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Weak Pseudo-Invexity and Characterizations of Solutions in Multiobjective Programming

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Abstract: In this paper, we study Fritz John type optimality for nonlinear multiobjective programming problems under new classes of generalized invex vector functions. Relationships between these classes of vector functions are established by giving several examples. Furthermore, optimality conditions and characterizations of efficient and weakly efficient solutions are obtained under weak pseudo-invexity and by using a concept of generalized Fritz John vector critical point. We have illustrated through various non-trivial examples that the results obtained in this paper extend many previously known results in this area.

Keywords: Multiple objective programming, Weak pseudo-invexity, Weak FJ-pseudo-invexity, Generalized Fritz-John vector critical point, (Weakly) efficient solution.

1 Introduction

Convexity and generalized convexity play a fundamental role in various fields such as mathematical economics, engineering, management science, and optimization theory. This led to consider the research on convexity and generalized convexity as one of the most important and attractive aspects in mathematical programming. Several new concepts concerning a generalized convex function have been proposed in the literature. Among these, the concept of invexity, for differentiable functions, introduced by Hanson in [13] has received a great extent of attention. A differentiable function $f: D \subseteq \mathbb{R}^n \to \mathbb{R}$ is said to be *invex* at $x_0 \in D$ with respect to $\eta: D \times D \to \mathbb{R}^n$, if for each $x \in D$, $f(x) - f(x_0) \geqq [\nabla f(x_0)]^t \eta(x, x_0)$. Craven and Glover [10] and Ben-Israel and Mond [8] stated that the class of invex functions are all those functions whose stationary points are global minima. Hanson [13] noted that there are simple extensions of invex functions, the pseudo-invex and quasi-invex functions. Furthermore, in the scalar case, Ben-Israel and Mond [8] proved that the classes of invex and pseudo-invex functions coincide.

For the classical mathematical programming problem (P), defined by

$$(P) \quad \begin{array}{c} \text{Minimize } f(x), \\ \text{subject to } g_j(x) \leqq 0, \ j \in K = \{1, \dots, k\}, \end{array}$$

with differentiable functions $f, g_j : D \subseteq \mathbb{R}^n \to \mathbb{R}, \ j \in K$, Hanson [13] showed that, under the invexity requirement for f and $g_j, \ j \in K$ (with respect to the same η), every Kuhn-Tucker critical point is a global minimizer of (P). Martin [18] remarked that the converse is not true in general, and he proposed a weaker notion, called KT-invexity, which assures that every Kuhn-Tucker critical point is a minimizer of problem (P) if and only if problem (P) is KT-invex.

Later, researchers have extended these results to multiobjective problems. So, Ruíz-Canales and Rufián-Lizana [27] have characterized weakly efficient solutions in the case of nondifferentiable functions. In the differentiable case, Osuna-Gómez et al. [25, 26] have defined new kind of vector pseudo-invex functions and they have characterized the weakly efficient solutions for unconstrained and constrained multiobjective programming problems. Arana-Jiménez et al. [4,5] have extended the study of Osuna-Gómez et al. [25,26] to provide necessity and sufficiency results for efficient solutions under new kind of functions. They called these functions pseudo-invex II in difference to pseudo-invex of Osuna-Gómez et al. which is called pseudo-invex I by Arana-Jiménez et al. Further sufficient optimality conditions and duality results for multiobjective problems have been obtained, with different approaches, under

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generalized invexity with respect to the same η, by Antczak [2,3], Batista Santos et al. [7], Hanson et al. [14], Kaul et al. [16], Mishra et al. [19,20], Niculescu [22], Nobakhtian [23], Nobakhtian and Pouryayevali [24], Ruiz-Garzón et al. [28], and others. By considering the invexity with respect to different $(\eta_i)_i$ (each function occurring in the studied problem is considered with respect to its own function η_i instead of a same function η), Slimani and Radjef [29,30,31] have obtained necessary and sufficient optimality conditions and duality results for nonlinear scalar and (nondifferentiable) multiobjective problems. Ahmad [1] has considered a nondifferentiable multiobjective problem and by using generalized univexity with respect to different $(\eta_i)_i$, he has obtained optimality conditions and duality results. Arana-Jiménez et al. [6] have used the concept of semidirectionally differentiable functions introduced in [31] to derive characterizations of solutions and duality results by means of generalized pseudoinvexity for nondifferentiable multiobjective programming. Kharbanda et al. [17] have considered a class of constrained nonsmooth multiobjective programming problem involving semi-directionally differentiable functions. They have obtained sufficient optimality conditions and various duality theorems by using a new generalized class of $(d_I-\rho-\sigma)$-V-type I univex functions with respect to different $(\eta_i)_i$.

In parallel to all these developments and advances of the invexity and its extensions in theory, some applications in practice begin to take place. Recently, Dinuzzo et al. [11] have obtained some kernel function in Machine Learning which is not quasi-convex (and hence also neither convex nor pseudoconvex) but it is invex. Nickisch and Seeger [21] have studied a multiple kernel learning problem and have used the invexity to deal with the optimization which is non convex. Syed et al. [32] have considered Minimization of Error Entropy (MEE) and Minimization of Error Entropy with Fiducial points (MEEF) and optimization properties are given involving invexity. In particular, they have shown that by varying the kernel parameter of the MEE and/or MEEF objective function in general leads to an invex problem.

In the present paper, we consider new concepts of generalized invex vector functions with respect to different $(\eta_i)_i$ and we extend the studies of Osuna-Gómez et al. [25,26] and Arana-Jiménez et al. [4,5]. We establish relationships between these classes of vector functions and we obtain necessary and sufficient optimality conditions for a feasible point to be weakly efficient or efficient solution for a multiobjective programming problem with inequality constraints. Moreover, we use a concept of Fritz John type vector critical point to establish characterizations of efficient and weakly efficient solutions.

2 Preliminaries and definitions

The following conventions for equalities and inequalities will be used. If $x=(x_1,\dots,x_n)$, $y=(y_1,\dots,y_n)\in\mathbb{R}^n$, then
$x=y \Leftrightarrow x_i=y_i$, i=1,...,n;
$x<y \Leftrightarrow x_i<y_i$, i=1,...,n;
$x\leqq y \Leftrightarrow x_i\leqq y_i$, i=1,...,n;
$x\leq y \Leftrightarrow x\leqq y$ and $x\neq y$.
We also note $\mathbb{R}^q_{\geqq}$ (resp. $\mathbb{R}^q_{\geq}$ or $\mathbb{R}^q_{>}$) the set of vectors $y\in\mathbb{R}^q$ with $y\geqq 0$ (resp. $y\geq 0$ or $y>0$).

Invex functions were introduced to optimization theory by Hanson [13] (and named by Craven [9]) as a very broad generalization of convex functions.

Definition 1. *(Craven, [9], Hanson, [13]) Let D be a nonempty open set of $\mathbb{R}^n$ and $\eta: D\times D\to\mathbb{R}^n$ be a vector function. A function $f: D\to\mathbb{R}$ is said to be (def) at $x_0\in D$ on D with respect to η, if the function f is differentiable at x_0 and for each $x\in D$, (cond) holds.*

(i)def: invex,
cond:

$$f(x)-f(x_0)\geqq[\nabla f(x_0)]^t\eta(x,x_0). \tag{1}$$

(ii)def: pseudo-invex,
cond:

$$[\nabla f(x_0)]^t\eta(x,x_0)\geqq 0\Rightarrow f(x)-f(x_0)\geqq 0. \tag{2}$$

(iii)def: quasi-invex,
cond:

$$f(x)-f(x_0)\leqq 0\Rightarrow[\nabla f(x_0)]^t\eta(x,x_0)\leqq 0. \tag{3}$$

If the inequality in (1) (resp. second (implied) inequality in (3)) is strict ($x\neq x_0$), we say that f is strictly invex (resp. strictly quasi-invex) at x_0 on D with respect to η. f is said to be (strictly) invex (resp. pseudo-invex or (strictly) quasi-invex) on D with respect to η, if f is (strictly) invex (resp. pseudo-invex or (strictly) quasi-invex) at each $x_0\in D$ on D with respect to the same η.

Remark. When the function $\eta(x,x_0)=x-x_0$, the definition of (strict) invexity (resp. pseudo-invexity and quasi-invexity) reduces to the definition of (strict) convexity (resp. pseudo-convexity and quasi-convexity).

