Semilinear hyperbolic boundary value problem for linear elasticity equations

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Abstract: Consider a semilinear hyperbolic boundary value problem associated to the linear elastic equations. Then, existence of a weak solution is established through compactness method. The uniqueness and the regularity of the solution have been gotten by eliminating some hypotheses that have been imposed by other authors for different particular problems.

Keywords: Compactness, Existence, Gronwall’s inequality, linear elasticity, Uniqueness of solution, Regularity, Semilinear hyperbolic equation, Variational problem.

1. Introduction

In [8], Lions considered a semilinear boundary value problem associated to the Laplace operator with Neumann boundary conditions:

\[
\begin{align*}
\frac{\partial^2 u}{\partial t^2} - \Delta u + |u|^\nu u &= f \quad \text{in } \Omega \times (0,T), \\
\sigma (u) &= F(\varepsilon (u)) \quad \text{in } \Omega \times (0,T), \\
u(x,0) &= u_0(x), u'(x,0) = u_1(x), \quad x \in \Omega.
\end{align*}
\]

(1)

Using the compactness method and Faedo Galerkin techniques, the existence of a weak solution has been proved. Assuming that the condition \( \nu \leq \frac{2}{n-2} \) holds, then it follows the uniqueness and the regularity of the solution.

In this work, we consider a semilinear hyperbolic boundary value problem governed by partial differential equations that describe the evolution of linear elastic materials with Dirichlet and Neumann boundary conditions as follows:

\[
\begin{align*}
\frac{\partial^2 u}{\partial t^2} - \text{div}\sigma (u) + |u|^\nu u &= f \quad \text{in } \Omega \times (0,T), \quad \sigma (u) = F(\varepsilon (u)) \quad \text{in } \Omega \times (0,T), \\
u(x,0) &= u_0(x), u'(x,0) = u_1(x), \quad x \in \Omega,
\end{align*}
\]

(2)

where \( F \) is a linear function. Assume certain hypotheses on the data functions. Then, by using Faedo Galerkin techniques and compactness method, we will prove the existence of the solution. Our main goal is, without taking into account the condition on \( \nu \), to prove the uniqueness and the regularity of the solution.

2. Problem statement

Let \( \Omega \) be an open and bounded domain in \( \mathbb{R}^n \), recall that the boundary \( \Gamma \) of \( \Omega \) is assumed to be regular and is composed of two relatively closed parts : \( \Gamma_1, \Gamma_2 \), with mutually disjoint relatively open interiors. We assume that \( \text{meas} (\Gamma_i) > 0 \). We pose \( \Sigma_i = \Gamma_i \times (0,T) , i=1,2 \), where \( T \) is a finite real. To simplify the writing one will put \( u' = \frac{\partial u}{\partial t}, u'' = \frac{\partial^2 u}{\partial t^2} \).

\( \sigma \) is a stress field. To simplify the notations, we do not indicate explicitly the dependence of the functions \( u \) and \( \sigma \) with respect to \( x \in \Omega \) and \( t \in (0,T) \). Let \( \eta \) be the unit outward normal vector on \( \Gamma \). Here and throughout this work, the summation convention over repeated indices is used. The classical formulation of the problem is as follows. Find a displacement field \( \mathbf{u} : \Omega \times (0,T) \rightarrow \mathbb{R}^n \), a stress field \( \sigma : \Omega \times (0,T) \rightarrow S_n \), such that

\[
\begin{align*}
u'' - \text{div}\sigma (u) + |u|^\nu u &= f \quad \text{in } \Omega , \nu \in \mathbb{N},
\end{align*}
\]

(3)

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\( \sigma(u) = F(x, \varepsilon(u)) \) in \( Q \), \( u = g \) on \( \Sigma_1 \), \( \sigma(u)n = 0 \) on \( \Sigma_2 \),

\[
\begin{aligned}
    &\begin{cases}
      u(x, 0) = u_0(x) \text{in} \Omega,
      
      u'(x, 0) = u_1(x) \text{in} \Omega.
    \end{cases}
\end{aligned}
\]

Where \( S_n \) will denote the space of second-order symmetric tensors on \( \mathbb{R}^n \). \( u \), \( f \) and \( \sigma(u) \) represent the displacement field, the density of volume forces and the tensor of constraints, respectively. \( \text{div} \) denotes the divergence operator of the tensor valued functions and \( \sigma = (\sigma_{ij}) \), stands for the stress tensor field. The latter is obtained from the displacement field \( u \) by the constitutive law of linear elasticity defined by (4). \( F \) is a linear elastic constitutive law, and \( \varepsilon(u) = \frac{1}{2}(\nabla u + \nabla^T u) \) is the linearized strain tensor. The equation (3), without the non linear term \( |u|^2 \) \( u \), describes the evolution of linear elastic materials, while (5) and (6) are the mixed boundary conditions on \( \Sigma_i, i = 1,2 \) and initial conditions, respectively.

Now we define the space:

\[
\mathcal{H} = L^2(\Omega)^n = \{ \sigma = (\sigma_{ij}) \in S_n : \sigma_{ij} = \sigma_{ji} \in L^2(\Omega) \},
\]

which is a 
Hilbert space endowed with the inner product

\[
(\sigma, \tau) = \int_{\Omega} \sigma_{ij}\tau_{ij}dx,
\]

and the associated norm is denoted \( \| \cdot \|_{\mathcal{H}} \). When no ambiguity is to fear, we will put:

\[
\| v \|_{L^2(\Omega)} = \| v \|_{\mathcal{H}} = \left( \int_{\Omega} v^2dx \right)^{\frac{1}{2}},
\]

and we will use the notation \( \| v \|_{L^2(\Omega)} \) in possible ambiguity case.

