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Modern Perspectives on Mathematical Modeling: The Power of the Sumudu Transform and Iterative Approaches

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Abstract: This work uses the Sumudu transform method to discover approximate analytical solutions for systems of fractional nonlinear equations of population dynamics model and a fractional mathematical model of honey bees. The Caputo sense is used to describe fractional derivatives. In order to get approximate analytical solutions of systems of nonlinear equations of these models, the Sumudu transform method has been developed. Since it provides us with incredibly accurate solutions for both linear and nonlinear differential equations, this approach is simple to understand. It has also been investigated how the power-law kernel via Caputo derivative affects things. To demonstrate the simulations of the answers, we present some figures.

Keywords: Power-law kernel, Sumudu transform, population dynamics model, honey bee mathematical model.

1 Introduction

Fractional order derivatives refer to derivatives of any real or complex order, broadening the idea of conventional derivatives to situations where the order isn't a whole number. Because of the memory characteristic of fractional derivatives, the theory and uses of fractional calculus have become highly significant in modeling biological processes, applied mathematics, physics, human disease, economic growth, and engineering [1,2,3]. The focus of research in modeling real-world issues is transitioning towards the application of fractional order derivatives. As a result, in recent decades, the application of fractional order derivatives has gained significant interest in various areas of science and engineering for modeling various types of issues. It is widely recognized that finding analytical solutions for these types of problems is often challenging. Hence, methods that approximate the solutions of these equations are essential and beneficial. In recent decades, the creation of effective numerical methods for estimating the solutions of fractional differential equations (FDEs) has been a significant concern [4,5,6]. Consequently, techniques including the fractional finite volume method, Haar wavelet method, radial basis functions, operational matrix methods, Fourier spectral methods, Adomian decomposition methods, and variational iterative methods were suggested [7,8].

It is important to highlight that there are just two primary definitions of the fractional derivative; the initial one was introduced by Riemann and Liouville and represents the derivative of the convolution between a specific function and a power law kernel, while the second was put forward by Caputo [9] in 1967 and is the convolution of the local derivative of a specific function with a power law function. The Caputo fractional derivative, characterized by singular and non-local features, is employed to model phenomena by considering past interactions of everyday issues. The Caputo derivative is a type of fractional derivative that accommodates conventional initial and boundary conditions in problem formulations, making it especially suitable for differentiable functions. The calculation is based on a particular integral formula and contrasts with the Riemann-Liouville derivative in that the Caputo derivative of a constant function equals zero [10].

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Integral transforms are valuable for the simplification they provide, particularly when addressing differential equations with certain boundary conditions. Selecting suitable integral transforms aids in transforming differential and integral equations into algebraic equations that are simpler to solve. Numerous integral transforms within the Laplace transform category have been presented, including Sumudu, Elzaki, Natural, Aboodh, Pourreza, Mohand, G-transform, Sawi, and Kamal transforms. These transformations have been utilized for addressing various kinds of integral equations, ordinary differential equations (ODEs), partial differential equations (PDEs), and FDEs [11].

The analytical tools presented can be directly leveraged in quantum information by providing efficient, well-conditioned methods to obtain closed-form or rapidly convergent solutions for time-evolution, coherence decay and phase-space dynamics in Jaynes–Cummings–type systems [12]. Using Sumudu-transform based solutions and iterative schemes to treat generalized and multi-photon interactions [13] enables precise tracking of quasiprobability distributions and entropy squeezing under nonlinear media and multiphoton processes, which are crucial for state engineering and error characterization. For dissipative, dispersive two-atom cavities [14] the transform/iterative framework can incorporate fractional or memory kernels and deliver analytic expressions for entanglement decay and long-lived coherence, informing robust entanglement-preservation protocols. Likewise, for entropic uncertainty and second-harmonic effects [15] the approach yields compact representations of uncertainty measures and their parametric dependence, aiding optimization of measurement strategies and squeezing resources for quantum metrology. Sumudu-transform and iterative methods offer practical routes to derive controllable, computable models that connect microscopic interaction parameters to operational quantum-information metrics (entanglement, entropy squeezing, fidelity), enabling better design of gates, sensors and decoherence-mitigation schemes in cavity QED and related platforms [16].

Population dynamics models are generally expressed as an ordinary differential equation:

$$\frac{d\mathscr{F}}{dt} = \mu(k),$$

where k like air, food, hunter and etc.

The mentioned ordinary differential equation model is in the form of an exponential growth reaction. One of the most popular of these exponential growth reaction models is the Malthusian model. The Malthusian model is an exponential model and is useful when the population of organisms is small [17]. The Malthus model is defined as [18]:

$$\aleph'(t) = \curlywedge\aleph(t), \quad \aleph(0) = \aleph_0.$$

The solution to this initial value problem is given as:

$$\aleph(t) = \aleph_0 e^{\curlywedge t}.$$

Differential equations provide a natural framework for investigating the effects of organisms on population dynamics. Many mathematical models have been developed using these differential equations. These mathematical models contribute to a better understanding of population and resource dynamics in general and facilitate the application of the model framework to relevant conditions. Therefore, many population models have been developed [19,20] in. Two of the models developed are population growth models [21] and small hive bettle (SHB)-honey bee models [22].

The logistic growth model developed by Verhulst in 1838 is defined as follows:

$$\aleph'(t) = \curlywedge\aleph(t)\left(1 - \frac{\aleph(t)}{\Upsilon}\right),$$

where $\curlywedge$ is the rate of increase of the population and Υ is the carrying capacity, Υ and $\curlywedge > 0$. The solution of the Verhulst model is given as follows:

$$\aleph(t) = \frac{\aleph_0\Upsilon}{\aleph_0 + (\Upsilon - \aleph_0)e^{-\curlywedge t}}.$$

When $t \to \infty$, $\aleph(t)$ gets closer to Υ and increases by taking the competition. The logistic growth model shows a rapid increase when $\aleph(t)$ is smaller than Υ. It turns out that the growth decreases as $\aleph(t)$ approaches Υ. If $\aleph_0 = \Upsilon$, the population will remain $\aleph(t) = \Upsilon$, which is an equilibrium point.

The mathematical model developed by [22], which shows the relationship between small hive beetles (SHB) and honeybees, discovered by Andrew Murray in 1867 in Central Africa, is a small invasive parasite species that inhabits areas of the African continent south of the Sahara [23,24]. No mathematical model has yet been made for SHB, which has these properties and threatens honey bees. SHB is destructive for honey bee colonies, and the increase in temperature due to climate change increases the population of SHBs. Although studies have been conducted on the development and survival of SHBs with temperature, a mathematical model has not been put forward to explain this relationship. It is necessary to model how increasing temperatures affect the increase in SHB and the resulting honey bee population [25, 26, 27].

