Fixed Point Theorems In Dislocated Quasi-Metric Spaces

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1 Introduction

The notion of Dislocated quasi-metric space has been introduced by F.M. Zeyada et al and proved a version of Banach Contraction principle in such spaces(see [5]). In [4] C.T. Aage and J. N. Salunke proved dislocated quasi-metric version of Kannan mapping theorem. In this paper we present some interesting properties of dislocated quasi-metric spaces. Using these properties some fixed point theorem for contractions and Kannan mappings are derived dropping continuity condition imposed by F.M. Zeyada and C.T. Aage. In what follows \( \mathbb{N}, \mathbb{R}, \mathbb{Q} \) denote the sets of natural, real and rational numbers respectively.

Definition 1.[5] Let \( X \) be a nonempty set and \( d : X \times X \to [0, \infty) \) satisfy the following conditions:

(d1) \( d(x,y) = d(y,x) = 0 \) implies \( x = y \),
(d2) \( d(x,y) \leq d(x,z) + d(z,y) \) for all \( x, y, z \in X \).

Then the function \( d \) is called dislocated quasi-metric on \( X \) and the pair \( (X,d) \) is called a dislocated quasi-metric space(in short dq-metric space).

In addition, if \( d \) satisfies

(d3) \( d(x,y) = d(y,x) \) for all \( x, y \in X \),

then \( (X,d) \) is called a dislocated metric space (d-metric space in short).

Throughout this paper \( (X,d) \) will denote a dq-metric space.

Definition 2.[5] A sequence \( (x_n) \) dislocated quasi-converges (for short dq-converges) to \( x \) in \( X \) if

\[ \lim_{n \to \infty} d(x_n,x) = \lim_{n \to \infty} d(x,x_n) = 0. \]  \hspace{1cm} (1.1)

In this case \( x \) is called a dq-limit of \( (x_n) \) and we write \( \lim_{n \to \infty} x_n = x \) in \( (X,d) \).

Remark. Dq-limits of a dq-convergent sequence in dq-metric space are unique.

Definition 3.[5] A sequence \( (x_n) \) in \( X \) is called Cauchy (resp. Bicauchy) if for each \( \varepsilon > 0 \) there exists a positive integer \( N_\varepsilon \) such that for all \( m,n \geq N_\varepsilon \), \( d(x_n,x_m) < \varepsilon \) or \( d(x_n,x_m) \leq \varepsilon \) (resp. \( \max\{d(x_n,x_m),d(x_m,x_n)\} < \varepsilon \)). \( (X,d) \) is called complete dq-metric space if every Cauchy sequence in \( X \) is a dq-convergent in \( X \).

Definition 4.[5] Let \( (X,d_1) \) and \( (Y,d_2) \) be a dq-metric spaces. Then \( f : X \to Y \) is continuous if for each sequence \( (x_n) \) which is \( d_1q \)-convergent to \( x_0 \) in \( X \), the sequence \( (f(x_n)) \) is \( d_2q \)-convergent to \( f(x_0) \) in \( Y \).

Definition 5. A mapping \( T : X \to X \) is called a contraction if

\[ d(Tx,Ty) \leq \alpha d(x,y) \]  \hspace{1cm} (1.2)

for all \( x, y \in X \) and \( 0 \leq \alpha < 1 \). \( \alpha \) is called a contracting constant.

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Definition 6.[4] A mapping \( T : X \to X \) is called a Kannan mapping if

\[
d(Tx, Ty) \leq \alpha \{d(x, Tx) + d(y, Ty)\}
\]

(1.3)

for all \( x, y \in X \) and \( 0 \leq \alpha < \frac{1}{2} \).

The following proposition is a natural generalization of result in metric space (See [3]) to dq-metric space.

Proposition 1. If \( T \) is a contraction on a dq-metric space \( (X, d) \) with contracting constant \( \alpha \), then

\[
d(T^n x, T^n y) \leq \frac{\alpha^n}{1-\alpha^2} \left\{ d(x, Tx) + d(y, Ty) + d(Tx, Ty) \right\}
\]

and \( 0 \leq \frac{\alpha^n}{1-\alpha^2} < \frac{1}{2} \) for some positive integer \( n \). Further, if \( d(x, y) = d(y, x) \) for all \( x, y \in X \), then \( T^n \) is a Kannan mapping on \( X \) for some \( n \in \mathbb{N} \).

Examples 1. It is clear that metric spaces and d-metric spaces are dq-metric spaces.

1. Define \( d \) on \( \mathbb{Q} \times \mathbb{Q} \) by \( d(x, y) = |x - y| \). Then \( d \) is a complete dq-metric but not a metric on \( X \) and the sequence \( \left\{ \frac{1}{n} \right\}_{n \in \mathbb{N}} \) dq-converges to 0.

