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Weingarten Isotropic Embankment Surfaces According to Adapted Frame

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Abstract: The Tubembankmentlike surfaces with adapted frames had discussed in this study. We get the parametric description of the Isotropic Tubembankmentlike Surfaces and give a computational application to evident these surfaces by introducing them in isotropic 3−space. Weingarten of Isotropic Tubembankmentlike Surfaces is also calculated in terms of Gaussian and mean curvatures. Mathematica 3D visualizations are using to create these curvatures. We also present new applications for Tubembankmentlike surfaces in Isotropic space, as well as surface characterization.

Keywords: Embankment surface, Isotropic space, Weingarten surface, Frenet-Serret frame

1 Introduction

In this paper, we investigate the $(x,y)-$Weingarten Tubembankmentlike surfaces in Isotropic $3-$space that satisfy that all surfaces under discussion are smooth, regular, and topologically connected unless otherwise stated. We clarify the fundamental concept of the Frenet frame in Isotropic $3-$space, as well as the parametric equation of the Tubembankmentlike surface, to prepare some fundamental facts about the first and second fundamental forms, the Gaussian curvature and mean curvature in the $(x,y)-$Weingarten according to Frenet frame in I_3^1 and $(x,y)-$Weingarten Tubembankmentlike surface.
The Jacobi equation is satisfied by a Weingarten surface, often known as a W−surface.

$$\Psi(\mathscr{K},\mathscr{H}) = det\begin{pmatrix} \mathscr{K}_\phi & \mathscr{K}_\vartheta \\ \mathscr{H}_\phi & \mathscr{H}_\vartheta \end{pmatrix} = 0 \tag{1}$$

where $\mathscr{K}$ denotes Gaussian curvature and $\mathscr{H}$ denotes the surface's mean curvature. A Weingarten surface is a surface with constant Gaussian curvature or constant mean curvature [1].
Several geometers ([1],[2],[3]) have examined the W-surface and come up with interesting results.
An embankment surface is a one-parameter family of cones enclosure that is very important for engineers drawing embankment construction plans (for more on embankment constructions, see [4]).
Future structural engineers, on the other hand, will need to be familiar with Gaussian and mean curvatures. For example, in a uniform state of tensile prestressing, a Tensile fabric structure (such as a membrane roof) behaves like a soap film stretched over a wire bent in the shape of a closed space curve. Soap film takes on a shape that has the smallest area of all the other surfaces stretched over the same wire; this surface is therefore called minimal surface [5].

2 Isotropic space

Metrics and motion Isotropic geometry is based on the following affine transformation group G_6 of affine transformations $(x,y,z) \longrightarrow (x',y',z')$ in $\mathbf{R^3}$

$$\begin{aligned} x' &= d + x\cos(\zeta) - y\sin(\zeta), \\ y' &= e + x\sin(\zeta) + y\cos(\zeta), \\ z' &= f + c_1 x + c_2 y + z, \end{aligned} \tag{2}$$

where d, e, f, c_1, c_2, $\zeta \in \mathbf{R}$.
Isotropic congruence transformations or isotropic motions are examples of affine transformations.

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The distance between two points that is isotropic $P=(x_1,y_1,z_1)$ and $Q=(x_2,y_2,z_2)$ is defined as the Euclidean distance,

$$d(P,Q)_i=\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$$

Let $X=(x_1,y_1,z_1)$ and $Y=(x_2,y_2,z_2)$ be vectors in I_3^1. The isotropic inner product of X and Y is defined by

$$\langle X,Y\rangle_i=\begin{cases} z_1z_2 & if \quad x_i=y_i=0 \\ x_1x_2+y_1y_2 & if \quad otherwise \end{cases} \tag{3}$$

Also,consider $\gamma : I \longrightarrow I_3^1$ as a regular curve with $\gamma'=\frac{d\gamma}{dt}(t)\neq 0$. If γ has unit tangent vector field, unit principal normal vector field, and unit bi-normal vector field, respectively. The Frenet-Serret frame is defined by $\mathscr{T},\mathscr{N},\mathscr{B}$ and its formulas are provided by

$$\begin{bmatrix}\mathscr{T}'\\ \mathscr{N}'\\ \mathscr{B}'\end{bmatrix}=\begin{bmatrix}0&\kappa&0\\ -\kappa&0&\tau\\ 0&-\tau&0\end{bmatrix}\begin{bmatrix}\mathscr{T}\\ \mathscr{N}\\ \mathscr{B}\end{bmatrix} \tag{4}$$

where

$$\begin{aligned} h(\mathscr{T},\mathscr{T})=h(\mathscr{N},\mathscr{N})=h(\mathscr{B},\mathscr{B})=1,\\ h(\mathscr{T},\mathscr{N})=h(\mathscr{T},\mathscr{B})=h(\mathscr{N},\mathscr{B})=0.\end{aligned} \tag{5}$$

Isotropic curvature and isotropic torsion of γ are represented by κ and τ, respectively, [10].
A surface **S** immersed in I_3^1 is called admissible if it has no isotropic tangent planes.
A surface Γ in I_3^1 by

$$\Gamma_{tel}(\phi,\vartheta)=(\Gamma_{tel1}(\phi,\vartheta),\Gamma_{tel2}(\phi,\vartheta),\Gamma_{tel3}(\phi,\upsilon)) \tag{6}$$