Therefore, the mathematical model created by [28] for Varroa was reconstructed by [22] under certain assumptions, resulting in a new mathematical model. This newly developed model and its assumptions are presented as follows [22]:

$$\frac{dp}{dt} = vg_p(p) - \gamma qp - \delta_1 p, \quad p(0) = p_0, \quad g_p(p) = \frac{p^2}{\mathscr{K}^2 + p^2},$$
$$\frac{dq}{dt} = r(T)g_q(p,q)q - \delta_2 q, \quad q(0) = q_0, \quad g_q(p,q) = 1 - \frac{q}{\alpha p},$$
$$r(T) = r_m e^{\frac{-1}{2}(\frac{T-T_{opt}}{T_{sig}})^2}$$

where p is the number of honey bees, and q is the number of SHBs at time t. The $r(T)$ function indicates the developmental process and survival of SHB according to the temperature value in the Pradhan–Taylor (Gaussian) model [29,30]. r_m is the maximum growth rate, T (between 32 and 36 degrees) is the growing temperature, and T_{opt} and T_{sig} are the optimum temperature and shape parameters with the highest growth.

The linear T function is chosen as follows to calculate the growth of SHB in the brood between the appropriate temperature.

$$T = T_0, \quad t < t_1$$

$$T = T_0 + w(t - t_1), \quad t \geq t_1$$

t_1 is the moment when the brood temperature starts to rise. T_0 is the lowest temperature of brood at which SHB hatches and develops. w is the quantitative temperature rise rate. For numerical simulations of the honey bee-SHB population system, the system parameters of these equations are as follows; $v = 1500$ (the highest possible birth rate), $\mathscr{R}_0 = 1500$ (initial value for honey bees), $\gamma = 10^{-5}$ (the loss of honey bees as a result of SHB invasion), $\delta_1 = 0.004$ (denotes the rate of natural mortality of honey bees), $\mathscr{K} = 1400$ (the size of the bee colony), $q_0 = 60$ (initial value for SHBs), $\delta_2 = 0.01$ (indicates the rate of natural mortality rate of SHBs), $\alpha = 0.5$ (denotes the average number of SHBs that can be maintained for each bee), $T_{opt} = 37$, $T_{sig} = 20$, $r_m = 0.08$ (the maximum developmental rate), $w = 0.0067$ (quantifies the rate of temperature growth). The graphs of r(T) (Growth of SHB) and T (Brood Temperature) are presented in Figure (5) and Figure (6). New applicable methods have been developed that provide improvements in cases where applying some analytical methods to the models is impractical. The aim is to obtain approximate solutions for nonlinear fractional order differential equations.

2 Mathematical background

Definition 1. *The Sumudu transform of the function $\zeta(t)$ is presented as follows [31]:*

$$\mathscr{S}[\zeta(t)] = \int_0^\infty \zeta(ut)e^{-t}dt, \quad u \in (-\tau_1, \tau_2),$$

where in this study, $\mathscr{S}[\zeta(t)]$ will be denoted by $\mathscr{G}(u)$.

Consider the set of functions $\mathscr{X}$ such that:

$$\mathscr{X} = \{\zeta(t)|\exists \mathscr{Q}, \tau_1, \tau_2 > 0, |\zeta(t)| < \mathscr{Q}e^{|t|/\tau_j}, if\ t \in (-1)^j \times [0,\infty)\},$$

Definition 2. *The Caputo derivative is defined as [9]:*

$${}_a^{\mathscr{X}}D_t^{\ltimes}\zeta(t) = \frac{1}{\gamma(n-\varepsilon)}\int_a^t (t-x)^{n-\ltimes-1}\zeta^{(n)}(x)dx$$

and

$${}_a^{\mathscr{X}}D_t^{\ltimes}\zeta(t) = \frac{(-1)^n}{\gamma(n-\ltimes)}\int_a^t (t-x)^{n-\ltimes-1}\zeta^{(n)}(x)dx,$$

where $\ltimes \in \mathbb{C}$, $Re(\ltimes) > 0$, $n = [Re(\ltimes)] + 1$.

Lemma 1. *[32] The Sumudu transform of the Caputo fractional derivative is given as:*

$$\mathscr{S}[{}_a^{\mathscr{X}}D_t^{\ltimes}\zeta(t)] = \frac{\mathscr{G}(u) - \zeta(a)}{u^{\ltimes}},$$

where $0 < \ltimes < 1$.

3 Description of the method

3.1 Adomian polynomials

The Adomian decomposition method is a common technique used in analytical solution steps for a wide range of linear and nonlinear functional equations, including differential equations, integral equations, and integer differential equations. Adomian polynomials are obtained as follows [33]:

$$A_n = \frac{1}{n!}\left[\frac{d^n}{dx^n} f\left(\sum_{n=0}^{\infty} x^n u_n\right)\right], \quad x = 0, \quad n = 0, 1, 2, \cdots.$$

The first few steps of the Adomian polynomials are obtained as:

$$\begin{aligned}
A_0 &= f(u_0),\\
A_1 &= f(u_0)' u_1,\\
A_2 &= f(u_0)' u_2 + \frac{1}{2!} f''(u_0) u_1^2,\\
A_3 &= f(u_0)' u_3 + f(u_0)'' u_1 u_2 + f(u_0)''' \frac{u_1^3}{3!}, \quad \cdots.
\end{aligned}$$

3.2 Variational iterative method

As a result of numerous applications and improvements, the variational iteration algorithm has become an effective mathematical solution for solving nonlinear differential equations. The accuracy of this method is high, and its approximate solution is valid in the entire domain. The variational iteration method is preferred by many people and can be applied to different types of nonlinear problems. The primary important feature of this method is its flexibility and its ability to solve nonlinear differential equations accurately and conveniently. Also, recently, the variational iteration method, among other analytical methods, has been used as an effective technique for solving several different nonlinear problems without general restrictive assumptions. The Sumudu transform can be applied to the function if the function f satisfies the Dirichlet conditions [34].

1) it is single valued function which may have a finite number of isolated discontinuities for $t > 0$;

2) it remains less than $\omega e^{-\lambda_0 t}$ as $t \to \infty$, where ω is a positive constant and λ_0 is a real positive number.

3.3 Solution method with Sumudu transform

Using the Sumudu transformation proposed by Watugala, the differential equation is considered as a fractional-order differential equation, and the solution steps are as follows [34]:

Consider a nonlinear problem:

$${}_0^{\mathscr{X}}D_t^{\ltimes} \wp(t) + \mathscr{R}[\wp(t)] + \mathscr{N}[\wp(t)] = \Psi(t),$$

where $\mathscr{R}$ is a linear expression and $\mathscr{N}$ is a non-linear expression. $\Psi(t)$ is a given continuous function.

