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A Graph-Theoretic Method to Representing a Concept Lattice

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Abstract: Concept lattices are indeed lattices. In this paper, we present a new relationship between lattices and graphs: given a binary relation I, we define an underlying graph D_I, and find out the constitution in the set of cover elements of the minimum element of the concept lattice of I using the properties of D_I. The following is to provide a way to establish a one-to-one correspondence between the set of covers of an element in the concept lattice and the set of covers of the minimum in a sublattice of the concept lattice. We apply the one-to-one correspondence to define a new underlying graph, and generate the elements of the lattice.

Keywords: concept lattice; graph; neighbor; cover; interval

1 Introduction and Preliminaries

We know that many problems of data analysis are naturally formulated in terms of formal concept lattice. As A.Berry and A.Sigayret said in [4], one of the important challenges in data handling is generating or navigating the concept lattice of a binary relation.

In this paper, we provide a graph-theoretic approach by determining the concept lattice of a binary relation with the underlying graph. An important step in connecting this graph-theoretic approach with lattices is to associate each binary relation with an underlying undirected bipartite graph. When the bipartite graph is obtained, for the minimum element in the concept lattice of the binary relation, all its cover elements are searched out by the graph-theoretic method. Additionally, a key aspect of this approach is that it equates the concepts of the lattice with the minimum element in a new binary relation.

In summary, our method presented in this paper is an initial structural results, which we expect will provide support for further advance in this direction.

The outline of the paper is as follows. After introducing some notions from concept lattice, graph theory and lattice theory, we define the underlying graph D_I which we use to represent a binary relation I, and describe some of its properties. Then, we present the main results regarding the method of obtaining the concept lattice of I in this paper. This is followed by a description of the manner in which the concept lattice can be computed and visualized in steps, at the same time, the diagram of the concept lattice is born. The final section gives an example for illustrating the graph-theoretic method presented in this paper.

Although some of the definitions appearing in this section do not require that the sets involved be finite, we make a standing assumption that all the discussions under consideration are finite. Originally, the terminologies of concept lattices are given below. After that, some known properties about concept lattices are shown.

Definition 1 [1,3] A triple (G,M,I) is called a *formal context*, if G and M are sets and $I \subseteq G\times M$ is a binary relation between G and M. We call the elements of G *objects*, those of M *attributes*, and I the *incidence* of (G,M,I). For $A\subseteq G$, we define

$A' = \{m\in M \mid (g,m)\in I$, for all $g\in A\}$, and dually, for $B\subseteq M$,

$B' = \{g\in G \mid (g,m)\in I$, for all $m\in B\}$.

(A,B) is a *formal concept* of (G,M,I) if and only if $A\subseteq G, B\subseteq M, A' = B$ and $A = B'$.

The concepts of a given context are naturally ordered by the relation defined by

$(A_1,B_1)\le(A_2,B_2)\Leftrightarrow A_1\subseteq A_2(\Leftrightarrow B_2\subseteq B_1)$.

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The ordered set of all formal concepts of (G,M,I) is denoted by $\mathscr{B}(G,M,I)$ and it is called the *concept lattice* of (G,M,I).

In this paper, a formal context and a formal concept will be simply said a context and a concept respectively; $(g,m) \in I$ is often written as gIm.

Lemma 1 [1,3] (1) For $A_1,A_2,A \subseteq G$ and $B_1,B_2,B \subseteq M$, there are the following statements:
(i) $A_1 \subseteq A_2 \Rightarrow A_2' \subseteq A_1'$. (i)' $B_1 \subseteq B_2 \Rightarrow B_2' \subseteq B_1'$.
(ii) $A \subseteq A''$ and $A' = A'''$. (ii)' $B \subseteq B''$ and $B' = B'''$.
(iii) $A \subseteq B' \Leftrightarrow B \subseteq A'$.
(2) The concept lattice $\mathscr{B}(G,M,I)$ is complete.

The graphs used here are finite and undirected. About graph theory's knowledge, we just show some of them next and the others are referred to [2]. A *graph* $D = (V(D),E(D))$ means that $V(D)$ is the *vertex set* and $E(D) \subseteq V(D)^2 = \{xy|x,y \in V(D)\}$ is the *edge set*.