In the study of mechanical problem involving elastic materials, we assume that the operator \( F : \Omega \times S_n \rightarrow S_n \) satisfies the following conditions:

\[
\begin{cases}
    &\begin{cases}
      (a) \exists m > 0; (F(x, \varepsilon), \varepsilon) \geq m \| \varepsilon \|^2, \\
      \forall \varepsilon \in S_n, a.e. x \in \Omega,
    
    (b) (F(x, \varepsilon), \tau) = (F(x, \tau), \varepsilon), \\
      \forall \varepsilon, \tau \in S_n, a.e. x \in \Omega,
    
    (c) F \text{ or any } \varepsilon \in S_n, x \rightarrow F(x, \varepsilon)
    
      \text{is measurable on } \Omega.
    \end{cases}
  \end{cases}
\]

And we assume that the given data \( f, u_0 \) and \( u_1 \) and \( g \) verify

\[
\begin{aligned}
    &f \in L^2(Q),
    
    &u_0 \in V \cap L^p(\Omega), p = \nu + 2, 
    
    &u_1 \in L^2(\Omega),
    
    &g \in H^\frac{3}{2}(\Sigma_1).
\end{aligned}
\]

Referring to [4], it is easy to verify the following result.

**Lemma 1.** Assume that hypotheses (10) holds. Then the function, still denoted by \( F : \mathcal{H} \rightarrow \mathcal{H} \), defined by

\[
F(\varepsilon(\cdot)) = F(\cdot, \varepsilon(\cdot)), \text{ a.e.on } \Omega,
\]

is continuous on \( \mathcal{H} \).

**Lemma 2.** Assume that (10)-(14) hold. Then (3)-(6) is equivalent to the following variational problem:

\[
\begin{aligned}
    &\text{Find } u \in V \cap L^p(\Omega) \text{such that }
    
    &\int_{\Gamma_1}(\sigma(u(t))\eta g(t))d\Gamma_1
    
    + \int_{\Omega}(|u|^p u, v) + \int_{\Omega}\nabla u \cdot \nabla v
    
    + \int_{\Gamma_1}a(u, v)
    
    \Omega, p = \nu + 2,
    
    &w \in V \setminus \{v \in H^1(\Omega), v = g \text{ on } \Sigma_1 \}
    
    \text{and } a(u, v) = \int_{\Omega} \sigma(u(\eta))\varepsilon(v)x dx.
\end{aligned}
\]

where \( V = \{ v \in H^1(\Omega), v = g \text{ on } \Sigma_1 \} \) and \( a(u, v) = \int_{\Omega} \sigma(u(\eta))\varepsilon(v)x dx \).

3. Existence and Uniqueness

Our main existence and uniqueness result concerning problem (3)-(6), which we establish in this section, is the following.

3.1. Existence

**Theorem 1.** Assume that (10)-(14) hold. Then there exists at least one solution to problem (3)-(6) and it satisfies

\[
u \in L^\infty(0, T; V \cap L^p(\Omega)), \quad p = \nu + 2,
\]

\[
u' \in L^\infty(0, T; L^2(\Omega)).
\]

**Remark.** Where as the problem is defined on the open interval \( (0, T) \), the relations (6) don’t have a sense, for that reason we must justify the definition of \( u(t) \) and \( u'(t) \) at point 0 in the initial conditions (6).

Using the result of the Theorem 1, we are going to demonstrate the following Lemma.

**Lemma 3.** Assume that (10)-(14) hold. Then the initial conditions in (6) have a sense.
Proof.
Using hypotheses (10)-(14), according to the Theorem 1, we have
\[ u \in L^\infty (0, T; V) \text{ and } u' \in L^\infty (0, T; L^2(\Omega)). \] (18)

Referring to [8], it results
\[ u : [0, T] \rightarrow L^2(\Omega), \] (19)
is continuous, possibly after a modification on a subset of [0, T] with zero measure, then \( u(0) \) is well defined, therefore the first condition in (6) has a sense.

On the other hand, (16) implies that \( \varepsilon(u) \in L^\infty (0, T; L^2(\Omega)) \). Thus, since \( F \) is continuous and \( L^2(\Omega) \subset V' \), we have
\[ F(\varepsilon(u)) \in L^\infty (0, T; L^2(\Omega)). \] (20)

Also, we have
\[
\int_{\Omega} |(|u|^p u')^{\frac{p'}{p}} \, dx \leq \int_{\Omega} |u|^{(p+1)p'} \, dx = \int_{\Omega} |u|^{p+1} \, dx = \int_{\Omega} |u|^p \, dx
\]
\[
= \|u\|^p_{L^p(\Omega), \frac{1}{p} + \frac{1}{p'} = 1},
\] (21)

which implies that
\[ |u|^p u \in L^\infty (0, T; L^p(\Omega)) ; \forall u \in L^p(\Omega). \] (22)

Then, from (3) we have
\[ u'' = f + \text{div} (u) - |u| u \in L^1 (0, T; V') + L^\infty (0, T; V' + L^p(\Omega)), \] (23)

where \( V' \) denotes the dual of \( V \) and
\[ V' + L^p(\Omega) = \left\{ u + v; \ u \in V' \text{ and } v \in L^p(\Omega) \right\}. \] (24)

Since \( L^2(\Omega) \subset V' + L^p(\Omega) \), in particular case we have
\[ u'' \in L^2 (0, T; V' + L^p(\Omega)). \] (25)

Then, referring to [8] and using (17) we conclude that
\[ u' : [0, T] \rightarrow V' + L^p(\Omega) \] (26)
is continuous, possibly after a modification on a subset of [0, T] with zero measure, then \( u' (0) \) is well defined, therefore the second condition in (6) has a sense.