Applying the Sumudu transform to both sides of this equation and using $\mathscr{S}[\wp(t)] = \Im(u)$, gives:

$$\frac{\Im(u)}{u^{\ltimes}} - \frac{\wp(0)}{u^{\ltimes}} = \mathscr{S}[\Psi(t) - \mathscr{R}[\wp(t) - \mathscr{N}[\wp(t)]]].$$

The sequence of iteration equation goes like this:

$$\Im_{n+1}(u) = \Im_n(u) + \psi(u)\left(\frac{\Im_n(u)}{u^{\ltimes}} - \frac{\wp(0)}{u^{\ltimes}} - \mathscr{S}[\Psi(t) - \mathscr{R}[\wp(t)] - \mathscr{N}[\wp(t)]]\right),$$

where $\psi(u)$ is a Lagrange multiplier. If the expression $\mathscr{S}[\mathscr{R}[x(t)] + \mathscr{N}[x(t)]$ is considered as a bounded term and iteration is applied to this equation, the following occurs.

$$\delta_1\Im_{n+1}(u) = \delta_1\Im_n(u) + \psi(u)\frac{1}{u^{\ltimes}}\delta_1\Im_n(u),$$

where $\psi(u)$ is obtained from the equation:

$$\psi(u) = -u^{\ltimes}.$$

The inverse Sumudu transformation is applied to this equation, and then,

$$\begin{aligned}\wp_{n+1}(t) &= \wp_n(t) + \mathscr{S}^{-1}\left[-u^{\ltimes}\left(\frac{\Im_n(u)}{u^{\ltimes}} - \frac{\wp(0)}{u^{\ltimes}} - \mathscr{S}[\Psi(t) - \mathscr{R}[\wp(t)] - \mathscr{N}[\wp(t)]]\right)\right] \\ &= \wp_1(t) + \mathscr{S}^{-1}[u^{\ltimes}\mathscr{S}[\Psi(t) - \mathscr{R}[\wp(t)] - \mathscr{N}[\wp(t)]]],\end{aligned}$$

where $\wp_1(t)$ represents the initial value.

4 Main result

4.1 Fractional population dynamics model with Caputo fractional derivative

The differential in the Verhults [21] model is treated as the Caputo fractional derivative

$${}_0^C D_t^{\ltimes}\aleph(t) = \leftthreetimes\aleph(t)\left(1 - \frac{\aleph(t)}{\Upsilon}\right), \quad t > 0, \quad \aleph(0) = \aleph_0.$$

We applied the Sumudu transform, and we obtained:

$$\mathscr{S}[{}_0^C D_t^{\ltimes}\aleph(t)] = \mathscr{S}\left[\leftthreetimes\aleph(t)\left(1 - \frac{\aleph(t)}{\Upsilon}\right)\right],$$

$$\frac{\mathscr{S}[\aleph(t)] - \aleph(0)}{u^{\ltimes}} = \mathscr{S}\left[\leftthreetimes\aleph(t)\left(1 - \frac{\aleph(t)}{\Upsilon}\right)\right],$$

where $\mathscr{S}[\aleph(t)] = \Theta(u)$. The variational iteration formula is acquired as follows:

$$\Theta_{n+1}(u) = \Theta_n(u) + \psi(u)\left(\frac{\Theta_n(u)}{u^{\ltimes}} - \frac{\aleph_0}{u^{\ltimes}} - \mathscr{S}\left[\leftthreetimes\aleph_n(t)\left(1 - \frac{\aleph_n(t)}{\Upsilon}\right)\right]\right).$$

The classic variation operators on both sides of this equation are detected, and the Lagrange multiplier is:

$$\psi(u) = -u^{\ltimes}.$$

The inverse Sumudu transform of both sides of this equation is attained as:

$$\aleph_{n+1}(t) = \aleph_n(t) + \mathscr{S}^{-1}\left[-u^{\ltimes}\left(\frac{\Theta_n(u)}{u^{\ltimes}} - \frac{\aleph_0}{u^{\ltimes}} - \mathscr{S}\left[\leftthreetimes\aleph_n(t)\left(1 - \frac{\aleph_n(t)}{\Upsilon}\right)\right]\right)\right] = \aleph_0 + \mathscr{S}^{-1}\left[\leftthreetimes u^{\ltimes}\left(\mathscr{S}[\aleph_n(t)] - \frac{1}{\Upsilon}\mathscr{S}[\aleph_n(t)\aleph_n(t)]\right)\right],$$

where the initial value for $n = 0$ turns out to be $\aleph_1(t) = \aleph(0) = \aleph_0$.

A nonlinear expression $\mathscr{Q}(\aleph)$ is represented as:

$$\mathscr{Q}(\aleph) = \sum_{i=0}^{\infty}\mathscr{X}_i = \frac{1}{i!}\left[\frac{d^i}{d\phi^i}f\left(\sum_{n=0}^{\infty}\phi^n v_n\right)\right], \quad \phi = 0,$$

where $\mathscr{X}$ is an Adomian polynomial [33].

Let $\aleph_n = \sum_{i=0}^{n} v_i$ and the non-linear term $'\aleph_n(t)\aleph_n(t)'$ expressed the Adomian series as follows:

$$\mathscr{X}_0 = v_0^2,$$

$$\mathscr{X}_1 = 2v_0v_1,$$

$$\mathscr{X}_2 = 2v_0v_2 + v_1^2,$$
$$\mathscr{X}_3 = 2v_0v_3 + 2v_1v_2.$$

These specified data are formulated

$$v_0(t) = v_0 = \aleph_0,$$
$$v_{n+1}(t) = \mathscr{S}^{-1}\left[\measuredangle u^{\ltimes}\left(\mathscr{S}[v_n(t)] - \frac{1}{\Upsilon}\mathscr{S}[\mathscr{X}_n]\right)\right]$$

and all iterations are found to be:

$$v_0 = \aleph_0,$$
$$v_1 = \mathscr{S}^{-1}\left[\measuredangle u^{\ltimes}\left(\mathscr{S}[v_0] - \frac{1}{\Upsilon}\mathscr{S}[\mathscr{X}_0]\right)\right] = \measuredangle v_0\mathscr{S}^{-1}[u^{\ltimes}] - \frac{\measuredangle v_0^2}{\Upsilon}\mathscr{S}^{-1}[u^{\ltimes}] = \aleph_0\left(1 - \frac{\aleph_0}{\Upsilon}\right)\frac{\measuredangle t^{\ltimes}}{\Gamma(\ltimes+1)},$$
$$v_2 = \mathscr{S}^{-1}\left[\measuredangle u^{\ltimes}\left(\mathscr{S}[v_1] - \frac{1}{\Upsilon}\mathscr{S}[\mathscr{X}_1]\right)\right] = \left[\aleph_0\left(1 - \frac{\aleph_0}{\Upsilon}\right) - \frac{2\aleph_0^2}{\Upsilon}\left(1 - \frac{\aleph_0}{\Upsilon}\right)\right]\frac{\measuredangle^2 t^{2\ltimes}}{\Gamma(2\ltimes+1)},$$
$$v_3 = \mathscr{S}^{-1}\left[\measuredangle u^{\ltimes}\left(\mathscr{S}[v_2] - \frac{1}{\Upsilon}\mathscr{S}[\mathscr{X}_2]\right)\right] = \left[\aleph_0\left(1 - \frac{\aleph_0}{\Upsilon}\right) - \frac{2\aleph_0^2}{\Upsilon}\left(1 - \frac{\aleph_0}{\Upsilon}\right)\right.$$
$$\left. - \frac{1}{k}\left(2\aleph_0\left[\aleph_0\left(1 - \frac{\aleph_0}{\Upsilon}\right) - \frac{2\aleph_0^2}{\Upsilon}\left(1 - \frac{\aleph_0}{\Upsilon}\right)\right] + \aleph_0^2\left(1 - \frac{\aleph_0}{\Upsilon}\right)^2\right)\right]\frac{\measuredangle^3 t^{3\ltimes}}{\Gamma(3\ltimes+1)}.$$

