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Characterizations of Nadaraja Haghighi (NH) Distribution by Truncation

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Abstract: In this paper, four characterization results for Nadaraja Haghighi (NH) distribution are obtained. These results are based on simple truncation, two truncated moments, truncated moment of smallest order statistics and truncated moment of largest order statistics.

Keywords: Truncated moment, characterization, order statistics, Nadaraja Haghighi (NH) distribution

1 Introduction

Characterization of a probability distribution is an important problem in probability and statistics. Thus, various methods have been established to characterize a probability distribution. Truncated moments are extensively used for characterization of the probability distribution. [1] began the general theory of characterizing a probability distribution by truncated moments. [2], [3], [4],[5] amongst the others who contributed to further development in this area, however, most of these characterizations are based on a simple relationship between two different moments truncated from the left at the same point. In recent years many researchers contributed to this area. For example, one can see [6], [7], [8], [9].
Let $X_1, X_2, ..., X_n$ be n random variables. Suppose random variables $X_1, X_2, ..., X_n$ are arranged in ascending order of magnitude such that $X_{1:n} \leq X_{2:n} \leq ... \leq X_{n:n}$ then $X_{r:n}$ is called the r^{th} order statistic. $X_{1:n} = min(X_1, X_2, ..., X_n)$ and $X_{n:n} = max(X_1, X_2, ..., X_n)$ are called extreme order statistics. [10].
The probability density function (pdf) and the distribution function (df) of $X_{1:n}$ are [11], [10]

$$f_{X_{1:n}} = n[\bar{F}(x)]^{n-1} f(x) \tag{1}$$

and

$$F_{X_{1:n}} = 1 - [\bar{F}(x)]^n \tag{2}$$

where $\bar{F}(x) = 1 - F(x)$.
The pdf and the df of $X_{n:n}$ are [11], [10]

$$f_{X_{n:n}} = n[F(x)]^{n-1} f(x) \tag{3}$$

and

$$F_{X_{n:n}} = [F(x)]^n \tag{4}$$

[12] introduced a new extension of the exponential distribution called Nadaraja Haghighi (NH) distribution, which can be used as an alternative of gamma, Weibull and exponentiated exponential (EE) distribution. NH distribution has closed form of survival functions and hazard rate functions. It has an attractive feature of always having the zero-mode and yet allowing for increasing, decreasing, and constant hazard rate functions. The pdf of NH distribution is given by

$$f(x;\alpha,\lambda) = \alpha\lambda(1+\lambda x)^{\alpha-1} e^{\{1-(1+\lambda x)^{\alpha}\}}, x > 0, \alpha, \lambda > 0 \tag{5}$$

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and the corresponding df is given by

$$F(x;\alpha,\lambda) = 1 - e^{\{1-(1+\lambda x)^{\alpha}\}}, x > 0, \alpha, \lambda > 0 \tag{6}$$

From (5) and (6), we have

$$\frac{f(x)}{\bar{F}(x)} = \alpha\lambda(1+\lambda x)^{\alpha-1} \tag{7}$$

[13] find single and product moments of NH distribution based on upper record values and also find BLUEs for parameters. [14] find relation for single and product moment based order statistics. [15] compare different method of estimation of the parameter for NH distribution. [16] calculated Bayesian and Non-Bayesian estimates of the parameter. [17] find Shannon entropy, moments and characterize NH distribution based on generalized order statistics. [18] drive the recurrence relation for moments of progressive type-II right censored order statistics. [19], [20], [21], [22], [23] amongst other who use NH distribution for their study.
In this paper, we have given four characterization results of NH distribution. The first result is based on a simple truncation of a function. The second result is based on two truncated moments. The third and fourth result is based on truncated moments of smallest and largest order statistics.

2 Characterization Theorems

Characterization by simple truncation

Theorem 1: Let $X : \Omega \to (0,\infty)$ be a continuous random variable with df $F(x)$, then for $\psi(x) = e^{\{1-(1+\lambda x)^{\alpha}\}}$

$$E[\psi(x)|X \geq x] = \frac{1}{2}e^{\{1-(1+\lambda x)^{\alpha}\}} \tag{8}$$

if and only if has df defined in (6).