We turn now to prove Theorem 1.

Proof of Theorem 1. 
It consists of four steps:

Step 1 : Approximated solution.

It introduces a sequence \( (u_n) \) of functions having the following properties:
* \( \forall j = 1, \ldots, m; \ w_j \in V \cap L^p(\Omega); \)
* The family \( \{w_1, w_2, \ldots, w_m\} \) is linearly independent;
* The \( V_m = \text{span} \{w_1, w_2, \ldots, w_m\} \) generated by \( \{w_1, w_2, \ldots, w_m\} \) is dense in \( V \cap L^p(\Omega). \)

Let \( u_m = u_m(t) \) be an approached solution such that
\[ u_m(t) = \sum_{i=1}^{m} K_{jm}(t) w_i. \] (27)

The \( K_{jm} \) being to be determined by the following expression:
\[
\begin{cases}
(u_m(t), w_j) + a(u_m(t), w_j) - \int_{t_1}^{t} \sigma(u_m(t)) \eta g(t) \, dt = 1,
\end{cases}
\] (28)
which is a nonlinear system of ordinary differential equations and will be completed by the following initial conditions
\[ u_m(0) = u_{0m} = \sum_{i=1}^{m} \alpha_i w_i \rightarrow u_0 \text{ in } V \cap L^p(\Omega), \] (29)
\[ u_m'(0) = u_{1m} = \sum_{i=1}^{m} \beta_i w_i \rightarrow u_1 \text{ in } L^2(\Omega). \] (30)

As the family \( \{w_1, w_2, \ldots, w_m\} \) is linearly independent, the system (28), (29) and (30) admits at least one solution \( u_m \in (0, T) \) having the following regularity
\[ u_m(t) \in L^2 (0, t_m; V_m), \ u_m'(t) \in L^2 (0, t_m; V_m). \] (31)

A-priori, the time interval \( (0, T) \) depends on \( m \) and thereafter we shall demonstrate that \( t_m \) does not depend on \( m \) based on the following a-priori estimates.

Step 2 : A priori estimates. Let
\[ \|u\|_{L^p}^2 = a(u, u) = \int_{\Omega} F(\varepsilon(u)) \varepsilon(u) \, dx. \] (32)

Then, using (10), it can be shown that \( \|u\|_1 \) is a norm on \( V \) equivalent to the norm \( \|u\|_1 \) on \( H^1(\Omega) \).

Multiplying the equation (28) by \( K'_{jm}(t) \) and performing the summation over \( j = 1 \) to \( m \), yields
\[
\int_{t_1}^{t} \left( (u_m(t), u_m(t)) + a(u_m(t), u_m(t)) - \int_{t_1}^{t} \sigma(u_m(t)) \eta g(t) \, dt \right) K'_{jm}(t) \, dt 
\]

\[ + \left( \int_{t_1}^{t} \|u_m\|_1^2 u_m(t), u_m(t) \right) = \int_{t_1}^{t} \left( f(t), u_m(t) \right). \] (33)

Since \( u_m \in L^2 (0, t_m; V_m), \ u_m' \in L^2 (0, t_m; V_m), \) then
\[ \varepsilon(u_m), \varepsilon(u_m') \in L^2 (0, T; L^2(\Omega)). \] (34)
On the other hand, we have
\[ \frac{d}{dt} (u_m(t), u_m(t)) = (F (c(u_m(t))), c(u_m(t))) + (F (c(u_m(t))), c(u_m(t))) = a(u_m(t), u'_m(t)) + a(u_m(t), u_m(t)). \] (35)

Then, using (10.b), we obtain
\[ 2a(u_m(t), u'_m(t)) = \frac{d}{dt} a(u_m(t), u_m(t)) = \frac{d}{dt} \|u_m(t)\|^2. \] (36)

Also, we have
\[ \frac{1}{p} \frac{d}{dt} \|u_m(t)\|^p = (u_m(t), u'_m(t)). \]
\[ \frac{1}{p} \frac{d}{dt} \|u_m(x,t)\|^p_{L^p(\Omega)} = (|u_m|^p, u'_m(t)), p = \nu + 2. \]

As the function of the injection \( u \rightarrow \sigma(u(t)) \) of \( H^1(\Omega) \) in \( H^\frac{\nu}{2} (\Gamma_2) \) is continuous and \( g'(t) \in H^\frac{\nu}{2} (\Gamma_1) \), then
\[ \int_{\Gamma_1} \sigma(u_m(t))g'(t) \, d\Gamma_1 \leq C \|u_m(t)\| \leq C_2 (1 + \|u_m(t)\|^2). \] (38)

Then from (33), by Cauchy-Schwarz’s inequality we may conclude that
\[ \frac{1}{2} \frac{d}{dt} \left[ \|u_m(t)\|^2 + C_1 \|u_m(t)\|^2 \right] + \frac{1}{p} \frac{d}{dt} \|u_m(x,t)\|^p_{L^p(\Omega)} \leq \|f(s)\| |u_m(s)| + \int_{\Gamma_1} |\sigma(u_m(t))g'(t)| \, d\Gamma_1. \] (39)