2. Let \( X = \mathbb{N} \) and \( d(x, y) = x \) for all \( x, y \in \mathbb{N} \). Then \( (X, d) \) is a complete dq-metric space.

2 Main Results

In this section \( A \) will denote the set \( \{ x \in X : d(x, x) = 0 \} \).

Example 1. The set \( A \) in \( (X, d) \) of Example 1(1) is \( \{0\} \) whereas in Example 1(2) it is empty.

Result 2. If \( (x_n) \) is a dq-convergent sequence in \( X \) with a dq-limit \( x \), then \( x \in A \).

Proof. Let \( x \) be a dq-limit of a sequence \( (x_n) \) in \( X \). Let \( \varepsilon > 0 \). Then there exists a positive integer \( N_{\varepsilon} \) such that for all \( n \geq N_{\varepsilon} \), \( d(x_n, x) \leq \varepsilon \). Since \( \varepsilon > 0 \) is arbitrary, \( d(x, x) = 0 \).

Result 3. Let \( A \) be a nonempty subset of complete dq-metric space \( X \), then \( (A, d) \) is complete subspace of \( X \).

Proof. Let \( (x_n) \) be a Cauchy sequence in \( A \). Since \( X \) is complete, there exists \( x \in X \) such that \( (x_n) \) dq-converges to \( x \) in \( X \). By Result 2, \( x \in A \). Therefore \( A \) is complete.

Lemma 1. If, we define \( D \) on \( X \times X \) by \( D(x, y) = \frac{d(x, y) + d(y, x)}{2} \), then

1. \( D \) is a d-metric on \( X \).
2. \( \lim D(x_n, x) = 0 \) if and only if \( \lim x_n = x \) in \( (X, d) \).
3. If \( A \) is a nonempty set, then \( D \) is a metric on \( A \).
4. If \( (X, d) \) is a complete dq-metric space, then \( (X, D) \) is a complete d-metric space.

Proof. The proof of (1), (2) and (3) are clear. Let \( (x_n) \) be a Cauchy sequence in \( (X, D) \) and let \( \varepsilon > 0 \). Then there exists a positive integer \( N_\varepsilon \) such that \( D(x_n, x_m) < \frac{\varepsilon}{2} \) whenever \( n, m \geq N_\varepsilon \). Thus for \( n, m \geq N_\varepsilon \), \( \min \{d(x_n, x_m), d(x_n, x_m)\} \leq D(x_n, x_m) + d(x_n, x_m) < \varepsilon \) and hence \( (x_n) \) is a Cauchy sequence in \( (X, D) \). Since \( (X, D) \) is complete, there exists \( x \) in \( X \) such that \( (x_n) \) dq-converges to \( x \). By (2), \( \lim D(x_n, x) = 0 \) and hence \( (X, D) \) is a complete d-metric space.

Lemma 2. Let \( f \) be a contraction on \( X \) with a contracting constant \( \lambda \). Then for each \( x \in X \) the sequence \( (f^n x) \) is a Cauchy sequence in \( X \).

Proof. Let \( x \in X \) and \( x_0 = f^n x \). Then

\[
d(x_n, x_{n+1}) = d(f^n x, f^{n+1} x) \leq \lambda d(f^{n-1} x, f^n x) \leq \ldots \leq \lambda^n d(x, f x).
\]

(2.1)

Therefore \( (f^n x) \) is a Cauchy sequence in \( X \). Let \( (f^n x) \) be convergent to \( y \) and \( \lambda = \lim d(x, f x) \). Then \( (f^n x) \) is a Cauchy sequence in \( X \).

Theorem 4. Let \( f \) be a contraction on a complete dq-metric space \( X \) with contracting constant \( \lambda \). Then \( f \) has a unique fixed point in \( X \).

Proof. Let \( x \in X \) and \( x_0 = f^n x \). By Lemma 2, the sequence \( (f^n x) \) is a Cauchy sequence in \( X \). Since \( X \) is complete, there exists \( u \) in \( X \) such that \( \lim f^n x = u \). By Result 2, \( u \in A \) and hence \( A \) is nonempty. Define \( D(x, y) = \frac{d(x, y) + d(y, x)}{2} \) for all \( x, y \in X \), then by Lemma 1 and Result 3, \( (A, D) \) is a complete metric space. Let \( x \in A \). Then \( d(f(x), f(x)) \leq \lambda d(x, x) = 0 \) and hence \( f(A) \subseteq A \). Also,

\[
D(fx, fy) = \frac{d(fx, fy) + d(fy, fx)}{2} \leq \lambda \frac{d(fx, fy) + d(fy, fx)}{2} = \lambda D(x, y).
\]

(2.2)

Therefore \( f \) is a contraction on \( (A, D) \). Hence by Banach contraction principle for metric spaces, \( f \) has a unique fixed point \( x \) in \( A \).