Proof: First, we will prove (6) implies (8).
we have

$$E[\psi(x)|X \geq x] = E[e^{\{1-(1+\lambda x)^{\alpha}\}}|X \geq x] = \frac{1}{1-F(x)}\int_x^{\infty} e^{\{1-(1+\lambda y)^{\alpha}\}} f(y)dy. \tag{9}$$

Putting $f(x)$ and $F(x)$ from (5) and (6), we have

$$E[\psi(x)|X \geq x] = \frac{\alpha\lambda}{e^{\{1-(1+\lambda x)^{\alpha}\}}}\int_x^{\infty}(1+\lambda y)^{\alpha-1}e^{2\{1-(1+\lambda y)^{\alpha}\}}dy. \tag{10}$$

Substituting $u = e^{\{1-(1+\lambda y)^{\alpha}\}}$ in (10), we have

$$E[\psi(x)|X \geq x] = \frac{1}{e^{\{1-(1+\lambda x)^{\alpha}\}}}\int_0^{e^{\{1-(1+\lambda x)^{\alpha}\}}} udu.$$

which gives

$$E[\psi(x)|X \geq x] = \frac{1}{2}e^{\{1-(1+\lambda x)^{\alpha}\}}. \tag{11}$$

and hence the if part.
Now we will prove (8) implies (6)
we have,

$$E[\psi(x)|X \geq x] = \frac{1}{2}e^{\{1-(1+\lambda x)^{\alpha}\}}.$$

or

$$\frac{1}{\bar{F}(x)}\int_x^{\infty} e^{\{1-(1+\lambda y)^{\alpha}\}} f(y)dy = \frac{1}{2}e^{\{1-(1+\lambda x)^{\alpha}\}}$$

which can be written as

$$\int_x^{\infty} e^{\{1-(1+\lambda y)^{\alpha}\}} f(y)dy = \frac{1}{2}e^{\{1-(1+\lambda x)^{\alpha}\}}\bar{F}(x). \tag{12}$$

Differentiating (12) *w.r.t.x*, we have

$$e^{\{1-(1+\lambda x)^{\alpha}\}}f(x)=\alpha\lambda(1+\lambda x)^{\alpha-1}e^{\{1-(1+\lambda x)^{\alpha}\}}\bar{F}(x)$$

or

$$\frac{f(x)}{\bar{F}(x)}=\alpha\lambda(1+\lambda x)^{\alpha-1}. \tag{13}$$

After simplification, (13) leads to

$$F(x)=1-e^{\{1-(1+\lambda x)^{\alpha}\}}$$

which is the distribution function of NH distribution and hence the theorem.

Characterization by two truncated moments

Theorem 2: Let $X:\Omega\rightarrow(0,\infty)$ be a continuous random variable with $df\ F(x)$ and let $\psi(x)=e^{\{1-(1+\lambda x)^{\alpha}\}}$ and $\phi(x)=e^{3\{1-(1+\lambda x)^{\alpha}\}}$, where $\psi(x)$ and $\phi(x)$ be two real functions defined over $(0,\infty)$ such that

$$E[\phi(x)|X\geq x]=E[\psi(x)|X\geq x]\eta(x),\quad 0<x<\infty \tag{14}$$

is defined with some real function η(x), then X has distribution function defined in (6) if and only if

$$\eta(x)=\frac{1}{2}e^{2\{1-(1+\lambda x)^{\alpha}\}} \tag{15}$$

Proof: First we will prove (6) implies (15)
we have

$$E[\psi(x)|X\geq x]=\frac{1}{\bar{F}(x)}\int_{x}^{\infty}e^{\{1-(1+\lambda y)^{\alpha}\}}f(y)dy.$$

For NH distribution

$$\bar{F}(x)E[\psi(x)|X\geq x]=\alpha\lambda\int_{x}^{\infty}(1+\lambda y)^{\alpha-1}e^{2\{1-(1+\lambda y)^{\alpha}\}}dy$$

or

$$\bar{F}(x)E[\psi(x)|X\geq x]=\frac{1}{2}e^{2\{1-(1+\lambda x)^{\alpha}\}}. \tag{16}$$

Similarly

$$\bar{F}(x)E[\phi(x)|X\geq x]=\frac{1}{4}e^{4\{1-(1+\lambda x)^{\alpha}\}}. \tag{17}$$

Now from (14), (16) and (17), we have

$$\eta(x)=\frac{E[\phi(x)|X\geq x]}{E[\psi(x)|X\geq x]}=\frac{1}{2}e^{2\{1-(1+\lambda x)^{\alpha}\}} \tag{18}$$

and hence the if part.