Therefore, from (38) we obtain that
\[ \frac{1}{2} \frac{d}{dt} \left[ \|u_m(t)\|^2 + C_1 \|u_m(t)\|^2 \right] + \frac{1}{p} \frac{d}{dt} \|u_m(x,t)\|^p_{L^p(\Omega)} \leq \|f(s)\| |u_m(s)| + \frac{1}{2} C_2 \|u_m(t)\|^2 + \frac{1}{2} C_2. \] (40)

Now, integrating inequality (40) over \((0,T)\), we deduce that
\[ \frac{1}{2} \left( \|u_m(t)\|^2 + C_1 \|u_m(t)\|^2 \right) + \frac{1}{p} \|u_m(t)\|^p_{L^p(\Omega)} \leq \frac{1}{2} |u_{m1}|^2 + \frac{1}{2} C_1 \|u_{m0}\|^2 + \frac{1}{2} \|f(s)\| |u_m(s)| \, ds + \frac{1}{2} C_2 \int_0^t \|u_m(s)\|^2 \, ds + \frac{1}{2} C_2. \] (41)

Then, from (41) by Young’s inequality, we have that
\[ \frac{1}{2} \left( \|u_m(t)\|^2 + C_1 \|u_m(t)\|^2 \right) + \frac{1}{p} \|u_m(t)\|^p_{L^p(\Omega)} \leq \frac{1}{2} |u_{m1}|^2 + \frac{1}{2} C_1 \|u_{m0}\|^2 + \frac{1}{2} \|f(s)\| |u_m(s)| \, ds + \frac{1}{2} \|u_m(s)\|^2 \, ds + \frac{1}{2} C_2 \] (42)

Since by assumptions, there exists a constant \( C_3 > 0 \) such that
\[ \frac{1}{2} |u_{m1}| + \frac{1}{2} C_1 \|u_{m0}\|^2 + \frac{1}{2} \|f(s)\| |u_m(s)| \, ds + \frac{1}{2} \|u_m(s)\|^2 \leq C_3, \forall m \in \mathbb{N}. \] (43)

It then follows from (42) that
\[ \left\{ \begin{array}{l} \frac{1}{2} \left( \|u'_m(t)\|^2 + C_1 \|u_m(t)\|^2 \right) + \frac{1}{p} \|u_m(t)\|^p_{L^p(\Omega)} \\
\leq C_3 + \frac{1}{2} \int_0^t \left( \|u'_m(s)\|^2 + C_2 \|u_m(s)\|^2 \right) \, ds. \end{array} \right. \] (44)

Hence
\[ \|u'_m(t)\|^2 + C_1 \|u_m(t)\|^2 \leq 2 C_3 + \frac{1}{2} \int_0^t \left( \|u'_m(s)\|^2 + C_2 \|u_m(s)\|^2 \right) \, ds. \forall t \in (0,T). \] (45)

Then, by Gronwall’s inequality, we have that
\[ \|u'_m(t)\| + \|u_m(t)\| \leq C \text{ (independent of } m). \] (46)

Then, using (44), we arrive at
\[ \|u_m(t)\|_{L^p(\Omega)} \leq C \text{ (independent of } m). \] (47)

From where, we deduce that \( t_m \) is independent of \( m \).
By passing to the limit where \( m \rightarrow \infty \), from (46) and (47) we conclude that
\[ \left\{ \begin{array}{l} (u_m) \text{ is bounded in } L^\infty (0,T; V \cap L^p(\Omega)) \\\n(u'_m) \text{ is bounded in } L^\infty (0,T; L^2(\Omega)). \end{array} \right. \] (48)

Step 3: Passage to the limit.
It follows from (48) that there exists a subsequence \( (u_{m_j}) \) of \( (u_m) \) such that
\[ u_{m_j} \rightarrow u \text{ in } L^\infty (0,T; V \cap L^p(\Omega)) \text{ weak star}, \] (49)
\[ u'_{m_j} \rightarrow u' \text{ in } L^\infty (0,T; L^2(\Omega)) \text{ weak star}. \] (50)

From (48), it is obtained that sequences \( (u_{m_j}) \), \( (u'_{m_j}) \) are bounded in \( L^2(0,T; V) \subset L^2(0,T; L^2(\Omega)) = L^2(Q), L^2(Q), \) respectively.

Then, in particular, \( (u_m) \) is a bounded sequence in \( H^1(Q) \).
It is known, see [8], that the injection of \( H^1(Q) \) in \( L^2(Q) \) is compact. Then, from (49) and (50) we have
\[ u_{m_j} \rightarrow u \text{ strongly in } L^2(Q). \] (51)

Setting \( \frac{1}{p} + \frac{1}{p'} = 1, p = \nu + 2, \) using (48) we have \( |u_m|^{p'} u_m \) is a bounded sequence in \( L^\infty (0,T; L^{p'}(\Omega)) \).
Therefore, we have
\[ |u_m|^{p'} u_m \rightarrow |u|^{p'} u \text{ in } L^\infty (0,T; L^{p'}(\Omega)) \text{ weak star}. \] (52)

Let \( j \) be fixed and \( \mu > j \). Then, by (28) we have
\[ a(u_{m_j}(t), w_j) + a(u_{m_j}(t), w_j) - \int_{\Gamma_1} \sigma(u_{m_j}(t))g(t) \, d\Gamma_1 + \int_{\Gamma_1} (|u_{m_j}|^{p'} u_{m_j}(t), w_j) = \langle f(t), w_j \rangle. \] (53)