In the following example, we give two scalar functions f_1 and f_2 such that each function f_i is invex at a point x_0 with respect to its own η_i, $i=1,2$. However, there exists no a function η for which the vector function $f=(f_1,f_2)$ is invex at x_0.

Example 1. The function $f_1:\ D=]0,\frac{\pi}{2}[\to\mathbb{R}$ defined by $f_1(x)=x+\sin x$ is invex at $x_0=\frac{\pi}{3}$ on D with respect to $\eta_1(x,x_0)=(\sin x-\sin x_0)/\cos x_0$, but f_1 is not invex at x_0 on D with respect to $\eta_2(x,x_0)=(\cos x_0-\cos x)/\sin x_0$ (take $x=\frac{\pi}{6}$).
On the other hand, the function $f_2:\ D\to\mathbb{R}$ defined by

$f_2(x) = \cos x$ is invex at $x_0 = \frac{\pi}{3}$ on D with respect to η_2, but f_2 is not invex at x_0 on D with respect to η_1 (take $x = \frac{\pi}{6}$). Furthermore, it is not difficult to prove that there exists no a function $\eta : D \times D \to \mathbb{R}$ for which the functions f_1 and f_2 are both invex at $x_0 = \frac{\pi}{3}$ on D (take $x = \frac{\pi}{6}$).

Now, we consider the invex and weakly pseudo-invex vector functions with respect to different $(\eta_i)_{i\in\mathcal{N}}$ such that $\mathcal{N} = \{1,...,N\}$.

Definition 2. *Let D be a nonempty open set of $\mathbb{R}^n$ and $\eta_i : D \times D \to \mathbb{R}^n$, $i \in \mathcal{N}$ be vector functions. A vector function $f : D \to \mathbb{R}^N$ is said to be invex at $x_0 \in D$ on D with respect to $(\eta_i)_{i\in\mathcal{N}}$, if the function f is differentiable at x_0 and for each $x \in D$:*

$$f_i(x) - f_i(x_0) \geqq [\nabla f_i(x_0)]^t \eta_i(x,x_0), \text{ for all } i \in \mathcal{N}. \quad (4)$$

In other terms, f is invex at $x_0 \in D$ on D with respect to $(\eta_i)_{i\in\mathcal{N}}$, if each of its components f_i is invex at x_0 on D with respect to its own η_i, $i \in \mathcal{N}$. f is said to be invex on D with respect to $(\eta_i)_{i\in\mathcal{N}}$, if f is invex at each $x_0 \in D$ on D with respect to the same $(\eta_i)_{i\in\mathcal{N}}$. If the inequalities in (4) are strict, we say that f is strictly invex at x_0 on D with respect to $(\eta_i)_{i\in\mathcal{N}}$.

Arana-Jiménez et al. [4,5] have defined two classes of functions generalizing the class of scalar pseudo-invex functions. They call them pseudo-invex I, pseudo-invex in the sense of Osuna-Gómez et al. [25,26], and pseudo-invex II (with respect to the same η). In the same manner, we introduce new kinds of functions which we will designate as weak pseudo-invex I and weak pseudo-invex II (with respect to different $(\eta_i)_{i\in\mathcal{N}}$).

Definition 3. *Let D be a nonempty open set of $\mathbb{R}^n$ and $\eta_i : D \times D \to \mathbb{R}^n$, $i \in \mathcal{N}$ be vector functions. A vector function $f : D \to \mathbb{R}^N$ is said to be (def) at $x_0 \in D$ on D with respect to $(\eta_i)_{i\in\mathcal{N}}$, if the function f is differentiable at x_0 and for each $x \in D$, (cond) holds.*

(i)def: weakly pseudo-invex I,
cond:

$$f(x) - f(x_0) < 0 \Rightarrow \exists\, \bar{x} \in D,\ [\nabla f_i(x_0)]^t \eta_i(\bar{x},x_0) < 0, \text{ for all } i \in \mathcal{N}. \quad (5)$$

(ii)def: weakly pseudo-invex II,
cond:

$$f(x) - f(x_0) \leq 0 \Rightarrow \exists\, \bar{x} \in D,\ [\nabla f_i(x_0)]^t \eta_i(\bar{x},x_0) < 0, \text{ for all } i \in \mathcal{N}. \quad (6)$$

If $\bar{x} = x$, in the relation (5) (resp. (6)), we say that f is pseudo-invex I (resp. II) at x_0 on D with respect to $(\eta_i)_{i\in\mathcal{N}}$. f is said to be (weakly) pseudo-invex I (resp. II) on D with respect to $(\eta_i)_{i\in\mathcal{N}}$, if f is (weakly) pseudo-invex I (resp. II) at each $x_0 \in D$ on D with respect to the same $(\eta_i)_{i\in\mathcal{N}}$.

Remark. (i)Note that, in Definition 3, $\bar{x}$ depends on x and x_0, *i.e.* $\bar{x} = \bar{x}(x,x_0)$, and it is easy to see that if the vector functions η_i, $i \in \mathcal{N}$ are equal to a same function η and $\bar{x} = x$, we obtain equivalently, with the condition (5), the pseudo-invexity given by Osuna-Gómez et al. [25,26] and, with the condition (6), the pseudo-invexity II given by Arana-Jiménez et al. [4,5].
(ii)In Definition 3, if $N = 1$ then weak pseudo-invexity I and II are equivalent and we will say that the function is weakly pseudo-invex at x_0 (that is the weak pseudo-invexity of scalar function). If further $\bar{x} = x$, we deduce the pseudo-invexity of Definition 1.

In the following example, we give a vector function which is not pseudo-invex with respect to the same η (in the sense of Osuna-Gómez et al. [25,26] and in the sense of Arana-Jiménez et al. [4,5]) but it is weakly pseudo-invex I (and II) with respect to different $(\eta_i)_i$ (in the sense of Definition 3).

Example 2. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ with $f(x) = (f_1(x), f_2(x)) = (x_1 - x_2 - x_1^2, -x_1 + x_2 - x_2^2)$. There exists no a function η for which the vector function f is pseudo-invex (in the sense of Osuna-Gómez et al. [25,26] and in the sense of Arana-Jiménez et al. [4,5]) at $x_0 = (0,0)$ on $\mathbb{R}^2$ (take $x = (0,2)$). But f is weakly pseudo-invex I at x_0 on $\mathbb{R}^2$ with respect to $\eta_1(x,x_0) = (x_1, -x_1)$ and $\eta_2(x,x_0) = (-x_2, x_2)$ (take $\bar{x}(x,x_0) = f(x) - f(x_0) \in \mathbb{R}^2$). Furthermore, f is weakly pseudo-invex II at x_0 on $\mathbb{R}^2$ with respect to the same η_1 and η_2 (take $\bar{x} = (a,b) < 0$).

We have seen that a vector function may be invex or weakly pseudo-invex I (II) with respect to different $(\eta_i)_{i\in\mathcal{N}}$ without it be with respect to the same η (Examples 1 and 2). However, conversely, if a vector function is invex or weakly pseudo-invex I (II) with respect to a given η, then it is invex or weakly pseudo-invex I (II) with respect to different $(\eta_i)_{i\in\mathcal{N}}$.

Proposition 1. *Let D be a nonempty open set of $\mathbb{R}^n$. If a function $f : D \to \mathbb{R}^N$ is invex or weakly pseudo-invex I (II) at $x_0 \in D$ on D with respect to a given η, then it is invex or weakly pseudo-invex I (II) at x_0 on D with respect to $(\eta_i)_{i\in\mathcal{N}}$ with $\eta_i(x,x_0) = \eta(x,x_0) - \nabla f_i(x_0)$, $i \in \mathcal{N}$.*

Remark. From Proposition 1, we conclude that the invex (resp. weakly pseudo-invex I (II)) functions set with respect to the same η is included in the invex (resp. weakly pseudo-invex I (II)) functions set with respect to different $(\eta_i)_{i\in\mathcal{N}}$ and from Examples 1 and 2, we deduce that the inclusions are strict.

3 Relationships between the classes of vector functions

In this section, we present relationships between the introduced classes of functions namely invex and weakly pseudo-invex I (II) functions with respect to different $(\eta_i)_i$.

Proposition 2.