The solution to the desired equation is obtained by writing the iterations found as sums.

$$\aleph(t) = \lim_{n\to\infty}\aleph_n = \lim_{n\to\infty}\sum_{i=0}^{n} v_i$$
$$= \aleph_0 + \aleph_0\left(1 - \frac{\aleph_0}{\Upsilon}\right)\frac{\measuredangle t^{\ltimes}}{\Gamma[\ltimes+1]} = \left[\aleph_0\left(1 - \frac{\aleph_0}{\Upsilon}\right) - \frac{2\aleph_0^2}{\Upsilon}\left(1 - \frac{\aleph_0}{\Upsilon}\right)\right]\frac{\measuredangle^2 t^{2\ltimes}}{\Gamma(2\ltimes+1)}$$
$$= \left[\aleph_0\left(1 - \frac{\aleph_0}{\Upsilon}\right) - \frac{2\aleph_0^2}{\Upsilon}\left(1 - \frac{\aleph_0}{\Upsilon}\right)\right.$$
$$\left. - \frac{1}{k}\left(2\aleph_0\left[\aleph_0\left(1 - \frac{\aleph_0}{\Upsilon}\right) - \frac{2\aleph_0^2}{\Upsilon}\left(1 - \frac{\aleph_0}{\Upsilon}\right)\right] + \aleph_0^2\left(1 - \frac{\aleph_0}{\Upsilon}\right)^2\right)\right]\frac{\measuredangle^3 t^{3\ltimes}}{\Gamma(3\ltimes+1)} + \cdots.$$

Figure (1) and Figure (2) illustrate the approximate solution for the Verhulst model.

4.2 Fractional honey bee mathematical model with Caputo fractional derivative

In the mathematical model developed by [22], the differential is considered as a fractional Caputo derivative and is written again as follows:

$${}_0^C D_t^{\ltimes} p(t) = v g_p(p) - \gamma q p - \delta_1 p, \quad p(0) = p_0, \quad g_p(p) = \frac{p^2}{\mathscr{K}^2 + p^2},$$
$${}_0^C D_t^{\ltimes} q(t) = r(T) g_q(p,q) q - \delta_2 q, \quad q(0) = q_0, \quad g_q(p,q) = 1 - \frac{q}{\alpha p},$$

Taking the Sumudu transformation of both sides of last equation, yields:

$$\mathscr{S}[{}_0^C D_t^{\ltimes} p(t)] = \mathscr{S}\left[v\frac{p^2}{\mathscr{K}^2 + p^2} - \gamma q p - \delta_1 p\right]$$
$$\mathscr{S}[{}_0^C D_t^{\ltimes} q(t)] = \mathscr{S}\left[r(T)\left(1 - \frac{q}{\alpha p}\right)q - \delta_2 q\right],$$

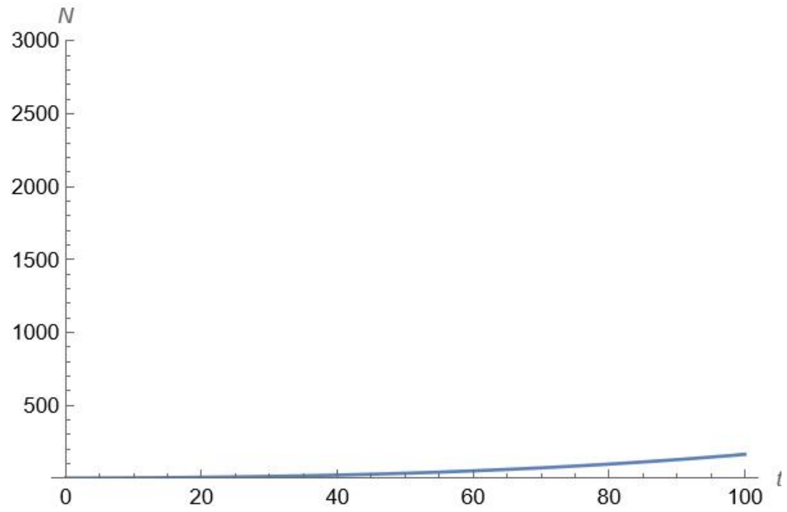

Fig. 1: Numerical simulation of population dynamics model with Caputo derivative for $\ltimes = 0.9$, $\aleph_0 = 1$, $\Upsilon = 3000$, $\curlywedge = 0.125$.

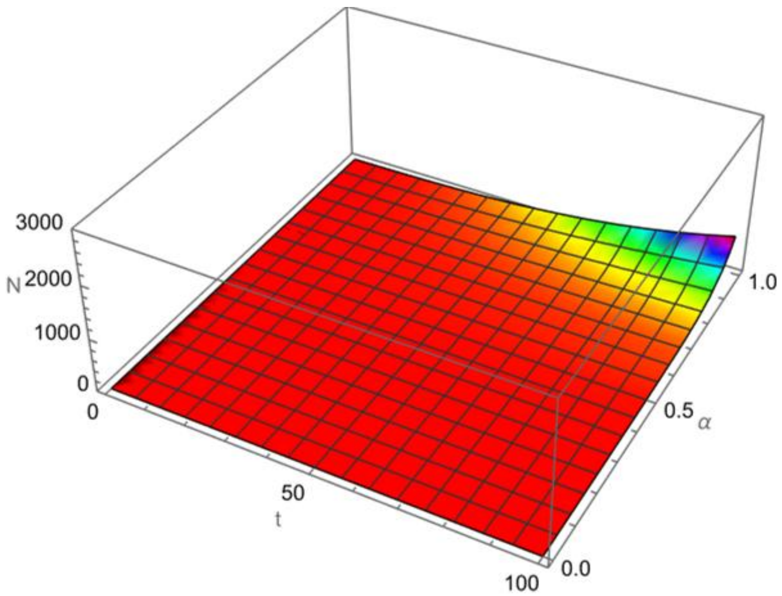

Fig. 2: Numerical 3D simulation of population dynamics model with Caputo derivative for $\ltimes = 0.9$, $\aleph_0 = 1$, $\Upsilon = 3000$, $\curlywedge = 0.125$.

$$\frac{\mathscr{S}[p(t)] - p(0)}{u^{\ltimes}} = \mathscr{S}\left[v\frac{p^2}{\mathscr{K}^2 + p^2} - \gamma q p - \delta_1 p\right],$$

$$\frac{\mathscr{S}[q(t)] - q(0)}{u^{\ltimes}} = \mathscr{S}\left[r(T)\left(1 - \frac{q}{\alpha p}\right)q - \delta_2 q\right].$$