Therefore, (49), (50) imply
\[ a(u_{m_j}, w_j) \rightarrow a(u, w_j) \text{ in } L^\infty (0,T) \text{ weak star}, \] (54)
\[ (u'_{m_j}, w_j) \rightarrow (u', w_j) \text{ in } L^\infty (0,T) \text{ weak star}, \] (55)
\[ \int_{\Gamma_1} \sigma(u(t))g(t) \, d\Gamma_1 \rightarrow \int_{\Gamma_1} \sigma(u(t))g(t) \, d\Gamma_1 \text{ in } L^\infty (0,T) \text{ weak star}. \] (56)
Hence
\[ (u''_{\mu}(t), w_j) \rightarrow (u''(t), w_j) \text{ in } \mathcal{D}'(0, T). \] (57)

Also, using (52) we have
\[ ([u_{\mu}]', u_{\mu}, w_j) \rightarrow ([u']', u, w_j) \text{ in } L^\infty(0, T) \text{ weak star}. \] (58)

Then (53) takes the form
\[ (u''(t), w_j) + a(u(t), w_j) - \int_0^t \sigma(u(t)) g_y(t) \, dt + ([u']' u(t), w_j) = (f(t), w_j). \] (59)

Finally, by using the density of \( V_m \) in \( V \cap L^p(\Omega) \) we obtain
\[ (u''(t), v) + a(u(t), v) - \int_0^t \sigma(u(t)) g_y(t) \, dt + ([u']' u(t), v) = (f(t), v), \quad \forall v \in V \cap L^p(\Omega). \] (60)

Then \( u \) satisfies (3).

Step 4 : Initial condition verifications.
It follows from (49) and (50) that
\[ u_{\mu}(0) \rightarrow u(0) \text{ weakly in } L^2(\Omega). \] (61)

Then, using (29) we deduce in particular that
\[ u_{\mu}(0) = u_0 \rightarrow u_0 \text{ in } V \cap L^p(\Omega). \] (62)

Thus, the first condition in (6) is obtained. On the other hand, by using (57) we have
\[ (u''_{\mu}(t), w_j) \rightarrow (u''(t), w_j) \text{ in } L^\infty(0, T) \text{ weak star}. \] (63)

Hence
\[ (u''_{\mu}(0), w_j) \rightarrow (u'(0), w_j). \] (64)

Since \( (u''_{\mu}(0), w_j) \rightarrow (u_1, w_j) \), we have \( (u'(0), w_j) = (u_1, w_j), \forall j \). Then the second condition in (6) is satisfied.

3.2. Uniqueness

Many authors, for some particular problems have showed the uniqueness of the solution basing on the condition \( \nu \leq \frac{2}{n-2} \). In this subsection the uniqueness of the solution will prove, by eliminating this condition.

Theorem 2. Assume the conditions of Theorem 1 and also
\[ \nu \leq \frac{2k}{n-2}, \quad k \in \mathbb{N}^+, \quad n \neq 2 (\nu \text{ any finished so } n = 2). \] (65)

Then, there exists a unique solution \( u \) to problem (3)-(6) and it satisfies (16), (17).

Proof.
Let \( u, \nu \) be two solutions of problem (3)-(6), to the sense of the Theorem 1.

Setting \( w = u - \nu \), since \( F \) is linear we have
\[ w'' - div F(\nu w) + (|u|'' u - |\nu|'' \nu) = 0 \text{ in } Q, \] (66)
\[ w(0) = w'(0) = 0 \text{ in } \Omega, \] (67)
\[ w = 0 \text{ on } \Sigma_1, \quad \sigma(w) \eta = 0 \text{ on } \Sigma_2, \] (68)
\[ w \in L^\infty(0, T; V \cap L^p(\Omega)), \quad p = \nu + 2, \] (69)
\[ w' \in L^\infty(0, T; L^p(\Omega)), \quad p = \nu + 2. \] (70)

Multiplying the equation (66) by \( \nu \) and integrating over \( \Omega \). Then, by using Green’s formula together with the conditions (67), (68), we obtain
\[ \frac{1}{2} \frac{d}{dt} \left( |w''(t)|^2 \right) + a(w(t), w'(t)) = \int_{\Omega} (|\nu|'' u - |u|'' \nu) \, u \, w' \, dx. \] (71)

Then by (10, b) we have
\[ a(w(t), w'(t)) = \frac{d}{dt} a(w(t), w(t)) - \int_0^t \frac{d}{dt} (F(\nu w)) \, v \, dx \geq C_1 \frac{d}{dt} ||w||^2 - \int_0^t (F(\nu w')) \, v \, dx = C_1 \frac{d}{dt} ||w||^2 - a(w(t), w(t)). \] (72)

In this case (71) takes the form
\[ \frac{1}{2} \frac{d}{dt} \left( |w''(t)|^2 + C_1 ||w||^2 \right) \leq \int_{\Omega} (|\nu|'' u - |u|'' \nu) \, u \, w' \, dx. \] (73)

Also, we have
\[ \int_{\Omega} (|\nu|'' u - |u|'' \nu) \, w' \, dx \leq (\nu + 1) \int_{\Omega} \sup (|\nu|'' \nu, |\nu|'' \nu) \, |v| \, w \, dx. \] (74)

Next, by using Holder’s inequality we have
\[ \int_{\Omega} (|\nu|'' u - |u|'' \nu) \, w' \, dx \leq C_2 \left( ||\nu||_{L^\infty(\Omega)} + ||\nu||_{L^\infty(\Omega)} \right) \, ||w(t)||_{L^p(\Omega)} \, ||w'\Omega||. \] (75)

where \( \frac{1}{p} + \frac{1}{q} + \frac{1}{2} = 1 \).