For a surface that is always totally isotropic, the coefficients $\mathsf{g}_{11},\mathsf{g}_{12},\mathsf{g}_{22}$ of the first fundamental forms for the induced metric and the coefficients L_{11}, $\mathsf{L}_{12},\mathsf{L}_{22}$ of the second fundamental forms for the normal vector field found. **S**'s 1st and 2nd fundamental forms are defined by

$$\begin{aligned} I=\ \mathsf{g}_{11}\,d\phi^2+\mathsf{g}_{12}\,d\phi d\vartheta\ +\mathsf{g}_{22}\,d\vartheta^2,\\ \Pi=\ \mathsf{L}_{11}\,d\phi^2+\ \mathsf{L}_{12}\,d\phi d\vartheta\ +\ \mathsf{L}_{22}\,d\vartheta^2\end{aligned} \tag{7}$$

where

$$\begin{aligned} &\mathsf{g}_{11}=z\left(\Gamma_\phi,\Gamma_\phi\right), && \mathsf{g}_{12}=z\left(\Gamma_\phi,\Gamma_\vartheta\right),\\ &\mathsf{g}_{22}=z\left(\Gamma_\vartheta,\Gamma_\vartheta\right), && \mathsf{L}_{11}=\frac{det\left(\Gamma_\phi,\Gamma_\vartheta,\Gamma_{\phi\phi}\right)}{\sqrt{\mathsf{g}_{11}\mathsf{g}_{22}-\mathsf{g}_{12}^2}},\\ &\mathsf{L}_{12}=\frac{det\ \left(\Gamma_\phi,\Gamma_\vartheta,\Gamma_{\phi\vartheta}\ \right)}{\sqrt{\mathsf{g}_{11}\mathsf{g}_{22}-\mathsf{g}_{12}^2}}, && \mathsf{L}_{22}=\frac{det\ \left(\Gamma_\phi,\Gamma_\vartheta,\Gamma_{\vartheta\vartheta}\ \right)}{\sqrt{\mathsf{g}_{11}\mathsf{g}_{22}-\mathsf{g}_{12}^2}}.\end{aligned} \tag{8}$$

since $\mathsf{g}_{11}\mathsf{g}_{22}-\mathsf{g}_{12}^2>0$.
$\varphi=(0,0,1)$ denotes the isotropic unit normal vector field.
The Gaussian curvature $\mathscr{K}$ and mean curvature $\mathscr{H}$ are defined by using the classical notation above.

$$\mathscr{K}=\frac{\mathsf{L}_{11}\mathsf{L}_{22}-\mathsf{L}_{12}^2}{\mathsf{g}_{11}\mathsf{g}_{22}-\mathsf{g}_{12}^2},\quad \mathscr{H}=\frac{\mathsf{g}_{11}\mathsf{L}_{22}-2\mathsf{g}_{12}\mathsf{L}_{12}+\mathsf{L}_{11}\mathsf{g}_{22}}{\mathsf{g}_{11}\mathsf{g}_{22}-\mathsf{g}_{12}^2}$$

The surface **S** is said to be isotropic flat if $\mathscr{K}$ (resp. $\mathscr{H}$) vanishes (resp. isotropic minimal) [[6]- [7]- [8]].

3 TubeEmbankmentlike surface in I_3^1

If a one-parameter family of regular implicit surfaces $\Psi_\imath : g(X,\imath)=0, \imath\in[\imath_1,\imath_2]$. The intersection curve of two neighbored surfaces $\Psi_\imath$ and $\Psi_{\imath+\triangle\imath}$ fulfills the two equations $g(X,\imath)=0$ and $g(X,\imath+\triangle\imath)=0$. We calculate the limit for $\triangle\imath\longrightarrow 0$ and we get

$$g_\imath(X,\imath)=\lim_{\triangle\imath\longrightarrow 0}\frac{g(X,\imath)-g(X,\imath+\triangle\imath)}{\triangle\imath}=0,$$

The following definition is based on this.

Definition 1.[9] *Let $\Psi_\imath : g(X,\imath)=0, \imath\in[\imath_1,\imath_2]$ be a one parameter family of regular implicit $\mathbf{C}^2-$ surfaces. The surface that the two equations define*

$$g(X,\imath)=0,\qquad g_\imath(X,\imath)=0$$

is called an envelope of the given family surfaces.

Definition 2.[9] *Let $\Gamma : X=r(s)=(a(s),b(s),\imath(s))$ be a regular space curve and $0<\ell\in\mathbb{R}$ with $\left|\ell\imath'\right|<\sqrt{a'^2+b'^2}$. The envelope of the one parameter family of cones*

$$g(X;s)=(x-a(s))^2+(y-b(s))^2-\ell^2(z-\imath(s))^2=0$$

is called an embankment surface and Γ its base curve.

