[6] D. D. Bainov and P. S. Simeonov, *Impulsive Differential Equations: Asymptotic Properties of the Solutions*, vol.**28** of Series on Advances in Mathematics for Applied Sciences, World Scientific, Singapore, (1995).

[7] D. Bainov and V. Covachev, *Impulsive Differential Equations with a Small Parameter*, vol. **24** of Series on Advances in Mathematics for Applied Sciences, World Scientific, Singapore, (1994).

[8] D. D. Bainov and P. S. Simeonov, Impulsive differential equations: periodic solutions and applications, Longman Scientific and Technical, Essex, England, (1993).

[9] D. D. Bainov, V. Lakshmikantham, and P. S. Simeonov, *Theory of Impulsive Differential Equations*, vol.**6** of Series in Modern Applied Mathematics, World Scientific, Singapore, (1989).

[10] D.D. Bainov and A. Dishliev, Population dynamics control in regard to minimizing the time necessary for the regeneration of a biomass taken away from the population, *Comp. Rend. Bulg. Scie.*, **42**, (1989), 29-32.

[11] D.D. Bainov and P.S. Simenov, *Systems with Impulse Effect Stability Theory and Applications*, Ellis Horwood Limited, Chichester, (1989).

[12] Erbe, L. H., Liu and Xinzhi, Existence results for boundary value problems of second order impulsive differential equations, *J. Math. Anal. Appl.*, **149**, (1990), 56-69.

[13] G. Jiang, Q. Lu, Impulsive state feedback control of a predator-prey model, *J. Comput. Appl. Math.*, **200**, (2007), 193-207.

[14] M. Choisy, J.F. Guegan and P. Rohani, Dynamics of infectious diseases and pulse vaccination: teasing apart the embedded resonance effects, *Phys. D.*, **223**, (2006), 26-35.

[15] M.U. Akhmet, On the general problem of stability for impulsive differential equations, *J. Math. Anal. Appl.*, **288**, (2003), 182-196.

[16] M. Frigon and D. O'Regan, Impulsive differential equations with variable times, *Nonl. Anal. Tma.*, **26**, (1996), 1913-1922.

[17] R.K. George, A.K. Nandakumaran and A. Arapostathis, A note on controllability of impulsive systems, *J. Math. Anal. Appl.*, **241**, (2000) , 276-283.

[18] S. Nenov, Impulsive controllability and optimization problems in population dynamics, *Nonl. Anal.*, **36**, (1999), 881-890.

[19] V.D. Mil'man and A.D. Myshkis, On the stability of motion in the presence of impulses, *Sib. Math. J.*, **1**, (1960), 233-237.

[20] Y.V. Rogovchenko, Impulsive evolution systems: main results and new trends, *Dyn. Contin. Discrete Impuls. Syst.*, **1**, (1997), 57-88.

[21] X. J. Ran, M. Z. Liu and Q. Y. Zhu, Numerical methods for impulsive differential equation , *Math. Comp. Mode.*, **48**, (2008), 46-55.

[22] A. M. A. El-Sayed and I. Ameen, Continuation of a parameterized impulsive differential equation to an internal nonlocal Cauchy problem, *Alex. Jour. Math.*, **2**, (2011), 1-7.

[23] A. M. A. EL-Sayed, A. S. Abdel-Rady, S. Z. Rida and I. Ameen, Continuation of a Parameterized Impulsive Differential Equation to an Initial Value Problem, *J. Math. Comput. Sci.*, **2**, (2012), 847-852.