Also, by referring to [6] we have
\[ ||v||_{L^{\infty}(\Omega)} = ||v||_{H^1(\Omega)}^{\frac{1}{k}} \quad \forall k, q \in \mathbb{N}^*. \] (76)

Therefore by (76) we have
\[ ||v||_{L^\infty(\Omega)} = ||v||_{L^\infty(\Omega)} \quad \forall \nu \in \mathbb{N}. \] Using (65) we have
\[ \nu \leq \frac{2k}{n-2}, \quad k \in \mathbb{N}^+, \quad n \neq 2 (\nu \text{ any finished so } n = 2). \] (65)

Therefore, we have
\[ \frac{1}{n} \leq \frac{k}{n-2}, \quad k \in \mathbb{N}^+, \quad n \neq 2 (\nu \text{ any finished so } n = 2). \] (65)

Since \( \frac{1}{n} + \frac{1}{q} + \frac{1}{2} = 1 \), by referring to [7] we have \( H^1(\Omega) \subset L^2(\Omega) \), from where
\[ \left| \left| v \right| \right|_{L^2(\Omega)} \leq \left| \left| v \right| \right|_{L^\infty(\Omega)} \leq C \left| \left| v \right| \right|_{H^1(\Omega)} \] (77)

which implies that
\[ \int_{\Omega} (|\nu|'' u - |u|'' \nu) \, w' \, dx \leq C_3 \left( ||u||_{L^\infty} + ||v||_{L^\infty} \right) \, ||w|| \, ||w'||. \] (78)
Since \( u, v \in L^\infty(0, T; V \cap L^p(\Omega)) \) we obtain
\[
\int_\Omega | |w|^\nu - |u|^\nu| |w'| \, dx \leq C_4 ||w|| | |w'| .
\] (79)
Then, by Young's inequality from (73) we conclude that
\[
\frac{1}{2} \frac{d}{dt} \left( | |w'(t)|^2 + C_1 ||w(t)||^2 \right) \leq C_4 ||w|| | |w'| \leq \frac{1}{2} C_4 \left( | |w'(t)|^2 + ||w(t)||^2 \right).
\] (80)
Integrating equation above together with the initial conditions (67), we obtain
\[
\left( | |w'(t)|^2 + C_1 ||w(t)||^2 \right) \leq C_4 \int_0^t \left( | |w'(s)|^2 + ||w(s)||^2 \right) \, ds.
\] (81)
Finally, use Gronwall's inequality to find \( w = 0 \). \( \square \)

**Lemma 4.** Assume the conditions of Theorem 1. Then, for all \( \nu \in \mathbb{N} \) the solution \( u \) found to the Theorem 1. is unique.

**Proof.**
For all \( n > 2 \), setting
\[
k = E \left( \frac{\nu(n-2)}{2} \right) + 1,
\] (82)
where \( E(x) \) denotes the integer part of \( x \).
Then, we have
\[
\nu \leq \frac{2k}{n-2}, \quad k \in \mathbb{N}^*, \quad n \neq 2(\text{varyfinedashedson }= 2).
\] (83)
Thus, using Theorem 2, there exists a unique solution satisfies (16), (17).

4. **Regularity of the solution**

**Theorem 3.** Under the conditions stated in Theorem 1, and the additional assumptions
\[
f' \in L^2(Q),
\] (84)
\[
u_0 \in V \cap H^2(\Omega),
\] (85)
\[
u_1 \in V,
\] (86)
\[
\nu \leq \frac{2k}{n-2}, \quad k \in \mathbb{N}^* \quad n \neq 2(\nu \text{ any finished so } n = 2).
\] (87)
Then, there exists a unique solution \( u \) to problem (3)-(6) and it satisfies the following regularities:
\[
u \in L^\infty(0, T; V \cap H^2(\Omega)),
\] (88)
\[
u' \in L^\infty(0, T; V),
\] (89)
\[
u'' \in L^\infty(0, T; L^2(\Omega)).
\] (90)

**Proof.**
Consider the sequence of functions \( w_n \) such that
* \( \forall j = 1, \ldots, m; w_j \in V \cap H^2(\Omega); \)
* The family \( \{w_1, w_2, \ldots, w_m\} \) is linearly independent;
* The \( V_m = \text{span } \{w_1, w_2, \ldots, w_m\} \) generated by \( \{w_1, w_2, \ldots, w_m\} \) is dense in \( V \cap H^2(\Omega). \)
Let \( u_m = u_m(t) \) be an approached solution satisfies (27) and (28).
Also, we assume that the initial data satisfy
\[
u_0 \to u_0 \text{ in } V \cap H^2(\Omega),
\] (91)
\[
u_1 \to u_1 \text{ in } V.
\] (92)
Then, it follows from (28) that
\[
u_m(0), w_j) = (f(0) + \text{div } F(v(u_m))) - |u_m|^{\nu}u_m, w_j), \quad 1 \leq j \leq m.
\] (93)
Since \( F \) is continuous, then we conclude from (91) that
\[
|\text{div } F(v(u_m))| \leq C.
\] (94)
By Hölder's inequality,
\[
\int_0^\infty \left( |u_m|^{\nu}u_m \right) \, dx \leq \left( \int |u_m|^{\nu} \right)^\frac{2\nu}{\nu-1} \left( \int \right)^{\nu-1} \leq C \left( \int |u_m|^{2\nu} \right) \, dx.
\] (95)
From (76) it follows
\[
\left( \int |u_m|^{\nu} \right)^\frac{2k}{\nu} \leq q, \text{ then } \left( \int |u_m|^{2\nu} \right) \leq C \left( \int |u_m|^{2\nu} \right).
\] (96)
Also, from (87) it results \( \frac{\nu}{\nu-1} \leq q, \text{ then } \left( \int |u_m|^{2\nu} \right) \leq C \left( \int |u_m|^{2\nu} \right)^\frac{2k}{\nu}. \)
Consequently,
\[
\int_\Omega \left( |u_m|^{\nu}u_m \right) \, dx \leq C \left( \int |u_m|^{2\nu} \right)^\frac{2k}{\nu} \, dx.
\] (97)
Then, from (91) we conclude that
\[
|u_m|^{\nu}u_m \text{ is bounded in } L^2(\Omega).
\] (98)
Multiplying the equation (93) by \( K_m''(0) \) and performing the summation over \( j = 1 \) to \( m \), yields
\[
|u_m''(0)|^2 \leq \left( |f(0) + |\text{div } F(v(u_m))| + |u_m|^{\nu-1} \right)^{u_m''(0)}. \]
(100)
Therefore, using (11) and (84), we obtain \( f(0) \in L^2(\Omega). \)
Then,
\[
|u_m''(0)| \leq C_2.
\] (101)
On the other hand, by derivating to time, (28) takes the form
\[
(u_m''(0), w_j) + (u_m''(0), w_j) - \int_\Omega f(u_m') |w_j| \, dt + \nu (u_m''(0), w_j) = (f(0), w_j).
\] (102)
Multiplying (102) by $K''_{jm}(t)$ and performing the summation over $j = 1$ to $m$, yields