Now we will prove (15) implies (6)
we have from equation (15)

$$\eta(x)=\frac{E[\phi(x)|X\geq x]}{E[\psi(x)|X\geq x]}=\frac{1}{2}e^{2\{1-(1+\lambda x)^{\alpha}\}}$$

differentiating above equation *w.r.t.x*, we have

$$\eta'(x)=-\alpha\lambda(1+\lambda x)^{\alpha-1}e^{2\{1-(1+\lambda x)^{\alpha}\}} \tag{19}$$

Now let a function $s(x)$ such that $s'(x)=\frac{\eta'(x)\psi(x)}{\eta(x)\psi(x)-\phi(x)}$
From (15), (16), (17) and (19), we have

$$s'(x)=2\alpha\lambda(1+\lambda x)^{\alpha-1}$$

Now

$$e^{-s(x)} = e^{-\int_0^x s'(u)du} = e^{2\{1-(1+\lambda x)^\alpha\}}. \quad (20)$$

Using the result given in [7]

$$F(x) = \int_0^x C\left|\frac{\eta'(u)}{\eta(u)\psi(u)-\phi(u)}\right| e^{-s(u)}du$$

From (19), (20) and functions $\psi(x)$, $\phi(x)$ in theorem 2, we have

$$F(x) = C\int_0^x \left|\frac{-\alpha\lambda(1+\lambda u)^{\alpha-1}e^{2\{1-(1+\lambda u)^\alpha\}}}{-\frac{1}{2}e^{3\{1-(1+\lambda u)^\alpha\}}}\right| e^{2\{1-(1+\lambda u)^\alpha\}}du \quad (21)$$

where C is a constant such that $F(0) = 0$ and $F(\infty) = 1$.
After simplifying (21), we have

$$F(x) = 2C\alpha\lambda\int_0^x (1+\lambda u)^{\alpha-1}e^{\{1-(1+\lambda u)^\alpha\}}du$$

which gives

$$F(x) = 2C[1-e^{\{1-(1+\lambda x)^\alpha\}}] \quad (22)$$

after calculating C and putting in equation (22), we have

$$F(x) = 1-e^{\{1-(1+\lambda x)^\alpha\}}$$

which is the df of NH distribution and hence the theorem.

Characterization by smallest order statistics

Theorem 3: Let $X : \Omega \to (0,\infty)$ be a continuous random variable with df $F(x)$, then for $\psi(x) = e^{\{1-(1+\lambda x)^\alpha\}}$

$$E[\psi(X_{1:n})|X_{1:n} > t] = \frac{n}{n+1}e^{\{1-(1+\lambda t)^\alpha\}} \quad (23)$$

if and only if X has distribution function defined in (6).

Proof: First, we will prove (6) implies (23).
we have

$$E[\psi(X_{1:n})|X_{1:n} > t] = \frac{1}{[1-F_{X_{1:n}}]}\int_t^\infty e^{\{1-(1+\lambda x)^\alpha\}}f_{X_{1:n}}dx \quad (24)$$

Using (1), (2), (5) and (6), (24) reduces to

$$E[\psi(X_{1:n})|X_{1:n} > t] = \frac{\alpha\lambda n}{[1-F(t)]^n}\int_t^\infty (1+\lambda x)^{\alpha-1}e^{(n+1)\{1-(1+\lambda x)^\alpha\}}dx \quad (25)$$

Substituting $u = e^{\{1-(1+\lambda x)^\alpha\}}$ in (25), we have

$$E[\psi(X_{1:n})|X_{1:n} > t] = \frac{n}{e^{n\{1-(1+\lambda t)^\alpha\}}}\int_0^{e^{\{1-(1+\lambda t)^\alpha\}}} u^n du$$

which gives

$$E[\psi(X_{1:n})|X_{1:n} > t] = \frac{n}{(n+1)}e^{\{1-(1+\lambda t)^\alpha\}} \quad (26)$$

and hence if part.