We'll look at a special case of embankment surface:

*Remark.*Let $\gamma : I\subseteq\mathbb{R}\longrightarrow I_3^1$ be a non-zero curvature Tubemankmentlike surface with a unit speed directrix curve Γ_{tel}

$$\Gamma_{tel}=\gamma(\phi)+\imath\left(\cos(\vartheta)\mathscr{N}(\phi)+\sin(\vartheta)\mathscr{B}(\phi)\right) \tag{9}$$

where $\imath=\sqrt{\ell^2+1}$.
The Tubemankmentlike surface equation in I_3^1 is given by

$$\Gamma_{tel}=\gamma(\phi)+\imath\,\delta(\phi,\vartheta) \tag{10}$$

The base curve γ and the director curve δ mentioned, where γ is a differentiable curve parametrized by its arc length, and δ is a differentiable curve parametrized by its arc length, i.e., $\left\langle\gamma',\gamma'\right\rangle_i=1$ and $\langle\delta,\delta\rangle_i=1$.
The curve δ is orthogonal to the tangent vector field $\mathscr{T}$ of the base curve γ, i.e., $\left\langle\delta',\mathscr{T}_\gamma\right\rangle$. We have two types:

1.The plane curves γ and δ are non isotropic parametrized by $\gamma=(\sin(\phi),0,\cos(\phi))$ and $\delta=(0,\cos(\vartheta),\sin(\vartheta))$. Then the surface **S** is parameterized by

$$\begin{aligned}\Gamma_{tel}&=\gamma(\phi)+\ \imath\,\delta(\phi,\vartheta)\\ &=(\sin(\phi),\imath\cos(\vartheta),\cos(\phi)+\ \imath\ \sin(\vartheta)).\end{aligned} \tag{11}$$

2.The isotropic curve $\gamma = (0,0,\sin(\phi))$ and non isotropic space curve δ parametrized by $\delta = (0,\cos(\vartheta),\sin(\vartheta))$, where $\langle\delta,\delta\rangle_i = 1$.Then the surface **S** is parametrized by

$$\begin{aligned} \Gamma_{tel} &= \gamma(\phi) + \imath\,\delta(\phi,\vartheta) \\ &= (0,\ \imath\cos(\vartheta),\sin(\phi) + \imath\sin(\vartheta)). \end{aligned} \tag{12}$$

If we refer to the surfaces generated by (11) and (12) of Type 3 and Type 4 in isotropic Tubemankmentlike surfaces I_3^1, respectively ([11],[12]).

4 Tubemankmentlike surfaces Satisfying $\Delta\Gamma_{tel_i} = \lambda_i \Gamma_{tel_i}$

4.1 Tubemankmentlike surfaces of type 3

In this section, we discussed the Tubembankmentlike surface of Type 3 in I_3^1 that fulfills the equation

$$\Delta\Gamma_{tel_i} = \lambda_i \Gamma_{tel_i} \tag{13}$$

where $\lambda_i \in \mathbb{R}, i = 1,2,3$ and

$$\Delta\Gamma_{tel_i} = (\Delta\Gamma_{tel_1}, \Delta\Gamma_{tel_2}, \Delta\Gamma_{tel_3})$$

where

$$\Gamma_{tel_1} = \sin\phi, \quad \Gamma_{tel_2} = \imath\cos(\vartheta), \quad \Gamma_{tel_3} = \cos(\phi) + \imath\sin(\vartheta).$$

For the Tubembankmentlike surface given by (11), the coefficients of the first and second fundamental forms are

$$\begin{aligned} \mathrm{g}_{11} &= (\cos(\phi) - (\cos(\phi) + \imath\cos(\vartheta))\kappa^2(\phi)) \\ &\quad + (\sin(\phi) - \kappa(\phi)\sin(\phi) + \imath\sin(\vartheta)\tau(\phi) \\ \mathrm{g}_{12} &= \imath\sin(\vartheta)(\sin(\phi) - \kappa(\phi)\sin(\phi) + \imath\sin(\vartheta)\tau(\phi) \\ &,\mathrm{g}_{22} = \imath^2\sin^2(\vartheta) \end{aligned} \tag{14}$$

$$\begin{aligned} \mathrm{L}_{11} &= (-2+\kappa(\phi))\sin(\phi)\tau(\phi) - \imath\sin(\vartheta)\tau^2(\phi) \\ &\quad + (\cos(\phi) + \imath\cos(\vartheta))\tau'(\phi) \\ \mathrm{L}_{12} &= -\imath\sin(\vartheta)\tau(\phi), \quad \mathrm{L}_{22} = -\imath\sin(\vartheta), \end{aligned} \tag{15}$$

respectively. The Gaussian curvature $\mathscr{K}$ and the mean curvature $\mathscr{H}$ are

$$\mathscr{K} = -\frac{(-2+\kappa(\phi))\sin(\phi)\tau(\phi) + (\cos(\phi)+\imath\cos(\vartheta))\tau'(\phi)}{\imath\sin(\vartheta)(\cos(\phi) - (\cos(\phi)+\imath\cos(\vartheta))\kappa(\phi))^2} \tag{16}$$

$$\begin{aligned} \mathscr{H} &= -((\csc(\vartheta) + (\imath(2\cos(\phi) + \imath\cos(\vartheta))\cot(\vartheta) \\ &\quad + \csc(\vartheta))\kappa(\phi)^2 + 2\imath\sin(\phi)\tau(\phi) \\ &\quad - \kappa(\phi)(2(\imath\cos(\phi)\cot(\vartheta) + \csc(\vartheta)) + \imath\sin(\phi)\tau(\phi)) \\ &\quad - \imath(\cos(\phi) + \imath\cos(\vartheta))\tau'(\phi))/(2\imath(\cos(\phi)(-1+\kappa(\phi)) \\ &\quad + \imath\cos(\vartheta)\kappa(\phi))^2)) \end{aligned} \tag{17}$$

respectively.