$$\left\{ \frac{1}{2} \| u''_{m}(t) \|^2 + a(u'_{m}(t), u'_{m}(t)) = \int \sigma(u_{m}(t))q_{j}(t) \, dt + (f'_{j}(t), u'_{m}(t)) - (\nu_{m}(t), u''_{m}(t), u''_{m}(t)). \right\} \tag{103}$$

As the function of the injection $u \rightarrow (\sigma(u(t)))$ of $H^{1/2}(\Omega)$ is continuous and $q_{j}(t) \in H^{-1/2}(\Omega)$, then

$$\int_{\Gamma_{3}} \sigma(u_{m}(t))q_{j}(t) \, d\Gamma \leq C \| u'_{m}(t) \|^{2} \leq \frac{1}{2} C_{3} \left( 1 + \| u''_{m}(t) \|^{2} \right). \tag{104}$$

Since $\frac{1}{n} + \frac{3}{q} = \frac{1}{2}$, using Hölder’s inequality we conclude that

$$\left\{ (v + 1) \| u''_{m}(t) \|^2 + (u'_{m}(t), u'_{m}(t)) \right\} \leq C \left\| u_{m}(t) \right\| L^{n}(\Omega). \tag{105}$$

Then, using (16) and as $\nu_{m} \leq kq$ by (77) we obtain

$$\| u_{m}(t) \|^2 \leq C \| u_{m}(t) \|^2 \leq C_{3}. \tag{106}$$

We also have

$$(v + 1) \| u_{m}(t) \|^2 + (u'_{m}(t), u'_{m}(t)) \leq C_{4} \| u_{m}(t) \| L^{n}(\Omega). \tag{107}$$

Since

$$\frac{d}{dt} \sigma(u'(t), u'(t)) = a(u'(t), u'(t)) + a(u'(t), u'(t)) = 2a(u''(t), u''(t)) = \frac{d}{dt} \| u'(t) \|^{2}, \tag{108}$$

we have

$$a(u''(t), u''(t)) = \frac{1}{2} \| u'(t) \|^{2} \geq \frac{1}{2} C_{1} \| u''(t) \|^{2}. \tag{109}$$

Then, by using (108), (109) from (103) it follows

$$\int_{\Gamma_{3}} \sigma(u'(t))q_{j}(t) \, d\Gamma + (f'_{j}(t), u'_{m}(t)) \leq (f'_{j}(t), u'_{m}(t)) \tag{110}$$

But by the inequalities of Cauchy Schwartz and Young the second member of (110) is raised in absolute value by

$$\left\{ \left| (f'_{j}(t), u''_{m}(t)) + \int \sigma(u''_{m}(t))q_{j}(t) \, dt \right| \right\} \leq \left| f'_{j}(t) \right| \left| u''_{m}(t) \right| + \frac{1}{2} C_{3} \left( 1 + \left| u''_{m}(t) \right|^{2} \right) + C_{4} \left( \left| u''_{m}(t) \right| + \left| u''_{m}(t) \right|^{2} \right) \tag{111}$$

where $C_{3} = 1 + C_{2} + 2C_{4}$. It then follows from (110) that

$$\int_{\Omega} \left| u''_{m}(t) \right|^{2} + C_{1} \left| u''_{m}(t) \right|^{2} \leq \frac{1}{2} \left| f'_{j}(t) \right|^{2} + \frac{1}{2} C_{3} \left( 1 + \left| u''_{m}(t) \right|^{2} \right) + \frac{1}{2} C_{4} \left( 1 + \left| u''_{m}(t) \right|^{2} \right). \tag{112}$$

Integrating equation (112) over $(0, t)$, we obtain

$$\frac{1}{2} \left| u''_{m}(0) \right|^{2} + C_{1} \left| u''_{m}(0) \right|^{2} \leq \frac{1}{2} \int_{0}^{t} \left| f'(s) \right|^{2} \, ds + \frac{1}{2} \left| u''_{m}(0) \right|^{2} \tag{113}$$