Definition 2 [2] A *bipartite graph* is one whose vertex set can be partitioned into two subsets X and Y, so that each edge has one end in X and one end in Y.

The *degree* $d_D(u)$ of $u \in V(D)$ is the number of edges of D incident with u.

Two vertices u and v of D are said to be *connected* if there is a (u,v)-path in D. Connection is an equivalence relation on the vertex set V. Thus, there is a partition of V into nonempty subsets $V_1,V_2,\dots,V_w$ such that two vertices u and v are connected if and only if both u and v belong to the same set V_i.

The subgraphs $D[V_1],D[V_2],\dots,D[V_w]$ are called the *components* of D. If D has exactly one component, G is *connected*; otherwise, D is *disconnected*.

For any set S of vertices in D, we define the *neighbor set* of S in D to be the set of all vertices adjacent to vertices in S; this set is denoted by $N_D(S)$.

We will use $d(u)$ and $N(S)$ instead of $d_D(u)$ and $N_D(S)$ respectively if it does not cause confusion throughout the rest of the paper.

By Lemma 1, we see that $\mathscr{B}(G,M,I)$ is indeed a lattice. We know that the main goal of data analysis is just to find the lattice construction for a given context. Hence, we also need lattice theory to finish this duty. We just write out some known literature about lattice theory, the others of lattice theory are seen [3].

Definition 3 [3] The *diagram* of a poset $(P,\leq)$ represents the elements with small circles; the circle representing two elements x,y are connected by a straight line if and only if one covers the other; if x covers y, then the circle representing x is higher than the circle representing y.

Let L be a lattice. For $a,b \in L$ and $a \leq b$, the *interval* $[a,b] = \{x \in L \mid a \leq x \leq b\}$.

In view of the results in [3], we know that $[a,b]$ is a sublattice of L.

2 The bipartite graph underlying a binary relation

In the previous works, [4] introduces to represent a given context by a graph constructed on the complement of the relation; the breath first search graph partitions method is shown in [5]. To benefit by the ideas in [4,5], we just construct a graph on a given context to obtain the concept lattice and the diagram of the concept lattice. Of course, our graph is quite different from that in [4,5] and the other materials such as the references in [4,5]. Thus the method here is a new approach.

We should point out that the relations we work on are considered as non-empty.

Definition 4 Let (G,M,I) be a context; we will define an associated underlying graph, denoted D_I, as follows:
· The vertex set of D_I is $G \cup M$.
· For $x,y \in G$, there is not any edge to incident with x and y.
· For $x,y \in M$, there is not any edge to incident with x and y.
· For a vertex x of G and a vertex y of M, there is an edge in D_I if and only if (x,y) is in I.

Note that only the vertices between a vertex of G and a vertex of M are possible to be incident with an edge; the vertices between G or between M are not possible to be incident with an edge. Thus, D_I is a bipartite graph and undirected graph and $V(D_I) = G \cup M$ and $E(D_I) = \{xy \mid x \in G, y \in M, (x,y) \in I\} = \{yx \mid y \in M, x \in G, (x,y) \in I\}$. The graphs of this class have several remarkable properties, such as hereditary: any subgraph of bipartite graph which has more than one vertex is again bipartite graph. Moreover, since the relations we work on are considered as non-empty, D_I is always hereditary.

We present several nice properties on D_I, which makes our construction $\mathscr{B}(G,M,I)$ easier to handle than on more general graphs.

Lemma 2 Let $D_1,D_2,\dots,D_\gamma$ be all the components of D_I associated with $\mathscr{B}(G,M,I)$ and $(A,B) \in \mathscr{B}(G,M,I) \setminus \{M_\emptyset = min\{C \mid C \in \mathscr{B}(G,M,I)\}, G_\emptyset = max\{C \mid C \in \mathscr{B}(G,M,I)\}\}$. Then there is one and only one D_{γ_0} satisfying $A \cap V(D_{\gamma_0}) \neq \emptyset$. Similarly, there is one and only one D_{γ_1} satisfying $B \cap V(D_{\gamma_1}) \neq \emptyset$. Further, there is one and only one $D_{\gamma_{01}}$ such that A,B belong to $V(D_{\gamma_{01}})$.