Theorem 4.1.1 The isotropic-Tubembankmentlike surface given by (11) are isotropic flat $\mathscr{K} = 0$ under the following condition

$$\tau(\phi) = -\frac{(\cos(\phi) + \imath\cos(\vartheta))\csc(\vartheta)\tau'(\phi)}{-2+\kappa(\phi)}.$$

Proof. A Tubembankmentlike surface satisfies the equation $\mathscr{K} = 0$ from Eq. (16), We have

$$(-2+\kappa(\phi))\sin(\phi)\tau(\phi) + (\cos(\phi) + \imath\cos(\vartheta))\tau'(\phi) = 0$$

By solving this equation we have that $\tau(\phi) = -\dfrac{(\cos(\phi) + \imath\cos(\vartheta))\tau'(\phi)}{-2+\kappa(\phi)}$

Theorem 4.1.2 The isotropic-Tubembankmentlike surface (11) is harmonic $\mathscr{H} = 0$ under the following condition

$$\tau(\phi) = -\frac{\csc(\phi)\csc(\vartheta) - \imath\cos(\phi)\tau'(\phi) - \imath^2\cos(\vartheta)\tau'(\phi)}{2\imath}.$$

Proof. A Tubembankmentlike surface satisfies the equation $\mathscr{H} = 0$ from Eq. (17), We have

$$\begin{aligned} &\csc(\vartheta) + (\imath(2\cos(\phi) + \imath\cos(\vartheta))\cot(\vartheta) + \csc(\vartheta))\kappa^2(\phi) \\ &+ 2\imath\sin(\phi)\tau(\phi) - \kappa(\phi)(2(\imath\cos(\phi)\cot(\vartheta) + \csc(\vartheta)) \\ &+ \imath\sin(\phi)\tau(\phi)) - \imath(\cos(\phi) + \imath\cos(\vartheta))\tau'(\phi) = 0 \end{aligned}$$

then,

$$\tau(\phi) = -\frac{\csc(\phi)\csc(\vartheta) - \imath\cos(\phi)\tau'(\phi) - \imath^2\cos(\vartheta)\tau'(\phi)}{2\imath} \tag{18}$$

Theorem 4.1.3 The isotropic-Tubembankmentlike surface given by (11) is Weingarten surface iff the curve is a straight line and satisfies one of two conditions

1.$\tau(\phi) = 0$

2.$\tau(\phi) = -\left(\frac{2\cos^2(\phi)\csc(\vartheta)}{\imath\,(7\sin(\phi) - \sin(3\phi))}\right)$.

Proof. Weingarten Tubembankmentlike surface satisfies Jacobi equation

$$\begin{aligned} \mathscr{K}_\phi &= \Big(\csc(\vartheta)(2(-\sin(\phi) + \kappa(\phi)\sin(\phi) \\ &- (\cos(\phi) + \imath\cos(\vartheta))\kappa'(\phi))((-2+\kappa(\phi))\sin(\phi)\tau(\phi) \\ &+ (\cos(\phi) + \imath\cos(\vartheta))\tau'(\phi)) - (\cos(\phi) - (\cos(\phi) \\ &+ \imath\cos(\vartheta))\kappa(\phi))(\tau(\phi)(-2\cos(\phi) + \cos(\phi)\kappa(\phi) \\ &+ \sin(\phi)\kappa'(\phi)) + (-3+\kappa(\phi))\sin(\phi)\tau'(\phi) + \tau''(\phi)(\cos(\phi) \\ &+ \imath\cos(\vartheta)))\Big)/(\imath(\cos(\phi) - (\cos(\phi) + \imath\cos(\vartheta))\kappa(\phi))^3), \end{aligned}$$

$$\begin{aligned} \mathscr{K}_\vartheta &= \Big((\cos(\phi) - (\cos(\phi) + \imath\cos(\vartheta))\kappa(\phi))\tau'(\phi) + 2\kappa(\phi) \\ &((-2+\kappa(\phi))\sin(\phi)\tau(\phi) + (\cos(\phi) + \imath\cos(\vartheta))\tau'(\phi)) \\ &+ (1/\imath)\cot(\vartheta)\cos(\vartheta)(\cos(\phi) - (\cos(\phi) + \imath\cos(\vartheta))\kappa(\phi)) \\ &((-2+\kappa(\phi))\sin(\phi)\tau(\phi) + (\cos(\phi) \\ &+ \imath\cos(\vartheta))\tau'(\phi))\Big)/(\cos(\phi) - (\cos(\phi) + \imath\cos(\vartheta))\kappa(\phi))^3 \end{aligned}$$