(i)It is clear that if a function $f : D \to \mathbb{R}^N$ is invex at x_0 on D with respect to $(\eta_i)_{i\in\mathcal{N}}$, then it is pseudo-invex I at x_0 on D with respect to the same $(\eta_i)_{i\in\mathcal{N}}$.

(ii)If f is (weakly) pseudo-invex II at x_0 on D with respect to $(\eta_i)_{i\in\mathcal{N}}$, then f is (weakly) pseudo-invex I at x_0 on D with respect to the same $(\eta_i)_{i\in\mathcal{N}}$.

(iii)If f is pseudo-invex I (resp. II) at x_0 on D with respect to $(\eta_i)_{i\in\mathcal{N}}$, then it is weakly pseudo-invex I (resp. II) at x_0 on D with respect to the same $(\eta_i)_{i\in\mathcal{N}}$ (with $\bar{x}(x,x_0) = x$). However, if f is weakly pseudo-invex I (resp. II) at x_0 on D with respect to $(\eta_i)_{i\in\mathcal{N}}$, then f may not be pseudo-invex I (resp. II) at x_0 on D with respect to the same $(\eta_i)_{i\in\mathcal{N}}$ but it will be pseudo-invex I (resp. II) at x_0 on D with respect to $(\tilde{\eta}_i)_{i\in\mathcal{N}}$ with $\tilde{\eta}_i(x,x_0) = \eta_i(\bar{x}(x,x_0),x_0)$, $\forall\, x \in D$, $\forall\, i \in \mathcal{N}$ (see Examples 3 and 4). Thus, the classes of pseudo-invex I (resp. II) functions and weakly pseudo-invex I (resp. II) functions coincide.

Example 3.(f weakly pseudo-invex I $\not\Rightarrow f$ pseudo-invex I). Consider the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ with $f(x) = (f_1(x), f_2(x)) = (x_1 + \sin x_2, \sin x_2)$. f is weakly pseudo-invex I at $x_0 = (\frac{\pi}{6}, \frac{\pi}{3})$ on $\mathbb{R}^2$ with respect to $\eta_1(x,x_0) = x - x_0$ and $\eta_2(x,x_0) = x$ (take $\bar{x} = f(x) - f(x_0) \in \mathbb{R}^2$). But, f is not pseudo-invex I at x_0 on $\mathbb{R}^2$ with respect to the same $(\eta_i)_{i=1,2}$, because for $x = (\frac{\pi}{3}, 0)$, $f(x) - f(x_0) < 0$ and $[\nabla f_i(x_0)]^t \eta_i(x,x_0) = 0$, $\forall\, i = 1,2$. However, f is pseudo-invex I at x_0 on $\mathbb{R}^2$ with respect to $\tilde{\eta}_1(x,x_0) = f(x) - f(x_0) - x_0$ and $\tilde{\eta}_2(x,x_0) = f(x) - f(x_0)$.

Example 4.(f weakly pseudo-invex II $\not\Rightarrow f$ pseudo-invex II). Consider the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ with $f(x) = (f_1(x), f_2(x)) = (-x_1^2, x_2)$. f is weakly pseudo-invex II at $x_0 = (1,0)$ on $\mathbb{R}^2$ with respect to $\eta_1(x,x_0) = x_0 - x$ and $\eta_2(x,x_0) = -x$ (take $\bar{x} = (0,1) \in \mathbb{R}^2$). But f is not pseudo-invex II at x_0 on $\mathbb{R}^2$ with respect to the same $(\eta_i)_{i=1,2}$, because for $x = (1,-1)$, $f(x) - f(x_0) \leq 0$ and $[\nabla f_i(x_0)]^t \eta_i(x,x_0) \geqq 0$, $\forall\, i = 1,2$. However, f is pseudo-invex II at x_0 on $\mathbb{R}^2$ with respect to $\tilde{\eta}_1(x,x_0) = (1,-1)$, $\forall\, x \in \mathbb{R}^2$ and $\tilde{\eta}_2(x,x_0) = (0,-1)$, $\forall\, x \in \mathbb{R}^2$.

Let us continue the relationships between the concepts of invex and weakly pseudo-invex I (II) functions by giving the following examples.
From Proposition 2 (ii), we have the class of weakly pseudo-invex II functions is included in the class of weakly pseudo-invex I functions w.r.t. $(\eta_i)_{i\in\mathcal{N}}$. The converse in not true, as it is shown in Example 5.

Example 5.(f weakly pseudo-invex I $\not\Rightarrow f$ weakly pseudo-invex II). Consider the function $f : \mathbb{R} \to \mathbb{R}^2$ with $f(x) = (f_1(x), f_2(x)) = (x^2, 0)$. f is weakly pseudo-invex I on $\mathbb{R}$ with respect to any functions $(\eta_i)_{i=1,2}$ because $f(x) - f(x_0) \not< 0$, $\forall\, x, x_0 \in \mathbb{R}$. On the other hand, by choosing $x = 0$ and $x_0 = 1$, we have $f(x) - f(x_0) \leq 0$ and since $\nabla f_2(x_0) = 0$, it follows that $[\nabla f_2(x_0)]u = 0$, $\forall\, u \in \mathbb{R}$. Hence, there does not exist a function η_2 and $\bar{x} \in \mathbb{R}$ such that $[\nabla f_2(x_0)]\eta_2(\bar{x}, x_0) < 0$, and in consequence f is not weakly pseudo-invex II at x_0 on $\mathbb{R}$.

As in Arana-Jiménez et al. [4], the following Examples 6 and 7 show that the classes of invex functions and weakly pseudo-invex II functions w.r.t. $(\eta_i)_{i\in\mathcal{N}}$ are different.

Example 6.(f weakly pseudo-invex II $\not\Rightarrow f$ invex). Consider the function $f : \mathbb{R} \to \mathbb{R}^2$ with $f(x) = (f_1(x), f_2(x)) = (x^2, -x^2)$. We have that f_2 is not invex at $x_0 = 0$ on $\mathbb{R}$ because $\nabla f_2(x_0) = 0$ and x_0 is not a minimum of this function. We conclude that f is not invex at $x_0 = 0$ on $\mathbb{R}$. We now prove that f is weakly pseudo-invex II on $\mathbb{R}$. We have $f(x) - f(x_0) = (x^2 - x_0^2, x_0^2 - x^2) \leq 0 \Leftrightarrow$

$$\begin{cases} (i)\ \ x^2 - x_0^2 < 0 \ \text{ and } \ x_0^2 - x^2 \leqq 0; \\ \text{or} \\ (ii)\ \ x^2 - x_0^2 \leqq 0 \text{ and } x_0^2 - x^2 < 0. \end{cases}$$

If $x^2 - x_0^2 < 0$ then $x_0^2 - x^2 > 0$ and (i) is not verified. In the same way we prove that (ii) is not verified. Therefore, the inequality $f(x) - f(x_0) \leq 0$ is not verified for all $x, x_0 \in \mathbb{R}$, and we conclude that f is weakly pseudo-invex II on $\mathbb{R}$ with respect to any functions $(\eta_i)_{i=1,2}$.

Example 7.(f invex $\not\Rightarrow f$ weakly pseudo-invex II). Consider the function $f : \mathbb{R} \to \mathbb{R}^2$ with $f(x) = (f_1(x), f_2(x)) = (x^2, 0)$. From Example 5, we know that f is not weakly pseudo-invex II at $x_0 = 1$ on $\mathbb{R}$. However, we have that f_1 is convex and then it is invex on $\mathbb{R}$ with respect to $\eta_1(x,x_0) = x - x_0$, f_2 is invex on $\mathbb{R}$ with respect to any function $\eta_2(x,x_0)$. Therefore, the vector function f is invex on $\mathbb{R}$ with respect to $\eta_1(x,x_0) = x - x_0$ and any function $\eta_2(x,x_0)$.

From Proposition 2 (i), we conclude that the class of weakly pseudo-invex I functions contains the class of invex functions w.r.t. $(\eta_i)_{i\in\mathcal{N}}$. The converse in not true, as it is shown in Example 8.

Example 8.(f weakly pseudo-invex I $\not\Rightarrow f$ invex). Consider the function $f : \mathbb{R} \to \mathbb{R}^2$ with $f(x) = (f_1(x), f_2(x)) = (x^2, -x^2)$. From Example 6, we know that f is not invex at $x_0 = 0$ on $\mathbb{R}$. Besides, as f is weakly pseudo-invex II on $\mathbb{R}$ with respect to any functions $(\eta_i)_{i=1,2}$, it follows that, from Proposition 2, f is weakly pseudo-invex I on $\mathbb{R}$ with respect to any functions $(\eta_i)_{i=1,2}$.