Let $\mathscr{S}[p(t)] = P(u)$ and $\mathscr{S}[q(t)] = \mathscr{Q}(u)$. We get the variational iteration formula:

$$P_{n+1}(u) = P_n(u) + \psi_1(u)\left(\frac{P_n(u)}{u^{\ltimes}} - \frac{p_0}{u^{\ltimes}} - \mathscr{S}\left[v\frac{p_n^2(t)}{\mathscr{K}^2 + p_n^2(t)} - \gamma q_n(t) p_n(t) - \delta_1 p_n(t)\right]\right),$$

$$\mathscr{Q}_{n+1}(u) = \mathscr{Q}_n(u) + \psi_2(u)\left(\frac{\mathscr{Q}_n(u)}{u^{\ltimes}} - \frac{q_0}{u^{\ltimes}} - \mathscr{S}\left[r(T)\left(1 - \frac{q_n(t)}{\alpha p_n(t)}\right)q_n(t) - \delta_2 q_n(t)\right]\right).$$

The Lagrange multiplier is presented as follows:

$$\psi_1(u) = \psi_2(u) = -u^{\ltimes}.$$

The inverse Sumudu transform of both sides of the equations is taken, and the iteration form is obtained as:

$$\begin{aligned} p_{n+1}(t) &= p_n(t) + \mathscr{S}^{-1}\left[-u^{\ltimes}\left(\frac{P_n(u)}{u^{\ltimes}} - \frac{p_0}{u^{\ltimes}} - \mathscr{S}\left[v\frac{p_n^2(t)}{\mathscr{K}^2 + p_n^2(t)} - \gamma q_n(t)p_n(t) - \delta_1 p_n(t)\right]\right)\right] \\ &= p_0 + \mathscr{S}^{-1}\left[u^{\ltimes}\left(v\mathscr{S}\left[\frac{p_n^2(t)}{\mathscr{K}^2 + p_n^2(t)}\right] - \gamma\mathscr{S}[q_n(t)p_n(t)] - \delta_1\mathscr{S}[p_n(t)]\right)\right], \end{aligned}$$

$$\begin{aligned} q_{n+1} &= q_n + \mathscr{S}^{-1}\left[-u^{\ltimes}\left(\frac{\mathscr{Q}_n(u)}{u^{\ltimes}} - \frac{q_0}{u^{\ltimes}} - \mathscr{S}\left[r(T)\left(1 - \frac{q_n(t)}{\alpha p_n(t)}\right)q_n(t) - \delta_2 q_n(t)\right]\right)\right] \\ &= q_0 + \mathscr{S}^{-1}\left[u^{\ltimes}\left(\mathscr{S}[r(T)q_n(t)] - \mathscr{S}\left[\frac{r(T)q_n^2(t)}{\alpha p_n(t)}\right] - \delta_2\mathscr{S}[q_n(t)]\right)\right]. \end{aligned}$$

The nonlinear expressions is represented as;

$$\sum_{i=0}^{\infty}\mathscr{X}_i = f_1(p) = \frac{p_n^2}{\mathscr{K}^2 + p_n^2},$$

$$\sum_{i=0}^{\infty}\mathscr{Y}_i = f_2(q,p) = p_n q_n,$$

$$\sum_{i=0}^{\infty}\mathscr{Z}_i = f_3(q,p) = \frac{q_n^2}{\alpha p_n},$$

and

$$\mathscr{X}_i = \frac{1}{i!}\left[\frac{d^i}{d\phi^i}f_1\left(\sum_{n=0}^{\infty}\phi^n p_n\right)\right], \quad \phi = 0,$$

$$\mathscr{Y}_i = \frac{1}{i!}\left[\frac{d^i}{d\phi^i}f_2\left(\sum_{n=0}^{\infty}\phi^n p_n, \sum_{n=0}^{\infty}\phi^n q_n\right)\right], \quad \phi = 0,$$

$$\mathscr{Z}_i = \frac{1}{i!}\left[\frac{d^i}{d\phi^i}f_3\left(\sum_{n=0}^{\infty}\phi^n p_n, \sum_{n=0}^{\infty}\phi^n q_n\right)\right], \quad \phi = 0.$$

For nonlinear terms, the Adomian series are as follows:

$$\mathscr{X}_0 = \frac{p_0^2}{\mathscr{K}^2 + p_0^2}, \quad \mathscr{X}_1 = \frac{2p_0^3 p_1}{(\mathscr{K}^2 + p_0^2)^2} + \frac{2p_0 p_1}{\mathscr{K}^2 + p_0^2},$$

$$\mathscr{X}_2 = \frac{4p_0^4 p_1^2}{(\mathscr{K}^2 + p_0^2)^3} - \frac{5p_0^2 p_1^2}{(\mathscr{K}^2 + p_0^2)^2} + \frac{p_1^2}{\mathscr{K}^2 + p_0^2} - \frac{2p_0^3 p_2^2}{(\mathscr{K}^2 + p_0^2)^2} + \frac{2p_0 p_2}{\mathscr{K}^2 + p_0^2},$$

and

$$\mathscr{Y}_0 = q_0 p_0, \quad \mathscr{Y}_1 = q_1 p_0 + q_0 p_1, \quad \mathscr{Y}_2 = q_2 p_0 + q_1 p_1 + q_0 p_2,$$

and

$$\mathscr{Z}_0 = \frac{q_0^2}{\alpha p_0}, \quad \mathscr{Z}_1 = \frac{2q_0q_1}{\alpha p_0} - \frac{q_0^2 p_1}{\alpha p_0^2}, \quad \mathscr{Z}_2 = \frac{q_1^2}{\alpha p_0} + \frac{2q_0q_2}{\alpha p_0} - \frac{2q_0q_1p_1}{\alpha p_0^2} + \frac{q_0^2p_1^2}{\alpha p_0^3} - \frac{q_0^2p_2}{\alpha p_0^2}.$$

The equation is edited and rewritten

$$p_0(t) = p_0, \qquad p_{n+1}(t) = \mathscr{S}^{-1}[u^{\ltimes}(v\mathscr{S}[\mathscr{X}_n(t)] - \gamma\mathscr{S}[\mathscr{Y}_n(t)] - \delta_1\mathscr{S}[p_n(t)])],$$

and

$$q_0(t) = q_0, \qquad q_{n+1}(t) = \mathscr{S}^{-1}[u^{\ltimes}(\mathscr{S}[r(T)q_n] - \mathscr{S}[r(T)\mathscr{Z}_n(t)] - \delta_2\mathscr{S}[q_n])].$$