$$\begin{gathered}\mathscr{H}_\phi=\Big(-(2(\sin(\phi)-\kappa(\phi)\sin(\phi)+(\cos(\phi)\\+\imath\cos(\vartheta))\kappa'(\phi))(\csc(\vartheta)+(\imath(2\cos(\phi)\\+\imath\cos(\vartheta))\cot(\vartheta)+\csc(\vartheta))\kappa^2(\phi)+2\imath\sin(\phi)\tau(\phi)\\-\kappa(\phi)(2(\imath\cos(\phi)\cot(\vartheta)+\csc(\vartheta))\\+\imath\sin(\phi)\tau(\phi))-\imath(\cos(\phi)+\imath\cos(\vartheta))\tau'(\phi))\\-\cos(\phi)(-1+\kappa(\phi))\\+\imath\cos(\vartheta)\kappa(\phi))(-2\imath\cot(\vartheta)\kappa^2(\phi)\sin(\phi)+2\imath\cos(\phi)\tau(\phi)\\+2(\imath(2\cos(\phi)+\imath\cos(\vartheta))\cot(\vartheta)\\+\csc(\vartheta))\kappa(\phi)\kappa'(\phi)-(2(\imath\cos(\phi)\cot(\vartheta)\\+\csc(\vartheta))+\imath\sin(\phi)\tau(\phi))\kappa'(\phi)+3\imath\sin(\phi)\tau'(\phi)\\-\imath\kappa(\phi)(\cos(\phi)\tau(\phi)+\sin(\phi)(-2\cot(\vartheta)+\tau'(\phi)))\\-\imath(\cos(\phi)+\imath\cos(\vartheta))\\(\tau''(\phi))\Big)/(2\imath(\cos(\phi)(-1+\kappa(\phi))+\imath\cos(\vartheta)\kappa(\phi))^3)\end{gathered}$$

,and

$$\begin{gathered}\mathscr{H}_\vartheta=\Big(-2\kappa(\phi)\sin(\vartheta)(\csc(\vartheta)+(\imath(2\cos(\phi)\\+\imath\cos(\vartheta))\cot(\vartheta)+\csc(\vartheta))\kappa^2(\phi)+2\imath\sin(\phi)\tau(\phi)\\-\kappa(\phi)(2(\imath\cos(\phi)\cot(\vartheta)+\csc(\vartheta))+\imath\sin(\phi)\tau(\phi))\\-\imath(\cos(\phi)+\imath\cos(\vartheta))\tau'(\phi))+(1/\imath)\csc(\vartheta)^2\\(-\cos(\phi)+(\cos(\phi)+\imath\cos(\vartheta))\kappa(\phi))\\(\cos(\vartheta)-2(\imath\cos(\phi)+\cos(\vartheta))\kappa(\phi)\\+\kappa^2(\phi)(2\imath\cos(\phi)+\cos(\vartheta)(1+\imath^2+\imath^2\sin^2(\vartheta)))-\imath^2\sin^2(\vartheta)\\\tau'(\phi))\Big)/(2(\cos(\phi)(-1+\kappa(\phi))+\imath\cos(\vartheta)\kappa(\phi))^3)\end{gathered}$$

By apply $\mathscr{K}_\phi\mathscr{H}_\vartheta=\mathscr{K}_\vartheta\mathscr{H}_\phi$ Since this equation leads to one of two conditions:

$1.\tau(\phi)=0$

$2.\tau(\phi)=-(\frac{2\cos^2(\phi)\csc(\vartheta)}{\imath(7\sin(\phi)-\sin(3\phi))})$

4.2 Tubemankmentlike surfaces of type 4

In this section, we discussed the Tubembankmentlike surface of Type 4 in I_3^1 that fulfills the equation

$$\Delta\Gamma_{tel_i}=\lambda_i\Gamma_{tel_i} \tag{19}$$

where $\lambda_i\in\mathbb{R}, i=1,2,3$ and

$$\Delta\Gamma_{tel_i}=(\Delta\Gamma_{tel_1},\Delta\Gamma_{tel_2},\Delta\Gamma_{tel_3})$$

where

$$\Gamma_{tel_1}=0,\quad \Gamma_{tel_2}=\imath\cos(\vartheta),\quad \Gamma_{tel_3}=\sin(\phi)+\imath\sin(\vartheta).$$

For the Tubembankmentlike surface given by (12), the coefficients of the first and second fundamental forms are

$$\begin{gathered}\mathsf{g}_{11}=\imath^2\cos^2(\vartheta)\kappa^2(\phi)+(\sin(\phi)+\imath\sin(\vartheta))^2\tau^2(\phi)\\ \mathsf{g}_{12}=\imath\sin(\vartheta)(\sin(\phi)+\imath\sin(\vartheta)\tau(\phi),\quad \mathsf{g}_{22}=\imath^2\sin^2(\vartheta)\end{gathered} \tag{20}$$

$$\begin{gathered}\mathsf{L}_{11}=-\sin(\phi)-(\sin(\phi)+\imath\sin(\vartheta))\tau^2(\phi)+\imath\cos(\vartheta)\tau'(\phi),\\ \mathsf{L}_{12}=-\imath\sin(\vartheta)\tau(\phi),\quad \mathsf{L}_{22}=-\imath\sin(\vartheta),\end{gathered} \tag{21}$$

respectively. The Gaussian curvature $\mathscr{K}$ and the mean curvature $\mathscr{H}$ are

$$\mathscr{K}=\frac{\sin(\phi)+\sin(\phi)\tau^2(\phi)-\imath\cos(\vartheta)\tau'(\phi)}{\imath^3\kappa^2(\phi)\sin(\vartheta)\cos^2(\vartheta)} \tag{22}$$

$$\begin{gathered}\mathscr{H}=-\Big(\imath^2\csc(\vartheta)\kappa^2(\phi)+\sec(\vartheta)(\sec(\vartheta)\sin(\phi)(\imath+\csc(\vartheta)\\ \sin(\phi))\tau^2(\phi)+\imath(\sec(\vartheta)\sin(\phi)-\imath\tau'(\phi)))\Big)/(2\imath^3\kappa^2(\phi))\end{gathered} \tag{23}$$

respectively.