Let

$WPSI = \{f : D \subseteq \mathbb{R}^n \to \mathbb{R}^N /\ f$ is weakly pseudo-invex I w.r.t. $(\eta_i)_{i\in\mathcal{N}}\}$,

$WPSII = \{f : D \subseteq \mathbb{R}^n \to \mathbb{R}^N / f$ is weakly pseudo-invex II w.r.t. $(\eta_i)_{i\in\mathcal{N}}\}$,

$INV = \{f : D \subseteq \mathbb{R}^n \to \mathbb{R}^N /\ f$ is invex w.r.t. $(\eta_i)_{i\in\mathcal{N}}\}$.

From (i) and (ii) of Proposition 2, we conclude the following result.

Theorem 1. *$INV \cup WPSII \subset WPSI$.*

The above inclusion is strict and $INV \cup WPSII \neq WPSI$. To show this, the following example give a weakly pseudo-invex I function which is neither invex nor weakly pseudo-invex II.

*Example 9.*Consider the function $f:\mathbb{R}\to\mathbb{R}^2$ with $f(x)=(f_1(x),f_2(x))=(x^3,0)$. f is weakly pseudo-invex I on $\mathbb{R}$ with respect to any functions $(\eta_i)_{i=1,2}$ because $f(x)-f(x_0)\nless 0,\ \forall\, x,x_0\in\mathbb{R}$. On the other hand, f_1 is not invex at $x_0=0$ on $\mathbb{R}$ because $\nabla f_1(x_0)=0$ and x_0 is not a minimum for this function. We conclude that f is not invex at $x_0=0$ on $\mathbb{R}$. Furthermore, by choosing $x=0$ and $\tilde{x}=1$, we have $f(x)-f(\tilde{x})\leq 0$ and since $\nabla f_2(\tilde{x})=0$, it follows that $[\nabla f_2(\tilde{x})]u=0,\ \forall\, u\in\mathbb{R}$. Hence, there does not exist a function η_2 and $\overline{x}\in\mathbb{R}$ such that $[\nabla f_2(\tilde{x})]\eta_2(\overline{x},\tilde{x})<0$, and in consequence f is not weakly pseudo-invex II at $\tilde{x}=1$ on $\mathbb{R}$.

The intersection between invex functions set and weakly pseudo-invex II functions set (w.r.t. $(\eta_i)_{i\in\mathcal{N}}$) is a nonempty set, since a linear function is invex, weakly pseudo-invex I and weakly pseudo-invex II.

*Example 10.*Consider the function $f:\mathbb{R}\to\mathbb{R}^2$ with $f(x)=(f_1(x),f_2(x))=(x,-x)$. We have $f(x)-f(x_0)=(x-x_0,x_0-x)\leq 0 \Leftrightarrow (i)$ "$x-x_0<0$ and $x_0-x\leqq 0$" or (ii) "$x-x_0\leqq 0$ and $x_0-x<0$". If $x-x_0<0$ then $x_0-x>0$ and (i) is not verified. In the same way we prove that (ii) is not verified. Therefore, the inequality $f(x)-f(x_0)\leq 0$ is not verified for all $x,x_0\in X$, and we conclude that f is weakly pseudo-invex II (then weakly pseudo-invex I) on $\mathbb{R}$ with respect to any functions $(\eta_i)_{i=1,2}$. On the other hand, f is invex on $\mathbb{R}$ with respect to $\eta_1(x,x_0)=x-x_0-1$ and $\eta_2(x,x_0)=x-x_0+1$.

Consequently, the relationships between invex, weakly pseudo-invex I and weakly pseudo-invex II functions with respect to $(\eta_i)_{i\in\mathcal{N}}$ are as given in the following figure.

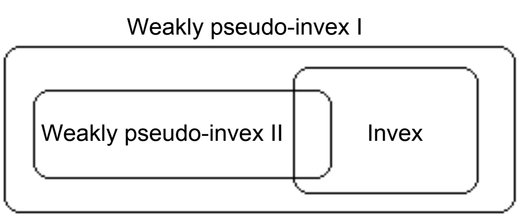

Fig. 1: Relationships between invex, weakly pseudo-invex I and weakly pseudo-invex II functions

According to Remark 2 and Proposition 2 (iii), Figure 1 above extends Figure 1 given in Arana-Jiménez et al. [4] to the wide classes of functions.

4 Optimality conditions

We consider the following multiobjective optimization problem

$$(VP)\qquad \begin{array}{l}\text{Minimize } f(x)=(f_1(x),\ldots,f_N(x)),\\ \text{subject to } g_j(x)\leqq 0,\ j\in K,\end{array}$$

where $f_i,g_j:D\to\mathbb{R},\ i\in\mathcal{N},\ j\in K$ and D is an open set of $\mathbb{R}^n$. Let $X=\{x\in D: g_j(x)\leqq 0,\ j\in K\}$ be the set of all feasible solutions of (VP). For $x_0\in D$, we denote by $J(x_0)$ the set $\{j\in K: g_j(x_0)=0\}$, $J_0=|J(x_0)|$ and by $\tilde{J}(x_0)$ (resp. $\bar{J}(x_0)$) the set $\{j\in K: g_j(x_0)<0$ (resp. $g_j(x_0)>0)\}$. We have $J(x_0)\cup\tilde{J}(x_0)\cup\bar{J}(x_0)=K$ and if $x_0\in X, \bar{J}(x_0)=\varnothing$.

We recall some optimality concepts, the most often studied in the literature, for the problem (VP). For other notions and their connections, see Yu [34].

Definition 4.*A point $x_0\in X$ is said to be a local weakly efficient solution of the problem (VP), if there exists a neighborhood $N(x_0)$ around x_0 such that*

$$f(x)\nless f(x_0),\ \text{for all } x\in N(x_0)\cap X. \tag{7}$$

Definition 5.*A point $x_0\in X$ is said to be a weakly efficient (resp. an efficient) solution of the problem (VP), if there exists no $x\in X$ such that*

$$f(x)<f(x_0)\ \ (\text{resp. } f(x)\leq f(x_0)). \tag{8}$$

Hayashi and Komiya [15] have proved an alternative lemma that we will use to prove Fritz John type necessary optimality conditions and to establish characterizations of efficient and weakly efficient solutions for (VP). Before giving this lemma, we recall the definition of convexlike vector function.

Definition 6.*[12] A function $f:D\to\mathbb{R}^N$ is a convexlike function if for any $x,y\in D$ and $0\leqq\lambda\leqq 1$, there exists $z\in D$ such that*

$$f(z)\leqq\lambda f(x)+(1-\lambda)f(y).$$

Lemma 1.*[15] Let S be a nonempty set in $\mathbb{R}^n$ and let $\psi: S\to\mathbb{R}^m$ be a convexlike function. Then either*

$\psi(x)<0$ has a solution $x\in S$,

or

$p^t\psi(x)\geqq 0$ for all $x\in S$, for some $p\in\mathbb{R}^m_{\geq}$,

but both alternatives are never true.

To prove necessary conditions for the problem (VP), we need to prove the following lemma.

Lemma 2.*Suppose that*

(i)x_0 is a (local) weakly efficient solution for (VP);
(ii)g_j is continuous at x_0 for $j\in\tilde{J}(x_0)$, the functions $f_i,\ i\in\mathcal{N},\ g_j,\ j\in J(x_0)$ are differentiable at x_0 and there exist vector functions $\eta_i: X\times D\to\mathbb{R}^n,\ i\in\mathcal{N}$, and $\theta_j: X\times D\to\mathbb{R}^n,\ j\in J(x_0)$ which satisfy at x_0 with respect to $\eta: X\times D\to\mathbb{R}^n$ the following inequalities,

$$[\nabla f_i(x_0)]^t\eta(x,x_0)\leqq[\nabla f_i(x_0)]^t\eta_i(x,x_0),\ \forall\, x\in X,\ \forall\, i\in\mathcal{N}, \tag{9}$$

$$[\nabla g_j(x_0)]^t\eta(x,x_0)\ \leqq[\nabla g_j(x_0)]^t\theta_j(x,x_0),\ \forall\, x\in X,\ \forall\, j\in J(x_0), \tag{10}$$

Then the system of inequalities

$$[\nabla f_i(x_0)]^t \eta_i(x,x_0) < 0,\ i \in \mathcal{N}, \tag{11}$$

$$[\nabla g_j(x_0)]^t \theta_j(x,x_0) < 0,\ j \in J(x_0), \tag{12}$$

has no solution $x \in X$.