The first three iteration terms are calculated as follows:

$$\begin{aligned}
p_0 &= p_0, \\
p_1 &= \mathscr{S}^{-1}[u^{\ltimes}(v\mathscr{S}[\mathscr{X}_0(t)] - \gamma\mathscr{S}[\mathscr{Y}_0(t)] - \delta_1\mathscr{S}[p_0(t)])] \\
&= \left[v\frac{p_0^2}{\mathscr{K}^2 + p_0^2} - \gamma q_0 p_0 - \delta_1 p_0\right]\frac{t^{\ltimes}}{\Gamma[\ltimes + 1]}, \\
p_2 &= \mathscr{S}^{-1}[u^{\ltimes}(v\mathscr{S}[\mathscr{X}_1(t)] - \gamma\mathscr{S}[\mathscr{Y}_1(t)] - \delta_1\mathscr{S}[p_1(t)])] \\
&= \Bigg(v\left(\frac{2p_0^3}{(\mathscr{K}^2 + p_0^2)^2}\left[v\frac{p_0^2}{\mathscr{K}^2 + p_0^2} - \gamma q_0 p_0 - \delta_1 p_0\right] + \frac{2p_0}{\mathscr{K}^2 + p_0^2}\left[v\frac{p_0^2}{\mathscr{K}^2 + p_0^2} - \gamma q_0 p_0 - \delta_1 p_0\right]\right) \\
&\qquad - \gamma\left(\left[r(T)q_0 - r(T)\frac{q_0^2}{\alpha p_0} - \delta_2 q_0\right]p_0 + q_0\left[v\frac{p_0^2}{\mathscr{K}^2 + p_0^2} - \gamma q_0 p_0 - \delta_1 p_0\right]\right) \\
&\qquad - \delta_1\left[v\frac{p_0^2}{\mathscr{K}^2 + p_0^2} - \gamma q_0 p_0 - \delta_1 p_0\right]\Bigg)\frac{t^{2\ltimes}}{\Gamma[2\ltimes + 1]}, \\
&\ \vdots
\end{aligned}$$

and

$$\begin{aligned}
q_0 &= q_0, \\
q_1 &= \mathscr{S}^{-1}[u^{\ltimes}(\mathscr{S}[r(T)q_0] - \mathscr{S}[r(T)\mathscr{Z}_0(t)] - \delta_2\mathscr{S}[q_0])] \\
&= \left[r(T)q_0 - r(T)\frac{q_0^2}{\alpha p_0} - \delta_2 q_0\right]\frac{t^{\ltimes}}{\Gamma[\ltimes + 1]}, \\
q_2 &= \mathscr{S}^{-1}[u^{\ltimes}(\mathscr{S}[r(T)q_1] - \mathscr{S}[r(T)\mathscr{Z}_1(t)] - \delta_2\mathscr{S}[q_1])] \\
&= \Bigg(r(T)\left[r(T)q_0 - r(T)\frac{q_0^2}{\alpha p_0} - \delta_2 q_0\right] \\
&\qquad - r(T)\left(\frac{2q_0}{\alpha p_0}\left[r(T)q_0 - r(T)\frac{q_0^2}{\alpha p_0} - \delta_2 q_0\right] - \frac{q_0^2}{\alpha p_0^2}\left[v\frac{p_0^2}{\mathscr{K}^2 + p_0^2} - \gamma q_0 p_0 - \delta_1 p_0\right]\right) \\
&\qquad - \delta_2\left[r(T)q_0 - r(T)\frac{q_0^2}{\alpha p_0} - \delta_2 q_0\right]\Bigg)\frac{t^{2\ltimes}}{\Gamma[2\ltimes + 1]}. \\
&\ \vdots
\end{aligned}$$

Solution of the model is obtained as:

$$
\begin{aligned}
p(t) &= p_0 + p_1 + p_2 + \cdots \\
&= p_0 + \left[v \frac{p_0^2}{\mathscr{K}^2 + p_0^2} - \gamma q_0 p_0 - \delta_1 p_0 \right] \frac{t^{\ltimes}}{\Gamma[\ltimes + 1]} \\
&\quad + \Bigg(v \left(\frac{2p_0^3}{(\mathscr{K}^2 + p_0^2)^2} \left[v \frac{p_0^2}{\mathscr{K}^2 + p_0^2} - \gamma q_0 p_0 - \delta_1 p_0 \right] + \frac{2p_0}{\mathscr{K}^2 + p_0^2} \left[v \frac{p_0^2}{\mathscr{K}^2 + p_0^2} - \gamma q_0 p_0 - \delta_1 p_0 \right] \right) \\
&\quad - \gamma \left(\left[r(T) q_0 - r(T) \frac{q_0^2}{\alpha p_0} - \delta_2 q_0 \right] p_0 + q_0 \left[v \frac{p_0^2}{\mathscr{K}^2 + p_0^2} - \gamma q_0 p_0 - \delta_1 p_0 \right] \right) \\
&\quad - \delta_1 \left[v \frac{p_0^2}{\mathscr{K}^2 + p_0^2} - \gamma q_0 p_0 - \delta_1 p_0 \right] \Bigg) \frac{t^{2\ltimes}}{\Gamma[2\ltimes + 1]},
\end{aligned}
$$

and

$$
\begin{aligned}
q(t) &= q_0 + q_1 + q_2 + \cdots \\
&= q_0 + \left[r(T) q_0 - r(T) \frac{q_0^2}{\alpha p_0} - \delta_2 q_0 \right] \frac{t^{\ltimes}}{\Gamma[\ltimes + 1]} \\
&\quad + \Bigg(r(T) \left[r(T) q_0 - r(T) \frac{q_0^2}{\alpha p_0} - \delta_2 q_0 \right] \\
&\quad - r(T) \left(\frac{2q_0}{\alpha p_0} \left[r(T) q_0 - r(T) \frac{q_0^2}{\alpha p_0} - \delta_2 q_0 \right] - \frac{q_0^2}{\alpha p_0^2} \left[v \frac{p_0^2}{\mathscr{K}^2 + p_0^2} - \gamma q_0 p_0 - \delta_1 p_0 \right] \right) \\
&\quad - \delta_2 \left[r(T) q_0 - r(T) \frac{q_0^2}{\alpha p_0} - \delta_2 q_0 \right] \Bigg) \frac{t^{2\ltimes}}{\Gamma[2\ltimes + 1]}.
\end{aligned}
$$

Figure (3) and Figure (4) illustrate the approximate solution for the honey bee-SHB model.

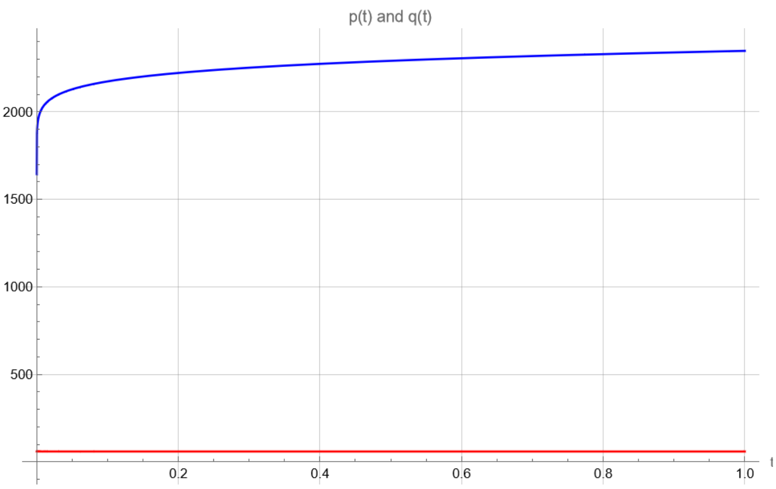

Fig. 3: Numerical simulation of honey bee model with Caputo derivative for $\ltimes$ = 0.1, p(t) (blue, BEE), q(t) (red, SHB).