Theorem 4.2.1 The isotropic-Tubembankmentlike surface given by (12) are developable $\mathscr{K}=0$ satisfies $\tau(\phi)=\pm\sqrt{-1+\imath\cos(\vartheta)\csc(\phi)\tau'(\phi)}$.

Proof. A Tubembankmentlike surface satisfies the equation $\mathscr{K}=0$ from Eq. (22), We have $\sin(\phi)+\sin(\phi)\tau^2(\phi)-\imath\cos(\vartheta)\tau'(\phi)=0$. We get $\tau(\phi)=\pm\sqrt{-1+\imath\cos(\vartheta)\csc(\phi)\tau'(\phi)}$.

Theorem 4.2.2 The isotropic-Tubembankmentlike surface (12) is isotropic minimal under one of two condition.

$$1.\kappa(\phi)=-\Big(\Im\sqrt{\frac{\sec(\vartheta)\sin(\phi)\tan(\vartheta)}{\imath}}\Big)$$
$$2.\kappa(\phi)=\Big(\Im\sqrt{\frac{\sec(\vartheta)\sin(\phi)\tan(\vartheta)}{\imath}}\Big)$$

Proof. A Tubembankmentlike surface satisfies the equation $\mathscr{H}=0$ from Eq. (23), We have

$$\begin{gathered}-(\imath^2\csc(\vartheta)\kappa^2(\phi)+\sec(\vartheta)(\sec(\vartheta)\sin(\phi)(\imath\\+\csc(\vartheta)\sin(\phi))\tau^2(\phi)+\imath(\sec(\vartheta)\sin(\phi)-\imath\tau'(\phi))))=0\end{gathered}$$

then we have one of these conditions

$$1.\kappa(\phi)=-\Big(\Im\sqrt{\frac{\sec(\vartheta)\sin(\phi)\tan(\vartheta)}{\imath}}\Big)$$
$$2.\kappa(\phi)=\Big(\Im\sqrt{\frac{\sec(\vartheta)\sin(\phi)\tan(\vartheta)}{\imath}}\Big)$$

Theorem 4.2.3 The Weingarten isotropic Tubembankmentlike surface given by (12) satisfies the following conditions:

$\kappa(\phi)=\pm\Big(\sqrt{\frac{\cos(\phi-\vartheta)-\cos(\phi+\vartheta)}{\imath+\imath\cos(2\vartheta)}}\Big)$

or

$\kappa(\phi)=2\tan(\phi)\kappa'(\phi)$

if the curve γ is the straight line.

Proof. Weingarten Tubembankmentlike surface satisfies Jacobi equation

$$\begin{gathered}\mathscr{K}_\phi=-(1/(\imath^3\kappa^3(\phi)))\csc(\vartheta)\sec(\vartheta)^2(2\kappa'(\phi)(\sin(\phi)\\+\sin(\phi)\tau^2(\phi)-\imath\cos(\vartheta)\tau'(\phi))-\kappa(\phi)(\cos(\phi)+\cos(\phi)\tau^2(\phi)\\+2\sin(\phi)\tau(\phi)\tau'(\phi)-\imath\cos(\vartheta)(\tau''(\phi)))\end{gathered}$$

,

$$\begin{gathered}\mathscr{K}_\vartheta=(\sec(\vartheta)(-\csc^2(\vartheta)+2\sec^2(\vartheta))\sin(\phi)(1+\tau^2(\phi))\\+4\imath\cot(2\vartheta)\csc(2\vartheta)\tau'(\phi))/(\imath^3\kappa^2(\phi))\end{gathered}$$

$$\begin{gathered}\mathscr{H}_\phi=(1/(2\imath^3\kappa^2(\phi)))\tan(\vartheta)(\imath^2\cot^2(\vartheta)\csc(\vartheta)\kappa^3(\phi)\\+\sin(\phi)(\sec^3(\vartheta)\sin(\phi)-2\sec^2(\vartheta)(\imath+\sec(\vartheta)\sin(\phi)))\\ \tau^2(\phi)+\imath\sec(\vartheta)(-2\sec(\vartheta)\sin(\phi)+\imath\tau'(\phi)))\end{gathered}$$

, and

$$
\begin{aligned}
\mathscr{H}_{\vartheta} = -\big(1/(\iota^3\kappa^3(\phi))\big)\csc(\vartheta)\sec^2(\vartheta)(2\kappa'(\phi)(\sin(\phi) \\
+\sin(\phi)\iota\,(\sec(\vartheta)\sin(\phi)-\iota\,\tau(\phi)^2 \\
-\iota\,\cos(\vartheta)\iota\,(\sec(\vartheta)\sin(\phi)-\iota\,\tau'(\phi)) \\
-\kappa(\phi)(\cos(\phi)+\cos(\phi)\iota\,(\sec(\vartheta)\sin(\phi)-\iota\,\tau^2(\phi) \\
+2\sin(\phi)\iota\,(\sec(\vartheta)\sin(\phi)-\iota\,\tau'(\phi) \\
-\iota\,\cos(\vartheta)(\iota\,(\sec(\vartheta)\sin(\phi)-\iota\,\tau''(\phi)))
\end{aligned}
$$