*Proof.*Let $x_0 \in X$ be a local weakly efficient solution for (VP) and suppose there exists $\tilde{x} \in X$ such that the inequalities (11)-(12) are true. For $i \in \mathcal{N}$, let $\varphi_{f_i}(x_0,\tilde{x},\tau) = f_i(x_0+\tau\eta(\tilde{x},x_0)) - f_i(x_0)$. We observe that this function vanishes at $\tau = 0$ and $\lim_{\tau\to 0^+} \tau^{-1}[\varphi_{f_i}(x_0,\tilde{x},\tau) - \varphi_{f_i}(x_0,\tilde{x},0)] = \lim_{\tau\to 0^+} \tau^{-1}[f_i(x_0+\tau\eta(\tilde{x},x_0)) - f_i(x_0)] = [\nabla f_i(x_0)]^t\eta(\tilde{x},x_0) < 0$ using (9) and (11).
It follows that, for all $i \in \mathcal{N}$, $\varphi_{f_i}(x_0,\tilde{x},\tau) < 0$ if τ is in some open interval $(0,\delta_{f_i})$, $\delta_{f_i} > 0$. Thus,

$$f_i(x_0+\tau\eta(\tilde{x},x_0)) < f_i(x_0),\ \tau \in (0,\delta_{f_i}),\ \forall\, i \in \mathcal{N}.$$

Similarly, by defining $\varphi_{g_j}(x_0,\tilde{x},\tau) = g_j(x_0+\tau\eta(\tilde{x},x_0)) - g_j(x_0)$, $j \in J(x_0)$ and using (10) with (12), we get

$$g_j(x_0+\tau\eta(\tilde{x},x_0)) < g_j(x_0) = 0,\ \tau \in (0,\delta_{g_j}),\ \forall\, j \in J(x_0),$$

where for all $j \in J(x_0)$, $\delta_{g_j} > 0$.
Now, since for $j \in \tilde{J}(x_0)$, $g_j(x_0) < 0$ and g_j is continuous at x_0, therefore, there exists $\delta_j > 0$ such that

$$g_j(x_0+\tau\eta(\tilde{x},x_0)) < 0,\ \tau \in (0,\delta_j),\ \forall\, j \in \tilde{J}(x_0).$$

Let $\delta_0 = \min\{\delta_{f_i},\ i \in \mathcal{N},\ \delta_{g_j},\ j \in J(x_0),\ \delta_j,\ j \in \tilde{J}(x_0)\}$. Then

$$(x_0+\tau\eta(\tilde{x},x_0)) \in N_{\delta_0}(x_0),\ \tau \in (0,\delta_0), \tag{13}$$

where $N_{\delta_0}(x_0)$ is a neighborhood of x_0 depending on δ_0. Now, for all $\tau \in (0,\delta_0)$ we have

$$f_i(x_0+\tau\eta(\tilde{x},x_0)) < f_i(x_0),\ i \in \mathcal{N}, \tag{14}$$

$$g_j(x_0+\tau\eta(\tilde{x},x_0)) < 0,\ j \in K. \tag{15}$$

By (13) and (15), we get $(x_0+\tau\eta(\tilde{x},x_0)) \in N_{\delta_0}(x_0) \cap X$, for all $\tau \in (0,\delta_0)$. Hence (14) is a contradiction to the assumption that x_0 is a (local) weakly efficient solution for (VP). Thus, there exists no $x \in X$ satisfying the system (11)-(12), and the lemma is proved.

In the next theorem, we obtain Fritz John type necessary optimality conditions with different functions $(\eta_i)_i$ and $(\theta_j)_j$ associated to the objective and constraint functions of (VP).

Theorem 2.*(Fritz John type necessary optimality conditions) Suppose that*

(i)x_0 is a weakly efficient solution for (VP);

(ii)g_j is continuous at x_0 for $j \in \tilde{J}(x_0)$, the functions f_i, $i \in \mathcal{N}$, g_j, $j \in J(x_0)$ are differentiable at x_0 and there exist vector functions $\eta_i : X \times D \to \mathbb{R}^n$, $i \in \mathcal{N}$, $\theta_j : X \times D \to \mathbb{R}^n$, $j \in J(x_0)$ which satisfy at x_0 with respect to $\eta : X \times D \to \mathbb{R}^n$ the inequalities (9) and (10);

(iii)$L(x) = ([\nabla f_i(x_0)]^t\eta_i(x,x_0),\ i \in \mathcal{N},\ [\nabla g_j(x_0)]^t\theta_j(x,x_0),\ j \in J(x_0)) \in \mathbb{R}^{N+J_0}$ is a convexlike function of x on X.

Then there exists $(\mu,\lambda) \in \mathbb{R}^{N+J_0}_{\geq}$ such that (x_0,μ,λ) satisfies the following generalized Fritz John condition

$$\sum_{i=1}^{N}\mu_i[\nabla f_i(x_0)]^t\eta_i(x,x_0) + \sum_{j\in J(x_0)}\lambda_j[\nabla g_j(x_0)]^t\theta_j(x,x_0) \geqq 0,\ \forall\, x \in X. \tag{16}$$

*Proof.*If the conditions (i) and (ii) are satisfied, then, by Lemma 2 the system (11)-(12) has no solution $x \in X$. Since, by hypothesis (iii), $L(x) = ([\nabla f_i(x_0)]^t\eta_i(x,x_0),\ i \in \mathcal{N},\ [\nabla g_j(x_0)]^t\theta_j(x,x_0),\ j \in J(x_0))$ is a convexlike function of x on X, therefore, by Lemma 1, there exists $(\mu,\lambda) \in \mathbb{R}^{N+J_0}_{\geq}$ such that the relation (16) is satisfied.

Now, using the generalized Fritz John condition (16), we establish sufficient conditions for a feasible point to be weakly efficient or efficient for (VP) under weak invexity with respect to different $(\eta_i)_{i\in\mathcal{N}}$.

Theorem 3.*Let $x_0 \in X$ and suppose that:*

1. *f is weakly pseudo-invex I at x_0 on X with respect to $\eta_i : X \times X \to \mathbb{R}^n$, $i \in \mathcal{N}$;*
2. *g is differentiable at x_0 and for all $j \in J(x_0)$, there exists a function $\theta_j : X \times X \to \mathbb{R}^n$ such that $[\nabla g_j(x_0)]^t\theta_j(x,x_0) < 0$, $\forall\, x \in X$.*

If there exists a vector $(\mu,\lambda) \in \mathbb{R}^{N+J_0}_{\geq}$ such that $(x_0,\mu,\lambda,(\eta_i)_i,(\theta_j)_j)$ satisfies the generalized Fritz John condition (16), then x_0 is a weakly efficient solution for (VP).

*Proof.*Let us suppose that x_0 is not a weakly efficient solution of (VP). Then there exists a feasible point x such that $f(x) - f(x_0) < 0$.
Since f is weakly pseudo-invex I at x_0 on X with respect to $(\eta_i)_{i\in\mathcal{N}}$, it follows that

$$\exists\, \bar{x} \in X,\ [\nabla f_i(x_0)]^t\eta_i(\bar{x},x_0) < 0,\ \forall\, i \in \mathcal{N}. \tag{17}$$

By hypothesis, we have

$$[\nabla g_j(x_0)]^t\theta_j(\bar{x},x_0) < 0,\ \forall\, j \in J(x_0). \tag{18}$$

As $(\mu,\lambda) \in \mathbb{R}^{N+J_0}_{\geq}$ and from (17) and (18), it follows that

$$\sum_{i=1}^{N}\mu_i[\nabla f_i(x_0)]^t\eta_i(\bar{x},x_0) + \sum_{j\in J(x_0)}\lambda_j[\nabla g_j(x_0)]^t\theta_j(\bar{x},x_0) < 0,$$

which contradicts (16), and therefore, x_0 is a weakly efficient solution of (VP).