5 Conclusion

The major goal of this research is to demonstrate that one of the most significant and straightforward approaches for solving both linear and nonlinear differential equations is the Sumudu transform method. This technique has been

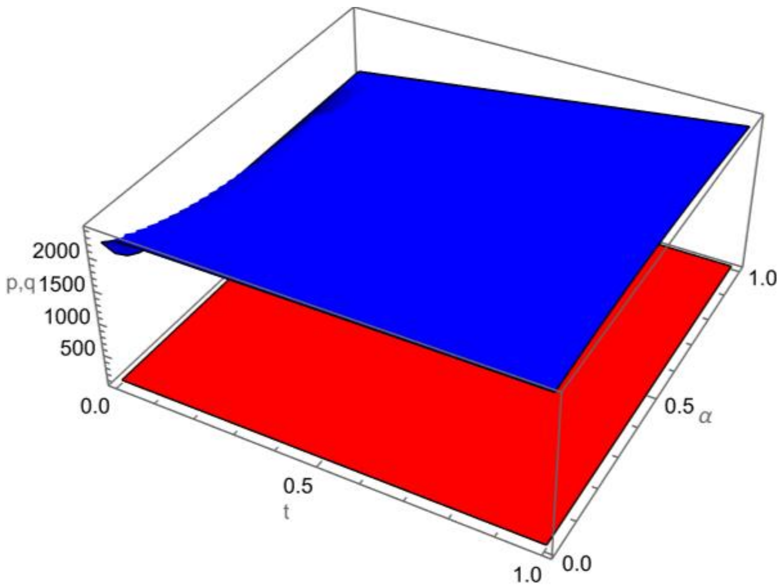

Fig. 4: Numerical 3D simulation of honey bee model with Caputo derivative for ⋉ = 0.1, p(t) (blue, BEE), q(t) (red, SHB).

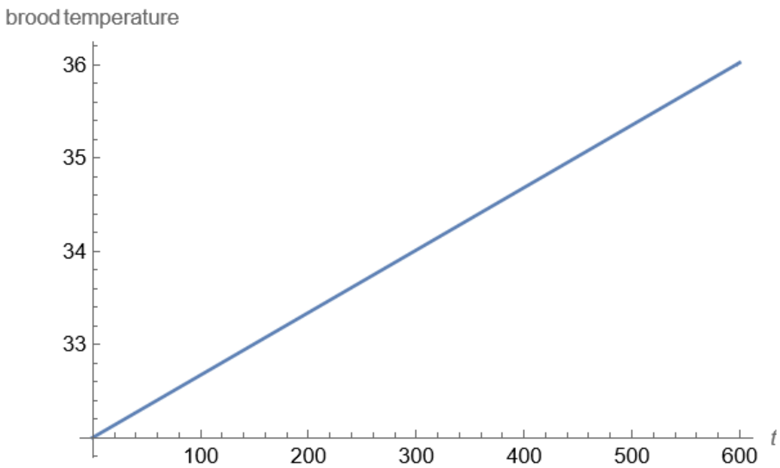

Fig. 5: Simulation of brood temperature versus time.

successfully used to solve systems of fractional nonlinear equations, including fractional population dynamics models and fractional mathematical models of honey bees. With this technique, we do not need to perform the challenging computation required to discover the Adomian polynomials. In terms of the theory of fractional calculus, the proposed method is generally promising and applicable to a wide range of linear and nonlinear situations. Additionally, the effect of the Caputo derivative via the power-law kernel has been investigated.

Availability of data and materials

All data that was used is included in the research.

Competing interests

On behalf of all authors, the corresponding author states that there is no conflict of interest.

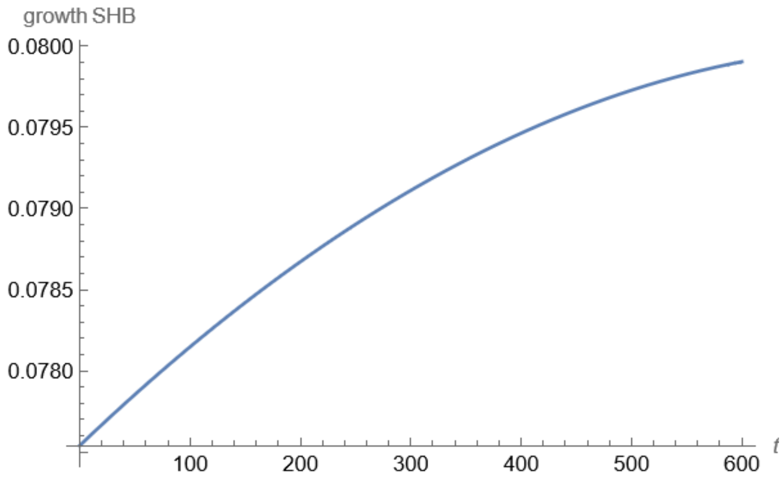

Fig. 6: Growth of SHB r(T).

Authors' contributions

All authors have contributed, read, and approved the manuscript.

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Ethics approval and consent to participate

Not applicable.

References

[1] M. Farman, A. Talib, K. S. Nisar, A. Sambas, M. Bayram, and M. Hafez, Investigation of ABPV predict dynamics infection in honeybee colony production: Soft patterns multiscale modeling with fractional approach, *Ain Shams Engineering Journal* **16**(10) (2025) 103626.

[2] H. Khan, W. F. Alfwzan, J. Alzabut, D. K. Almutairi, M. A. Azim, and R. Thinakaran, Artificial intelligence and neural networking for an analysis of fractal–fractional Zika virus model, *Fractals* **33**(08) (2025) 2540143.

[3] A. Khan, T. Abdeljawad, and R. Thinakaran, Computational analysis by artificial intelligence of the fractional-order plant virus spread model, *Fractals* **33**(08) (2025) 2540141.

[4] W. G. Alharbi, D. M. Almutairi, D. Abdelhamied, M. M. Alzubaidi, M. Areshi, M. El-Kady, and M. Abdelhakem, Pseudospectral fractional differentiation matrices for solving Riccati and Bagley–Torvik fractional models via mixed shifted polynomials, *Fractals* **34**(02) (2026) 2540214.