By apply $\mathscr{K}_{\phi}\mathscr{H}_{\vartheta}=\mathscr{K}_{\vartheta}\mathscr{H}_{\phi}$ Since this equation leads to

$$
\begin{aligned}
\sec(\vartheta)\Big((\sec(\vartheta)(-\csc^2(\vartheta)+2\sec^2(\vartheta))\sin(\phi) \\
(1+\tau^2(\phi))+4\iota\cot(2\vartheta)\csc(2\vartheta)\tau'(\phi)) \\
(2\kappa'(\phi)(-\sec(\vartheta)\sin(\phi)(\iota\,+(\iota\,+\csc(\vartheta)\sin(\phi))\tau^2(\phi)) \\
+\iota^2\tau'(\phi))+\kappa(\phi)(\sec(\vartheta)(\cos(\phi) \\
(\iota+(\iota+2\csc(\vartheta)\sin(\phi))\tau^2(\phi)) \\
+2\sin(\phi)(\iota+\csc(\vartheta)\sin(\phi))\tau(\phi)\tau'(\phi)) \\
-\iota^2\tau''(\phi))-\sec^2(\vartheta)(\iota^2\cot^2(\vartheta)\csc(\vartheta)\kappa^2(\phi) \\
+\sin(\phi)(\csc^3(\vartheta)\sin(\phi)-2\sec^2(\vartheta) \\
(\iota+\csc(\vartheta)\sin(\phi)))\tau^2(\phi)+\iota\sec(\vartheta)(-2\sec(\vartheta)\sin(\phi) \\
+\iota\tau'(\phi)))(2\kappa'(\phi)(\sin(\phi)+\sin(\phi)\tau^2(\phi) \\
-\iota\,\cos(\vartheta)\tau'(\phi))-\kappa(\phi)(\cos(\phi)+\cos(\phi)\tau^2(\phi) \\
+2\sin(\phi)\tau(\phi)\tau'(\phi)-\iota\,\cos(\vartheta)\tau'(\phi))\Big)\tau(\phi)=0
\end{aligned}
$$

Then we have one of these conditions:
$\kappa(\phi)=\pm\left(\sqrt{\frac{\cos(\phi-\vartheta)-\cos(\phi+\vartheta)}{\iota+\iota\cos(2\vartheta)}}\right)$
or
$\kappa(\phi)=2\tan(\phi)\kappa'(\phi)$

5 Visualizations for Tubembankmentlike Surfaces in I_3^1

Finally, consider the Visualizations below for Tubembankmentlike surfaces of types 3 and 4:

Application 5.1.
Let us take directrix as

$$\gamma(\phi)=(\phi,\cos(\phi),0), \tag{24}$$

which is an arbitrary in I_3^1 and $\ell=\sqrt{3}$.
Under these assumptions, the Tubembankmentlike surface (10) is produced by

$$\Gamma_{tel}=(\phi,\cos(\phi)+2\cos(\vartheta),2\sin(\vartheta)), \tag{25}$$

In figures 1 and 2, one can see the directrix (24) and Tubembankmentlike surface (25).

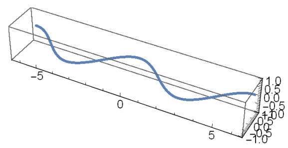

Fig. 1: The directrix (24)

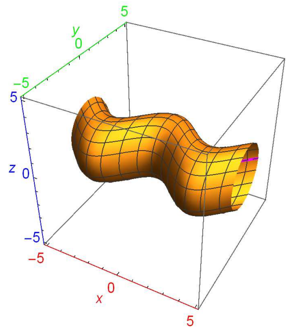

Fig. 2: Tubembankmentlike surface (25)

In Figures 3 and 4 we can see the Gaussian and mean curvatures functions' graphics above and the variations of Gaussian and mean curvatures on Tubembankmentlike surface (25) below.

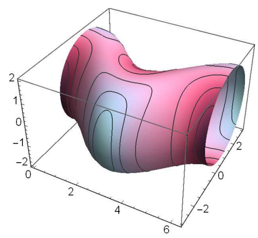

Fig. 3: Gaussian curvature function graphic above and the variations of Gaussian curvature on Tubembankmentlike surface (25)

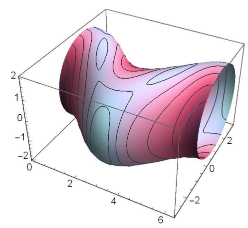

Fig. 4: mean curvature function graphic above and the variations mean curvature on Tubembankmentlike surface (25)

Application 5.2.
Let's look at the case of directrix

$$\gamma(\phi) = (\phi/2, \cos(\phi/2), 0), \tag{26}$$

In I_3^1 and $\ell = 1/2$, this is an arbitrary value.
The Tubembankmentlike surface (10) is created with these assumptions

$$\Gamma_{tel} = (\phi/2, \cos(\phi/2) + \sqrt{5}/2\cos(\vartheta), \sqrt{5}/2\sin(\vartheta)), \tag{27}$$

In figures 5 and 6 the directrix and Tubembankmentlike surface (26) can be shown in (27).