Proof The existence of D_{γ_0} for A is carried out by Definition 2 and Definition 4.

Suppose there are two components D_1 and D_2 of D_I satisfying $A \cap V(D_t) \neq \emptyset, (t = 1,2)$.

Let $a_t \in A \cap V(D_t) = A_t, (t = 1,2)$. Since $A' = B = \{b \in M \mid \forall x \in A, xIb\} = \{b \in M \mid \forall x \in A, xb \in E(D_I)\}$ by

Definition 1 and Definition 4. Especially, for $a_t \in A, (t = 1,2)$, there is $a_t b \in E(D_I)$ for any $b \in B, (t = 1,2)$. However, by the given, $B \neq \emptyset$. We could put $b_0 \in B$. There are $a_1 b_0, a_2 b_0 \in E(D_I)$. Further, $a_1(a_1 b_0) b_0 (b_0 a_2) a_2$ is an (a_1,a_2)-path.

Since $D_t = D[V(D_t)]$ is connected, i.e, for all $x_t \in V(D_t) \setminus a_t$, there is an (x_t,a_t)-path to connect x_t and a_t, $(t = 1,2)$. Thus, (x_1,a_1)-path, (a_1,a_2)-path, (a_2,x_2)-path taken together is an (x_1,x_2)-path to connect x_1 and x_2.

By the above, we have the connectivity of $D_1 \cup D_2$, a contradiction to the assumption.

Similarly, there is one and only one component D_{γ_1} satisfying $B \cap V(D_{\gamma_1}) \neq \emptyset$.

By the construction of D_I, $(A,B) \in \mathscr{B}(G,M,I)$ and Definition 1, we have, there is one and only one $D_{\gamma_{01}}$ satisfying $A,B \in V(D_{\gamma_{01}})$.

Suppose $(A,B),(X,Y) \in \mathscr{B}(G,M,I) \setminus \{M_\emptyset, G_\emptyset\}$ satisfy $(A,B) < (X,Y)$. Then by Definition 1, $A \subset X$ and $Y \subset B$. $Y \subset B$ tells us $Y \subseteq V(D_B)$, where $B \subseteq V(D_B)$ and D_B is a component of D_I. In virtue of Lemma 2, one gets $X \subseteq V(D_B)$ and $A \subseteq V(D_B)$. That is to say, if $(A,B),(X,Y) \in \mathscr{B}(G,M,I)$ and $(A,B),(X,Y)$ are comparable in $\mathscr{B}(G,M,I)$, it must have that A,B,X and Y belong to the same component in D_I.

About D_I, we have the following extreme statuses to explain.

Status 1. $y \in M$ and $N(y) = \emptyset$.

By Definition 1, for $(A,B) \in \mathscr{B}(G,M,I)$ and $B \neq M$, it has $y \notin B$. Namely, for $(A,B) \in \mathscr{B}(G,M,I)$, $y \in B$ if and only if $B = M$.

Status 2. $x \in G$ and $N(x) = \emptyset$.

In virtue of Definition 1, for $(A,B) \in \mathscr{B}(G,M,I)$ and $A \neq G$, it has $x \notin A$. Namely, for $(A,B) \in \mathscr{B}(G,M,I)$, $x \in A$ if and only if $A = G$.

Status 3. $\{g_{m_1}, g_{m_2}, \ldots, g_{m_s}\} \subseteq G$ satisfying $N(g_{m_1}) = N(g_{m_2}) = \ldots = N(g_{m_s}) = M$.

In light of Definition 1, for $(A,B) \in \mathscr{B}(G,M,I)$, it has $g_{m_i} \in A, (i = 1,2,\ldots,s)$. That is,

$(\{g_{m_1}, g_{m_2}, \ldots, g_{m_s}\}, M) \in \mathscr{B}(G,M,I)$.

Status 4. $\{m_{n_1}, m_{n_2}, \ldots, m_{n_t}\} \subseteq M$ satisfying $N(m_{n_1}) = N(m_{n_2}) = \ldots = N(m_{n_t}) = G$.