Proceeding in the same way as in the proof of the above result, we can prove the following theorem by using the weak pseudo-invexity II.

Theorem 4. *Let $x_0 \in X$ and suppose that:*

1. *f is weakly pseudo-invex II at x_0 on X with respect to $\eta_i : X \times X \to \mathbb{R}^n$, $i \in \mathcal{N}$;*
2. *g is differentiable at x_0 and for all $j \in J(x_0)$, there exists a function $\theta_j : X \times X \to \mathbb{R}^n$ such that $[\nabla g_j(x_0)]^t \theta_j(x, x_0) < 0$, $\forall\, x \in X$.*

If there exists a vector $(\mu, \lambda) \in \mathbb{R}^{N+J_0}_{\geq}$ such that $(x_0, \mu, \lambda, (\eta_i)_i, (\theta_j)_j)$ satisfies the generalized Fritz John condition (16), then x_0 is an efficient solution for (VP).

5 Characterization of weakly efficient and efficient solutions

Osuna-Gómez et al. [25,26] and Arana-Jiménez et al. [4,5] characterized the weakly efficient and efficient solutions of (VP) by using the concepts of Kuhn-Tucker (Fritz John) vector critical points under generalized invexity. In this section, we characterize the weakly efficient and efficient solutions of (VP) by means of a new concept of generalized Fritz John vector critical point and classes of generalized invex functions (with respect to different $(\eta_i)_i$ and $(\theta_j)_j$) which we present below.

Definition 7. *Let x_0 be a feasible point of (VP) and $\eta_i : X \times X \to \mathbb{R}^n$, $i \in \mathcal{N}$, $\theta_j : X \times X \to \mathbb{R}^n$, $j \in J(x_0)$ be vector functions. x_0 is said to be a generalized Fritz John (resp. Kuhn-Tucker) vector critical point with respect to $(\eta_i)_{i\in\mathcal{N}}$ and $(\theta_j)_{j\in J(x_0)}$, if the functions f and g are differentiable at x_0 and there exists a vector $(\mu, \lambda) \in \mathbb{R}^{N+J_0}_{\geq}$ (resp. there exist vectors $\mu \in \mathbb{R}^N_{\geq}$ and $\lambda \in \mathbb{R}^{J_0}_{\geqq}$), such that $(x_0, \mu, \lambda, (\eta_i)_{i\in\mathcal{N}}, (\theta_j)_{j\in J(x_0)})$ satisfies the relation (16) of Theorem 2.*

Osuna-Gómez et al. [25,26] have characterized the weakly efficient solutions for (VP) by using the concept of KT-pseudo-invexity (with respect to the same η) defined in the following way.

Definition 8. *Let $\eta : X \times X \to \mathbb{R}^n$ be a vector function. The problem (VP) is said to be KT-pseudo-invex on X with respect to η, if the functions f and g are differentiable on X and for each $x, x_0 \in X$,*

$$f(x) - f(x_0) < 0 \Rightarrow [\nabla f_i(x_0)]^t \eta(x, x_0) < 0,\ \forall\, i \in \mathcal{N}, \tag{19}$$

$$[\nabla g_j(x_0)]^t \eta(x, x_0) \leqq 0,\ \ \forall\, j \in J(x_0). \tag{20}$$

For the study of weakly efficient solutions and the generalized Fritz John vector critical points, we need a new kind of function which we define as follows.

Definition 9. *Let $\eta_i : X \times X \to \mathbb{R}^n$, $i \in \mathcal{N}$, $\theta_j : X \times X \to \mathbb{R}^n$, $j \in K$ be vector functions. The problem (VP) is said to be weakly FJ-pseudo-invex I at $x_0 \in X$ on X with respect to $(\eta_i)_{i\in\mathcal{N}}$ and $(\theta_j)_{j\in J(x_0)}$, if the functions f and g are differentiable at x_0 and for each $x \in X$,*

$$f(x) - f(x_0) < 0 \Rightarrow \exists\, \bar{x} \in X, \begin{cases} [\nabla f_i(x_0)]^t \eta_i(\bar{x}, x_0) < 0,\ \forall\, i \in \mathcal{N}, \\ [\nabla g_j(x_0)]^t \theta_j(\bar{x}, x_0) < 0,\ \forall\, j \in J(x_0). \end{cases} \tag{21}$$

If $\bar{x} = x$, in the relation (21), we say that (VP) is FJ-pseudo-invex I at x_0 on X with respect to $(\eta_i)_{i\in\mathcal{N}}$ and $(\theta_j)_{j\in J(x_0)}$. The problem (VP) is said to be (weakly) FJ-pseudo-invex I on X with respect to $(\eta_i)_i$ and $(\theta_j)_j$, if it is (weakly) FJ-pseudo-invex I at each $x_0 \in X$ on X with respect to the same $(\eta_i)_{i\in\mathcal{N}}$ and $(\theta_j)_{j\in J(x_0)}$.

Now, we establish a characterization of the weakly efficient solutions of (VP) by using the weak FJ-pseudo-invexity I with respect to different $(\eta_i)_i$ and $(\theta_j)_j$. Note that the following result is proved under weaker hypotheses than Theorem 3.3.12 given in [30]. The result remains true by using the concept of convexlikeness instead of the concepts of invexity and preinvexity. Thus, to prove the result we use the alternative Lemma 1 of Hayashi and Komiya [15] instead of the one given by Weir and Mond [33, Theorem 2.1].

Theorem 5. *Suppose that the functions f and g are differentiable on X and let $\eta_i : X \times X \to \mathbb{R}^n$, $i \in \mathcal{N}$ and $\theta_j : X \times X \to \mathbb{R}^n$, $j \in K$ be functions such that for all $x_0 \in X$, $L(x, x_0) = ([\nabla f_i(x_0)]^t \eta_i(x, x_0),\ i \in \mathcal{N},\ [\nabla g_j(x_0)]^t \theta_j(x, x_0),\ j \in J(x_0)) \in \mathbb{R}^{N+J_0}$ is a convexlike function of x on X. Then, every generalized Fritz John vector critical point with respect to $(\eta_i)_i$ and $(\theta_j)_j$ of problem (VP) is a weakly efficient solution if and only if (VP) is weakly FJ-pseudo-invex I on X with respect to $(\eta_i)_i$ and $(\theta_j)_j$.*

Proof. (**1**) (Sufficient condition) Let $x_0 \in X$ be a generalized Fritz John vector critical point with respect to $(\eta_i)_{i\in\mathcal{N}}$ and $(\theta_j)_{j\in J(x_0)}$ for (VP). If (VP) is weakly FJ-pseudo-invex I at x_0 on X with respect to $(\eta_i)_{i\in\mathcal{N}}$ and $(\theta_j)_{j\in J(x_0)}$, then, in the same manner as in Theorem 3, we obtain that x_0 is a weakly efficient solution for (VP).
(**2**) (Necessary condition) For the converse, suppose that every generalized Fritz John vector critical point with respect to $(\eta_i)_i$ and $(\theta_j)_j$ of problem (VP) is a weakly efficient solution.
Let us suppose that there exist two feasible points $\tilde{x}$ and x_0 such that

$$f(\tilde{x}) - f(x_0) < 0. \tag{22}$$

This means that x_0 is not a weakly efficient solution, and by using the initial hypothesis we deduce that x_0 is not a generalized Fritz John vector critical point with respect to $(\eta_i)_{i\in\mathcal{N}}$ and $(\theta_j)_{j\in J(x_0)}$ for (VP), *i.e.* the condition "$\sum_{i=1}^{N} \mu_i [\nabla f_i(x_0)]^t \eta_i(x, x_0) + \sum_{j\in J(x_0)} \lambda_j [\nabla g_j(x_0)]^t \theta_j(x, x_0) \geqq 0,\ \forall\, x \in X$" is not satisfied for all $(\mu, \lambda) \in \mathbb{R}^{N+J_0}_{\geq}$. Therefore, by Lemma 1, the system

$$\begin{cases} [\nabla f_i(x_0)]^t \eta_i(x,x_0) < 0, \ \forall\, i \in \mathscr{N}, \\ [\nabla g_j(x_0)]^t\, \theta_j(x,x_0) < 0, \ \ \forall\, j \in J(x_0). \end{cases}$$

has a solution $x = \bar{x} \in X$. In consequence, (VP) is weakly FJ-pseudo-invex I on X with respect to $(\eta_i)_i$ and $(\theta_j)_j$.