[5] M. Abdelhakem, Shifted Legendre Fractional Pseudo-spectral Integration Matrices for Solving Fractional Volterra Integro-Differential Equations and Abel's Integral Equations, *Fractals* **31**(10) (2023) 2340190.

[6] M. Abdelhakem, D. Abdelhamied, D. Baleanu, M. G. Alshehri, and M. El-Kady, Shifted Legendre fractional pseudospectral differentiation matrices for solving fractional differential problems, *Fractals* **30**(1) (2022) 2240038.

[7] S. S. Zeid, Approximation methods for solving fractional equations, *Chaos, Solitons & Fractals* **125** (2019) 171-193.

[8] M. Abdelhakem, M. R. Doha, D. Baleanu, and M. El-Kady, Shifted ultraspherical pseudo-Galerkin method for approximating the solutions of some types of ordinary fractional problems, *Advances in Difference Equations* **2021** (2021) 110.

[9] M. Caputo, Linear models of dissipation whose Q is almost frequency independent: II, *Geophysical Journal of the Royal Astronomical Society* **13**(5) (1967) 529–539.

[10] Y. Karaca and D. Baleanu, Computational fractional-order calculus and classical calculus AI for comparative differentiability prediction analyses of complex-systems-grounded paradigm, *Multi-Chaos, Fractal and Multi-Fractional Artificial Intelligence of Different Complex Systems* (2022) 149-168.

[11] H. Jafari, A new general integral transform for solving integral equations, *Journal of Advanced Research* **32** (2021) 133-138.

[12] M. Abdel-Aty, An investigation of entanglement and quasiprobability distribution in a generalized Jaynes-Cummings model, *Journal of Mathematical Physics*, **44** (2003) 1457–1471 DOI: http://dx.doi.org/10.1063/1.1559643

[13] M. Abdel-Aty, M. S. Abdalla, A.-S. F. Obada, Entropy and phase properties of isotropic coupled oscillators interacting with a single atom: one- and two-photon processes, *Journal of Optics B Quantum and Semiclassical Optics*, **4** (2002) S133–S141 DOI: http://dx.doi.org/10.1088/1464-4266/4/3/367

[14] J.-S. Zhang, A.-X. Chen, M. Abdel-Aty, Two atoms in dissipative cavities in dispersive limit: Entanglement sudden death and long-lived entanglement, *Journal of Physics B Atomic Molecular and Optical Physics*, **43** (2010) 025501 DOI: http://dx.doi.org/10.1088/0953-4075/43/2/025501

[15] M. Sebawe Abdalla, S. S. Hassan, M. Abdel-Aty, Entropic uncertainty in the Jaynes-Cummings model in presence of a second harmonic generation, *Optics Communications*, **244** (2005) 431–443 DOI: http://dx.doi.org/10.1016/j.optcom.2004.09.051

[16] M. Abdel-Aty, M. Sebawe Abdalla, A.-S.F. Obada, Entropy squeezing of a two-mode multiphoton Jaynes-Cummings model in the presence of a nonlinear medium, *J*ournal of Optics B Quantumand Semiclassical Optics, **4** (2002) 134–142 DOI: http://dx.doi.org/10.1088/1464-4266/4/2/309

[17] J. G. Skellam, Random dispersal in theoretical populations, *Bulletin of Mathematical Biology* **53** (1991) 135–165.

[18] T. R. Malthus, An essay on the principle of population, *J. Johnson, London* (1798).

[19] M. Chhaytle, R. Ouvrard, and T. Poinot, New population dynamics models to evaluate the climate impacts on wildlife populations, *IFAC-PapersOnLine* **55**(5) (2022) 54–59.

[20] E. E. Holmes, M. A. Lewis, J. E. Banks, and R. R. Veit, Partial-Differential Equations In Ecology - Spatial Interactions And Population-Dynamics, *Ecology* **75**(1) (1994) 17-29.

[21] P. F. Verhulst, Notice sur la loi que la population poursuit dans son accroissement, *Corresp. Math. Phys.* **10** (1838) 113-121.

[22] H. Tutun, Y. Sekerci, and S. Sevin, Future effects of small hive beetle, Aethina tumida, on honey bee colony in Turkey based on temperature factor using a mathematical model, *The European Zoological Journal* **89**(1) (2022) 1259-1270.

[23] R. A. Hayes, S. J. Rice, B. A. Amos, and D. M. Leemon, Increased attractiveness of honeybee hive product volatiles to adult small hive beetle, Aethina tumida, resulting from small hive beetle larval infestation, *Entomologia Experimentalis et Applicata* **155**(3) (2015) 240–248.

[24] N. D. Benda, D. Boucias, B. Torto, and P. Teal, Detection and characterization of Kodamaea ohmeri associated with small hive beetle Aethina tumida infesting honey bee hives, *Journal of Apicultural Research* **47** (2008) 194–201.

[25] M. N. Noor-ul-Ane and C. Jung, Temperature-dependent development and survival of small hive beetle, Aethina tumida (Coleoptera: Nitidulidae), *Journal of Apicultural Research* **59**(5) (2020) 807–816.

[26] Z. A. Jamal, H. F. Abou-Shaara, S. Qamer, M. A. Alotaibi, K. A. Khan, M. F. Khan, and M. J. Ansari, Future expansion of small hive beetles, Aethina tumida, towards North Africa and South Europe based on temperature factors using maximum entropy algorithm, *Journal of King Saud University - Science* **33**(1) (2021) 101242.

[27] S. M. Namin, Y. Koh, A. F. Osabutey, and C. Jung, Invasion pathway of the honeybee pest, small hive beetle, Aethina tumida (Coleoptera: Nitidulidae) in the Republic of Korea inferred by mitochondrial DNA sequence analysis, *Journal of Asia-Pacific Entomology* **22**(3) (2019) 963-968.

[28] V. Ratti, P. G. Kevan, and H. J. Eberl, A mathematical model of the honeybee–Varroa destructor–acute bee paralysis virus system with seasonal effects, *Bulletin of Mathematical Biology* **77**(8) (2015) 1493–1520.

[29] S. Pradhan, Rate of insect development under variable temperatures of the field, *Proceedings of the National Institute of Sciences of India* **11**(73) (1945) 80.

[30] F. Taylor, Ecology and evolution of physiological time in insects, *The American Naturalist* **117**(1) (1981) 1–23.

[31] G. K. Watugala, Sumudu transform: a new integral transform to solve differential equations and control engineering problems, *International Journal of Mathematical Education in Science and Technology* **24**(1) (1993) 35-43.

[32] E. Karatas and E. Toktas, Applications of Fractional Order Mathematical Models with an Effective Integral Transform, *Journal of Environmental Accounting and Management* **13**(3) (2025) 311-336.

[33] G. Adomian, A review of the decomposition method in applied mathematics, *Journal of Mathematical Analysis and Applications* **135**(2) (1988) 501–544.

[34] M. O. Aibinu, S. C. Thakur, and S. Moyo, Analyzing population dynamics models via Sumudu transform, *Journal of Mathematics and Computer Science* **29**(3) (2023) 283–294.