Integrating equation (112) over $(0, t)$, we obtain

$$\frac{1}{2} \left| u''_{m}(t) \right|^{2} + C_{1} \left| u''_{m}(t) \right|^{2} \leq \frac{1}{2} \int_{0}^{t} \left| f'(s) \right|^{2} \, ds + \frac{1}{2} \left| u''_{m}(0) \right|^{2} \tag{113}$$

Then, by (84), (101), (92), it follows from (113) that

$$\int_{0}^{t} \left| u''_{m}(s) \right|^{2} + C_{1} \left| u''_{m}(s) \right|^{2} \leq C_{0} \left( 1 + \int_{0}^{t} \left( \left| u''_{m}(s) \right|^{2} + \left| u''_{m}(s) \right|^{2} \right) \, ds \right). \tag{114}$$

where

$$C_{0} = \max \left( \frac{1}{2} \int_{0}^{t} \left| f'(s) \right|^{2} \, ds + \frac{1}{2} \left| u''_{m}(0) \right|^{2} + C_{1} \left| u''_{m}(0) \right|^{2} + C_{2} T^2 C_{3} \right).$$

Thus, by Gronwall’s inequality, it follows that

$$\left| u''_{m}(t) \right| + \left| u''_{m}(t) \right| \leq C \left( \text{independent of } m \right). \tag{115}$$

Therefore,

$$\left\{ \left( u_{m}(t) \right) \right. \text{ is bounded in } L^{\infty}(0, T; V), \tag{116}$$

$$\left\{ \left( u_{m}(t) \right) \right. \text{ is bounded in } L^{\infty}(0, T; L^{2}(\Omega)) \right\}.$$

Then, there exists a subsequence of $(u_{m}(t))$, denoted by $(u_{\mu})$ such that

$$u_{\mu} \rightarrow u \text{ in } L^{\infty}(0, T; L^{2}(\Omega)) \text{ weak star.} \tag{117}$$

But, by (51) we have that

$$u_{\mu} \rightarrow u \text{ strongly in } L^{2}(Q). \tag{118}$$

Also, by (52) we obtain

$$\left| u_{\mu} \right| \rightarrow \left| u \right| \text{ in } L^{\infty}(0, T; L^{2}(\Omega)) \text{ weak star.} \tag{119}$$

Let $j$ be fixed and $\mu > j$. Then, using (49), (117) and (52), we deduce that

$$a(u_{\mu}(t), u_{\mu}(t)) \rightarrow a(u(t), u(t)) \text{ in } L^{\infty}(0, T) \text{ weak star.} \tag{120}$$

$$\int_{\Omega} \sigma(u(t))q_{j}(t) \, dt \rightarrow \int_{\Omega} \sigma(u(t))q_{j}(t) \, dt \text{ in } L^{\infty}(0, T) \text{ weak star.} \tag{121}$$

$$\left\{ u_{\mu}(t), w_{\mu}(t) \right\} \rightarrow \left\{ u(t), w(t) \right\} \text{ in } L^{\infty}(0, T) \text{ weak star.} \tag{122}$$

$$\left\{ \left| u_{\mu} \right|, \left| w_{\mu} \right| \right\} \rightarrow \left\{ \left| u \right|, \left| w \right| \right\} \text{ in } L^{\infty}(0, T) \text{ weak star.} \tag{123}$$

Thus, it follows from (28) that

$$(u_{\mu}(t), w_{\mu}(t)) \rightarrow (u(t), w(t)) \text{ in } L^{\infty}(0, T; L^{2}(\Omega)) \text{ weak star.} \tag{124}$$

Again, using the density of $V_{m}$ in $V \cap H^{2}(\Omega)$ we find

$$\left( u(t), v \right) + \int_{\Gamma_{3}} \sigma(u(t))q_{j}(t) \, ds + \left( f(t), v \right) \rightarrow \left( u(t), v \right) + \int_{\Gamma_{3}} \sigma(u(t))q_{j}(t) \, ds + \left( f(t), v \right) \text{ for all } v \in V \cap H^{2}(\Omega), \tag{125}$$

which implies that $u$ satisfies (3), (89) and (90). On the other hand, using (91), $\left| u_{\mu} \right| \in L^{\infty}(0, T; L^{2}(\Omega))$ and $f \in L^{\infty}(0, T; L^{2}(\Omega))$, it then follows from (3) that

$$h = \text{div} \left( u \right) \in L^{\infty}(0, T; L^{2}(\Omega)). \tag{126}$$

Since $\text{div}(u)$, see [7], is an isomorphism from $V$ onto $V$. Let $G$ be its inverse. Then, since $u \in L^{\infty}(0, T; V)$ we have

$$u(t) = G h(t) \tag{127}.$$

As $\Omega$ is assumed regular. Then, by referring to [7] and [11] we have $G \in L(V^{2}; H^{2}(\Omega))$, which implies (88).
5. Conclusion

Consider the following function
\[ F(\varepsilon(u)) = 2\varepsilon(u) - \text{Trace}(\varepsilon(u))I, \]
where \( I \) denotes the identity operator and \( \text{Trace} \) denotes the trace operator. Then, problem (3)-(6), without the condition \( \sigma(u) = 0 \) on \( \Sigma \), is reduced to the following problem considered by Lions in [8]:
\[
\begin{align*}
\frac{d^2u}{dt^2} - \Delta u + |u|^\varepsilon u &= f \text{ in } Q, \\
u(x,0) &= u_0(x); \quad u'(x,0) = u_1(x), \ a.e. \ x \in \Omega.
\end{align*}
\]
(P)

Since \( F \) is linear and satisfies the hypotheses (10). Then, it is easy to verify that Theorems 1, 2 and 3 are verified for the problem (P).

References