*Remark.*Note that the hypothesis "for all $x_0 \in X$, $L(x,x_0) = ([\nabla f_i(x_0)]^t \eta_i(x,x_0),\ \ i \in \mathscr{N},\ \ [\nabla g_j(x_0)]^t \theta_j(x,x_0),\ \ j \in J(x_0)) \in \mathbb{R}^{N+J_0}$ is a convexlike function of x on X" is needed just to prove the necessary optimality condition of Theorem 5.

*Remark.*In Definition 9, it is easy to see that if the vector functions η_i, $i \in \mathscr{N}$ and θ_j, $j \in J(x_0)$ are equal to a same function η and $\bar{x} = x$, we obtain kind of functions which are contained in the KT-pseudo-invexity class given by Osuna-Gómez et al. [25,26]. On the other hand, the set of generalized Fritz John vector critical points is wider than the set of usual Kuhn-Tucker vector critical points. Thus, in this sense, Theorem 5 can be considered as an extension of Theorem 3.7 (resp. Theorem 2.3) given by Osuna-Gómez et al. [25] (resp. [26]).

In the following example, we show that there exist weakly efficient solutions which are not characterized by Theorem 3.7 (resp. Theorem 2.3) given by Osuna-Gómez et al. [25] (resp. [26]) but they are characterized by Theorem 5.

*Example 11.*We consider the following multiobjective optimization problem

$$\begin{aligned} \text{Minimize } \ & f(x) = (-(x_1+1)^2 - x_2, -x_1^2 + x_2^2 + x_1 x_2 - x_1), \\ \text{subject to } & g_1(x) = x_1^3 - x_2 \leqq 0, \\ & g_2(x) = x_2 \leqq 0, \\ & g_3(x) = -x_1 - 2 \leqq 0, \end{aligned} \tag{23}$$

where $f : \mathbb{R}^2 \to \mathbb{R}^2$ and $g = (g_1, g_2, g_3) : \mathbb{R}^2 \to \mathbb{R}^3$. The set of all feasible solutions of problem is $X = \{x = (x_1,x_2) \in \mathbb{R}^2 : x_1^3 - x_2 \leqq 0,\ x_2 \leqq 0 \text{ and } -x_1 - 2 \leqq 0\}$.
We have $x_0 = (0,0) \in X$ is not a Kuhn-Tucker vector critical point of problem (23), because the condition of Kuhn-Tucker at x_0 takes the form $\mu_1 \nabla f_1(x_0) + \mu_2 \nabla f_2(x_0) + \lambda_1 \nabla g_1(x_0) + \lambda_2 \nabla g_2(x_0) = (-2\mu_1 - \mu_2, -\mu_1 - \lambda_1 + \lambda_2) \neq (0,0),\ \forall\ (\mu_1,\mu_2) \geq 0,\ \forall\,(\lambda_1,\lambda_2) \geqq 0$. Thus, the point x_0 does not belong to the set of weakly efficient solutions characterized by Theorem 3.7 (resp. Theorem 2.3) given by Osuna-Gómez et al. [25] (resp. [26]).
However, the problem (23) is weakly FJ-pseudo-invex I on X with respect to $\eta_1(x,\tilde{x}) = (0, \eta_1'(x,\tilde{x}))$, $\eta_2(x,\tilde{x}) = (0, -x_2)$, $\theta_1(x,\tilde{x}) = (0, \theta_1'(x,\tilde{x}))$, $\theta_2(x,\tilde{x}) = (0, \theta_2'(x,\tilde{x}))$ and $\theta_3(x,\tilde{x}) = (\theta_3'(x,\tilde{x}), 0)$ such that η_1', θ_1' and θ_3' (resp. θ_2') can be any positive (resp. negative) functions on $X \times X$ (take $\bar{x}(x,\tilde{x}) = (\sqrt[3]{a}, a) \in X$, with $a \in [-8, 0[$).
Furthermore, x_0 is a generalized Fritz John vector critical point with respect to $(\eta_i)_{i=1,2}$ and $(\theta_j)_{j=1,2}$ (take $\mu_1 = 0$, $\mu_2 = 1$ and $\lambda_1 = \lambda_2 = 0$), it follows that, by using the sufficient condition of Theorem 5, x_0 is a weakly efficient solution of problem (23).
Note that x_0 is not an efficient solution of problem (23) because there exists $x = (-2, 0) \in X$ such that $f(x) - f(x_0) \leq 0$.

Arana-Jiménez et al. [5] have given characterizations of efficient solutions of (VP) by using the concepts of KT-pseudo-invexity II and FJ-pseudo-invexity II (with respect to the same η). They defined these concepts by effecting slight modifications on the KT-pseudo-invexity introduced by Osuna-Gómez et al. [25,26]. Similarly, we consider the following definition.

Definition 10.*Let $\eta_i : X \times X \to \mathbb{R}^n$, $i \in \mathscr{N}$, $\theta_j : X \times X \to \mathbb{R}^n$, $j \in K$ be vector functions. The problem (VP) is said to be weakly FJ-pseudo-invex II at $x_0 \in X$ on X with respect to $(\eta_i)_{i \in \mathscr{N}}$ and $(\theta_j)_{j \in J(x_0)}$, if the functions f and g are differentiable at x_0 and for each $x \in X$:*

$$f(x) - f(x_0) \leq 0 \Rightarrow \exists\, \bar{x} \in X, \begin{cases} [\nabla f_i(x_0)]^t \eta_i(\bar{x},x_0) < 0, \ \forall\, i \in \mathscr{N}, \\ [\nabla g_j(x_0)]^t\, \theta_j(\bar{x},x_0) < 0, \ \forall\, j \in J(x_0). \end{cases} \tag{24}$$

If $\bar{x} = x$, in the relation (24), we say that (VP) is FJ-pseudo-invex II at x_0 on X with respect to $(\eta_i)_{i \in \mathscr{N}}$ and $(\theta_j)_{j \in J(x_0)}$. The problem (VP) is said to be (weakly) FJ-pseudo-invex II on X with respect to $(\eta_i)_i$ and $(\theta_j)_j$, if it is (weakly) FJ-pseudo-invex II at each $x_0 \in X$ on X with respect to the same $(\eta_i)_{i \in \mathscr{N}}$ and $(\theta_j)_{j \in J(x_0)}$.

*Remark.*In Definition 10, it is easy to see that if the vector functions η_i, $i \in \mathscr{N}$ and θ_j, $j \in J(x_0)$ are equal to a same function η and $\bar{x} = x$, we obtain equivalently the FJ-pseudo-invexity II given by Arana-Jiménez et al. [5].

The relationship between this class of functions and those given in Definition 9 is as follows.

Proposition 3.*If (VP) is (weakly) FJ-pseudo-invex II at x_0 on X with respect to $(\eta_i)_{i \in \mathscr{N}}$ and $(\theta_j)_{j \in J(x_0)}$, then (VP) is (weakly) FJ-pseudo-invex I at x_0 on X with respect to the same $(\eta_i)_{i \in \mathscr{N}}$ and $(\theta_j)_{j \in J(x_0)}$.*

Following the same lines as the demonstration of Theorem 5 and using Theorem 4, we can prove the following characterization of efficient solutions of (VP) under weak FJ-pseudo-invexity II with respect to different $(\eta_i)_i$ and $(\theta_j)_j$, and the concept of convexlikeness.

Theorem 6.*Suppose that the functions f and g are differentiable on X and let $\eta_i : X \times X \to \mathbb{R}^n$, $i \in \mathscr{N}$ and $\theta_j : X \times X \to \mathbb{R}^n$, $j \in K$ be functions such that for all $x_0 \in X$, $L(x,x_0) = ([\nabla f_i(x_0)]^t \eta_i(x,x_0),\ i \in \mathscr{N},\ \ [\nabla g_j(x_0)]^t \theta_j(x,x_0),\ \ j \in J(x_0)) \in \mathbb{R}^{N+J_0}$ is a convexlike function of x on X. Then, every generalized Fritz John vector critical point with respect to $(\eta_i)_i$ and $(\theta_j)_j$ of problem (VP) is an efficient solution if and only if (VP) is weakly FJ-pseudo-invex II on X with respect to $(\eta_i)_i$ and $(\theta_j)_j$.*

In the following example, we show that there exist efficient solutions which are not characterized by Theorems 5 and 6 of Arana-Jiménez et al. [5] but they are characterized by Theorem 6 above.