For the sufficiency part, we have from (26)

$$E[\psi(X_{1:n})|X_{1:n} > t] = \frac{n}{(n+1)}e^{\{1-(1+\lambda t)^\alpha\}}$$

or

$$\frac{1}{[\bar{F}(t)]^n}\int_t^\infty e^{\{1-(1+\lambda x)^\alpha\}} n[\bar{F}(x)]^{n-1} f(x)dx = \frac{n}{(n+1)} e^{\{1-(1+\lambda t)^\alpha\}} \tag{27}$$

Integrating (27) by parts and arranging the terms, we have

$$\alpha\lambda\int_t^\infty (1+\lambda x)^{\alpha-1} e^{\{1-(1+\lambda x)^\alpha\}}[\bar{F}(x)]^n dx = \frac{1}{(n+1)} e^{\{1-(1+\lambda t)^\alpha\}}[\bar{F}(t)]^n. \tag{28}$$

Differentiating (28) *w.r.t.t* , we have

$$\frac{f(t)}{\bar{F}(t)} = \alpha\lambda(1+\lambda t)^{\alpha-1}$$

which gives

$$F(t) = 1 - e^{\{1-(1+\lambda t)^\alpha\}}$$

which is the df of NH distribution and hence the theorem.

Characterization by largest order statistics

Theorem 4: Let X be a continuous random variable with df $F(x)$, then for $\psi(x) = e^{\{1-(1+\lambda x)^\alpha\}}$

$$E[\psi(X_{n:n})|X_{n:n} < t] = \frac{1+ne^{\{1-(1+\lambda t)^\alpha\}}}{(n+1)} \tag{29}$$

if and only if has distribution function defined in (6).

Proof: First, we will prove (6) implies (29)
we have

$$E[\psi(X_{n:n})|X_{n:n} < t] = \frac{1}{F_{X_{n:n}}}\int_0^t e^{\{1-(1+\lambda x)^\alpha\}} f_{X_{n:n}} dx \tag{30}$$

Using (3), (4), (5) and (6), (30) reduces to

$$E[\psi(X_{n:n})|X_{n:n} < t] = \frac{\alpha\lambda n}{[1-e^{\{1-(1+\lambda t)^\alpha\}}]^n}\int_0^t (1+\lambda x)^{\alpha-1} e^{2\{1-(1+\lambda x)^\alpha\}}[1-e^{\{1-(1+\lambda x)^\alpha\}}]^{n-1} dx \tag{31}$$

Substituting $u = [1-e^{\{1-(1+\lambda x)^\alpha\}}$ in (31), we have

$$E[\psi(X_{n:n})|X_{n:n} < t] = \frac{n}{[1-e^{\{1-(1+\lambda t)^\alpha\}}]^n}\int_0^{[1-e^{\{1-(1+\lambda t)^\alpha\}}]^n} u^{n-1}(1-u)du$$

which gives

$$E[\psi(X_{n:n})|X_{n:n} < t] = \frac{1+ne^{\{1-(1+\lambda t)^\alpha\}}}{(n+1)} \tag{32}$$

hence the necessary part.

For Sufficient part, we have from (32)

$$E[\psi(X_{n:n})|X_{n:n} < t] = \frac{1+ne^{\{1-(1+\lambda t)^\alpha\}}}{(n+1)}$$

or

$$\frac{1}{[F(t)]^n}\int_0^t e^{\{1-(1+\lambda x)^\alpha\}} n[\bar{F}(x)]^{n-1} f(x)dx = \frac{1+ne^{\{1-(1+\lambda t)^\alpha\}}}{(n+1)} \tag{33}$$

Integration R.H.S. of (33) by parts and arranging the terms, we have

$$\alpha\lambda\int_0^t (1+\lambda x) e^{\{1-(1+\lambda x)^\alpha\}}[F(x)]^n dx = \frac{1}{(n+1)}[1-e^{\{1-(1+\lambda t)^\alpha\}}][F(t)]^n \tag{34}$$

Differentiating (34) $w.r.t.t$ and rearranging the terms, we have

$$\frac{f(t)}{F(t)} = \frac{\alpha\lambda(1+\lambda t)e^{\{1-(1+\lambda t)^{\alpha}\}}}{[1-e^{\{1-(1+\lambda t)^{\alpha}\}}]}$$

which gives

$$F(t) = [1-e^{\{1-(1+\lambda t)^{\alpha}\}}]$$

which is the distribution function of NH distribution and hence the theorem.

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