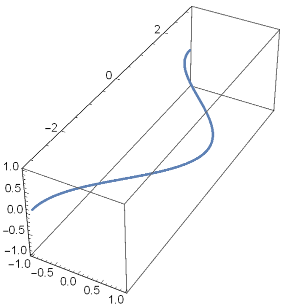

Fig. 5: The directrix (26)

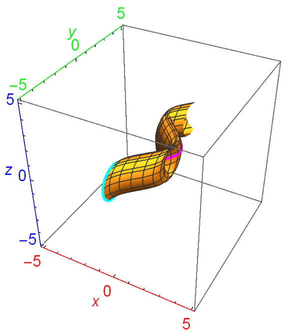

Fig. 6: Tubembankmentlike surface (27)

The graphics of the Gaussian and mean curvatures functions are shown above, and the variations of Gaussian and mean curvatures on the Tubembankmentlike surface (27) are shown below in Figures 7 and 8.

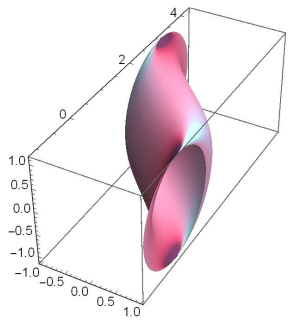

Fig. 7: Gaussian curvature function graphic above and the variations of Gaussian curvature on Tubembankmentlike surface (27)

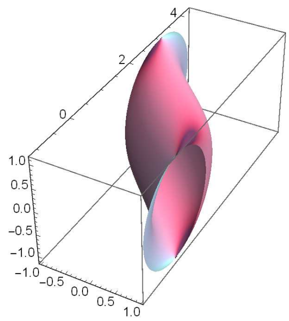

Fig. 8: mean curvature function graphic above and the variations mean curvature on Tubembankmentlike surface (27)

Application 5.3.
Consider the case of directrix.

$$\gamma(\phi) = (0, 0, \cos(\phi)), \tag{28}$$

This is an arbitrary value in I_3^1 and $\ell = 1$.
The Tubembankmentlike surface (10) is created with these assumptions

$$\Gamma_{tel} = (0, \sqrt{2}\cos(\vartheta), \cos(\phi) + \sqrt{2}\sin(\vartheta)), \tag{29}$$

In figures 9 and 10 the directrix and Tubembankmentlike surface (28) can be shown in (29).

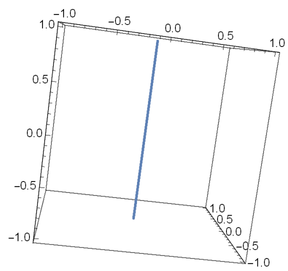

Fig. 9: The directrix (28)

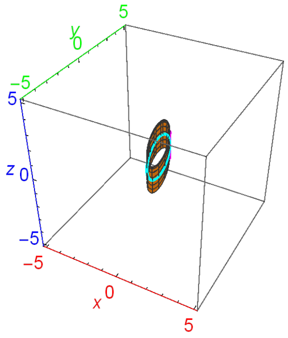

Fig. 10: Tubembankmentlike surface (29)

The Gaussian and mean curvatures functions are graphically depicted above, and changes of Gaussian and mean curvatures on the Tubembankmentlike surface (29) are depicted below in Figures 11 and 12.

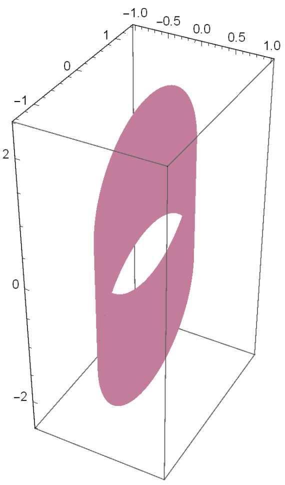

Fig. 11: The illustrations above show the Gaussian curvature function, as well as variation in Gaussian curvature on a Tubembankmentlike surface (29)

We can see from figures 11 and 12 that the mean curvature and Gaussian curvature are both equal.

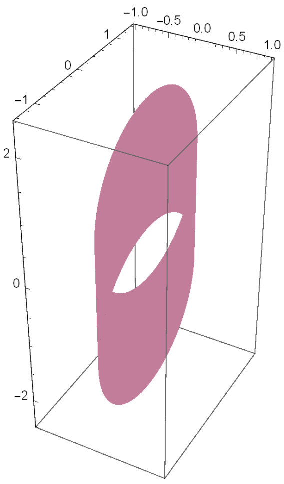

Fig. 12: The illustrations above show the mean curvature function, as well as variation in mean curvature on a Tubembankmentlike surface (29)

6 Perspective

This section discusses some research viewpoints of the Embankment surface in isotropic space according to the Frenet-Serret frame explained. Also, a case of Embankment and its differential geometric characteristics had studied. Finally, computational applications to establish our main results are presented and plotted. All calculations and figures in this paper had accomplished by using Wolfram Mathematica.

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