*Example 12.*We consider the following multiobjective optimization problem

$$\begin{array}{ll} \text{Minimize} & f(x) = (-x_1^2 + x_2^2, x_1 x_2), \\ \text{subject to} & g_1(x) = -x_1 + x_2 \leqq 0, \\ & g_2(x) = log(1+x_1) - x_2 \leqq 0, \\ & g_3(x) = x_1 + x_2^2 \leqq 0, \end{array} \tag{25}$$

where $f = (f_1, f_2) : \mathbb{R}^2 \to \mathbb{R}^2$ and $g = (g_1, g_2, g_3) : \mathbb{R}^2 \to \mathbb{R}^3$. The set of all feasible solutions of problem is $X = \{x = (x_1, x_2) \in \mathbb{R}^2 : -x_1 + x_2 \leqq 0,\ log(1+x_1) - x_2 \leqq 0 \text{ and } x_1 + x_2^2 \leqq 0\}$.
We have $x_0 = (0,0) \in X$ is a Kuhn-Tucker and then a Fritz John vector critical point of problem (25) (take $\mu_1 = \mu_2 = \lambda_1 = \lambda_2 = 1$ and $\lambda_3 = 0$), but there exists no a function $\eta : X \times X \to \mathbb{R}^2$ for which the problem (25) is KT-pseudo-invex II or FJ-pseudo-invex II in the sense of Arana-Jiménez et al. [5] (take $x' = (-\frac{1}{3}, -\frac{1}{3})$ and $x'' = (-\frac{1}{2}, -\frac{1}{2}) \in X$). Thus, even if the point x_0 is a Kuhn-Tucker and a Fritz John vector critical point of problem (25), it does not belong to the set of efficient solutions characterized by Theorems 5 and 6 given by Arana-Jiménez et al. [5].
However, the problem (25) is weakly FJ-pseudo-invex II on X with respect to $\eta_1(x,\tilde{x}) = (-x_1\tilde{x}_1, x_1\tilde{x}_2)$, $\eta_2(x,\tilde{x}) = (x_2\tilde{x}_2, x_2\tilde{x}_1)$, $\theta_1(x,\tilde{x}) = (\theta_1'(x,\tilde{x}), \theta_1''(x,\tilde{x}))$, $\theta_2(x,\tilde{x}) = (0, \theta_2'(x,\tilde{x}))$ and $\theta_3(x,\tilde{x}) = (\theta_3'(x,\tilde{x}), 0)$ such that θ_1' and θ_2' (resp. θ_1'' and θ_3') can be any positive (resp. negative) functions on $X \times X$ (take $\bar{x} = \bar{x}(x,\tilde{x}) = (\bar{x}_1(x,\tilde{x}), \bar{x}_2(x,\tilde{x}))$ with $\bar{x}_1 = \bar{x}_2 = \dfrac{\min\{f_1(x) - f_1(\tilde{x}), f_2(x) - f_2(\tilde{x})\}}{\sum_{i=1}^{2}[f_i(\tilde{x}) - f_i(x)] + 1}$ or $\bar{x} = (a,a)$ with $a \in\ ]-1,0[$. Furthermore, x_0 is a generalized Fritz John vector critical point with respect to $(\eta_i)_{i=1,2}$ and $(\theta_j)_{j=1,2,3}$ (take $\mu_1 = \mu_2 = 1$ and $\lambda_1 = \lambda_2 = \lambda_3 = 0$), it follows that, by using the sufficient condition of Theorem 6, x_0 is an efficient solution of problem (25).

In the following example, we present a problem which is not FJ-pseudo-invex II in the sense of Arana-Jiménez et al. [5] and which admits efficient solutions which are not Kuhn-Tucker vector critical points. Thus, we show that such efficient solutions are not characterized by Theorems 5 and 6 of Arana-Jiménez et al. [5] and they are not characterized as weakly efficient solutions by Theorem 3.7 (resp. Theorem 2.3) given by Osuna-Gómez et al. [25] (resp. [26]) but they are characterized by Theorem 6 above.

*Example 13.*We consider the following multiobjective optimization problem

$$\begin{array}{ll} \text{Minimize} & f(x) = (x_1(x_2+x_3) + 15x_3^3 - x_3, -x_1 + x_2^2 - x_3), \\ \text{subject to} & g_1(x) = x_2 \leqq 0, \\ & g_2(x) = x_3^3 - x_2 \leqq 0, \\ & g_3(x) = x_1 \leqq 0, \end{array} \tag{26}$$

where $f : \mathbb{R}^3 \to \mathbb{R}^2$ and $g = (g_1, g_2, g_3) : \mathbb{R}^3 \to \mathbb{R}^3$. The set of all feasible solutions of problem is $X = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : x_2 \leqq 0,\ x_3^3 - x_2 \leqq 0 \text{ and } x_1 \leqq 0\}$.

We have $x_0 = (0,0,0) \in X$ is not a Kuhn-Tucker vector critical point of problem (26), because the condition of Kuhn-Tucker at x_0 takes the form $\mu_1 \triangledown f_1(x_0) + \mu_2 \triangledown f_2(x_0) + \lambda_1 \triangledown g_1(x_0) + \lambda_2 \triangledown g_2(x_0) + \lambda_3 \triangledown g_3(x_0) = (-\mu_2 + \lambda_3, \lambda_1 - \lambda_2, -\mu_1 - \mu_2) \neq (0,0,0)$, $\forall\ (\mu_1, \mu_2) \geq 0$, $\forall\ (\lambda_1, \lambda_2, \lambda_3) \geqq 0$. Thus, the point x_0 does not belong to the set of weakly efficient solutions characterized by Theorem 3.7 (resp. Theorem 2.3) given by Osuna-Gómez et al. [25] (resp. [26]). Furthermore, x_0 does not belong to the set of efficient solutions characterized by Theorem 5 given by Arana-Jiménez et al. [5]. On the other hand, there exists no a function $\eta : X \times X \to \mathbb{R}^3$ for which the problem (26) is FJ-pseudo-invex II in the sense of Arana-Jiménez et al. [5] (take $x' = (-\frac{1}{2}, -\frac{1}{30}, -\frac{1}{3})$ and $x'' = (-1,0,0) \in X$). It follows that, also, x_0 does not belong to the set of efficient solutions characterized by Theorem 6 given by Arana-Jiménez et al. [5].
However, the problem (26) is weakly FJ-pseudo-invex II on X with respect to $\eta_1(x,\tilde{x}) = (-x_1, -x_1, 0)$, $\eta_2(x,\tilde{x}) = (\eta_2'(x,\tilde{x}), 0, 0)$, $\theta_1(x,\tilde{x}) = (0, \theta_1'(x,\tilde{x}), 0)$, $\theta_2(x,\tilde{x}) = (0, \theta_2'(x,\tilde{x}), 0)$ and $\theta_3(x,\tilde{x}) = (\theta_3'(x,\tilde{x}), 0, 0)$ such that η_2' and θ_2' (resp. θ_1' and θ_3') can be any positive (resp. negative) functions on $X \times X$ (take $\bar{x}(x,\tilde{x}) = (\sum_{i=1}^{3}[f_i(x) - f_i(\tilde{x})], 0, 0) \in X$ or $\bar{x} = (a,0,0)$ with $a < 0$). Furthermore, x_0 is a generalized Fritz John vector critical point with respect to $(\eta_i)_{i=1,2}$ and $(\theta_j)_{j=1,2,3}$ (take $\mu_1 = 1$, $\mu_2 = 0$ and $\lambda_1 = \lambda_2 = \lambda_3 = 0$), it follows that, by using the sufficient condition of Theorem 6, x_0 is an efficient solution of problem (26).

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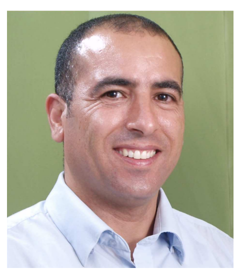

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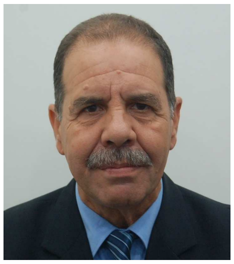

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