If \( y \) is a fixed point of \( f \) in \( X \), then

\[
d(x, y) = d(fx, fy) \leq \lambda d(x, y).
\]

Since \( 0 \leq \lambda < 1 \), \( d(x, y) = 0 \). By symmetry, \( d(y, x) = 0 \). Therefore \( x = y \) and \( f \) has a unique fixed point in \( X \).

In [4] C.T. Aage and J. N. Salunke, proved that every continuous Kannan mapping \( T \) on a complete dq-metric space has a unique fixed point. The following result shows that the assumption of continuity can be dropped to obtain the theorem under a less restrictive contractive condition.

Theorem 5. If \( T \) is mapping on a complete dq-metric space \( (X, d) \) into itself and if there is a constant \( \alpha \) such that \( 0 \leq \alpha < \frac{1}{2} \) and

\[
d(Tx, Ty) \leq \alpha \{d(x, Tx) + d(y, Ty)\},
\]

(2.3)

for all \( x, y \in X \), then \( T \) has a unique fixed point in \( X \).
Proof: Let \( x \in X \) and write \( x_n = T^n x \). Then
\[
d(x_1, x_2) = d(Tx, Tx) \leq \alpha \{d(x, Tx) + d(x_1, Tx_1)\}
\]
\[
(1 - \alpha)d(x_1, x_2) \leq \alpha d(x, x_1)
\]
\[
d(x_1, x_2) \leq \frac{\alpha}{1 - \alpha}d(x, x_1)
\]

An inductive argument yields the inequality
\[
d(x_n, x_{n+1}) \leq \lambda^n d(x, x_1)
\]
(2.4)
where \( \lambda = \frac{\alpha}{1 - \alpha} \).

Now
\[
d(x_n, x_{n+1}) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \ldots + d(x_{n+k-1}, x_{n+k})
\]
\[
\leq (\lambda^n + \lambda^{n+1} + \ldots + \lambda^{n+k-1})d(x, x_1)
\]
\[
< \frac{\lambda^n}{1 - \lambda}d(x, x_1).
\]
(2.5)

Since \( 0 \leq \lambda < 1 \), \( (x_n) \) is a Cauchy sequence and since \( X \) is complete there exists \( u \in X \) such that
\[
\lim_{n \to \infty} d(x_n, u) = \lim_{n \to \infty} d(x_n, x_{n+1}) = 0.
\]

Now we show that \( u \) is a fixed point of \( T \).

\[
d(u, Tu) \leq d(u, x_n) + d(x_n, Tu)
\]
\[
= d(u, x_n) + d(Tx_n, Tu)
\]
\[
\leq d(u, x_n) + \alpha \{d(x_{n-1}, x_n) + d(u, Tu)\}
\]
\[
\leq d(u, x_n) + \alpha d(u, Tu) + \alpha \lambda^{n-1}d(x, x_1)
\]
(2.6)
\[
\Rightarrow 0 \leq d(u, Tu) \leq \frac{1}{1 - \alpha}d(u, x_n) + \lambda^n d(x, x_1).
\]

Letting \( n \to \infty \), we get \( d(u, Tu) = 0 \). Now

\[
d(Tu, u) \leq d(Tu, x_n) + d(x_n, u)
\]
\[
= d(Tu, Tx_{n-1}) + d(x_n, u)
\]
\[
\leq \alpha \{d(u, Tu) + d(x_{n-1}, x_n)\} + d(x_n, u)
\]
(2.7)

\[
d(Tu, u) \leq \alpha d(x_{n-1}, x_n) + d(x_n, u) \quad \text{since} \quad d(u, Tu) = 0.
\]

Letting \( n \to \infty \), we get \( d(Tu, u) = 0 \). Therefore, \( Tu = u \).

**Uniqueness:** If \( a \) is a fixed point of \( T \), then \( d(a, a) = d(Ta, Ta) \leq \alpha \{d(a, Ta) + d(Ta, Ta)\} = \alpha \{d(a, a) + d(a, a)\} \) which implies that \( (1 - 2\alpha)d(a, a) \leq 0 \). Since \( 0 \leq \alpha < \frac{1}{2} \), \( d(a, a) = 0 \). If \( a, b \) are fixed points of \( T \), then \( d(a, b) = d(Ta, Tb) \leq \alpha \{d(a, Ta) + d(b, Tb)\} = \alpha \{d(a, a) + d(b, b)\} = 0 \). Therefore \( d(a, b) = 0 \). By symmetry, \( d(b, a) = 0 \). Hence \( a = b \).

References

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