Owing to Definition 1, for $(A,B) \in \mathscr{B}(G,M,I)$, it has $m_{n_i} \in A, (i = 1,2,\ldots,t)$.

Say, $(G, \{m_{n_1}, m_{n_2}, \ldots, m_{n_t}\}) \in \mathscr{B}(G,M,I)$.

According to the Statuses, we assure:

(1) Under the supposition of Status 3, $(A,B) \in \mathscr{B}(G,M,I)$ if and only if $(A \setminus \{g_{m_1}, g_{m_2}, \ldots, g_{m_s}\}, B) \in \mathscr{B}(G \setminus \{g_{m_1}, g_{m_2}, \ldots, g_{m_s}\}, M, I_1)$, where $(X,Y) \in I_1 \Leftrightarrow (X \cup \{g_{m_1}, g_{m_2}, \ldots, g_{m_s}\}, Y) \in I$.

(2) Under the supposition of Status 4, $(A,B) \in \mathscr{B}(G,M,I)$ if and only if $(A, B \setminus \{m_{n_1}, m_{n_2}, \ldots, m_{n_t}\}) \in \mathscr{B}(G, M \setminus \{m_{n_1}, m_{n_2}, \ldots, m_{n_t}\}, I_2)$, where $(X,Y) \in I_2 \Leftrightarrow (X, Y \cup \{m_{n_1}, m_{n_2}, \ldots, m_{n_t}\}) \in I$.

Therefore, in Section 3 and the part of 5.1 in Section 5, we only consider (G,M,I) with the property: for $\forall g \in G$, there is $m \in M$ satisfying $(g,m) \notin I$, and at the same time, for $\forall m_0 \in M$, there is $g_0 \in G$ satisfying $(g_0,m_0) \notin I$. Under such supposition, in $\mathscr{B}(G,M,I)$, $(\emptyset, M)$ and $(G, \emptyset)$ are known existed as the minimum $M_\emptyset$ and the maximum $G_\emptyset$ respectively.

In $\mathscr{B}(G,M,I)$, we call $M_\emptyset$ and $G_\emptyset$ *trivial elements* the others *nontrivial elements*.

3 The cover elements of the minimum in $\mathscr{B}(G,M,I)$

The main aim of studying on $\mathscr{B}(G,M,I)$ is to search nontrivial elements and the relationships among the members in $\mathscr{B}(G,M,I)$.

In this section, we only consider that D_I is connected. We will present a way to find all the cover elements of $(\emptyset, M)$ in the concept lattice $\mathscr{B}(G,M,I)$.

Let $G = \{g_1, g_2, \ldots, g_k\}$. In D_I, the degree sequence $(d(g_1), d(g_2), \ldots, d(g_k))$ satisfies $d(g_{1_1}) = d(g_{1_2}) = \ldots = d(g_{1_{t_1}}) = min\{d(g_1), d(g_2), \ldots, d(g_k)\} < d(g_{2_1}) = d(g_{2_2}) = \ldots = d(g_{2_{t_2}}) < \ldots < d(g_{i_1}) = d(g_{i_2}) = \ldots = d(g_{i_{t_i}}) < d(g_{(i+1)_1}) = d(g_{(i+1)_2}) = \ldots = d(g_{(i+1)_{t_{i+1}}}) < \ldots < d(g_{s_1}) = d(g_{s_2}) = \ldots = d(g_{s_{t_s}}) = max\{d(g_1), d(g_2), \ldots, d(g_k)\}$, where for any $g_j \in G, d(g_{i_1}) < d(g_j) < d(g_{(i+1)_1})$ is not true.

This sequence and D_I have some remarkable properties shown as follows.

Lemma 3 (1) $0 \leq d(g_j), (j = 1,2,\ldots,k)$.

(2) $d(g_j) = |N(g_j)|, (j = 1,2,\ldots,k)$.

(3) For any $g_j \in G$, if $G \ni g \in N(g_j)'$, then $N(g_j) \subseteq N(g), (j = 1,2,\ldots,k)$.

(4) $(X,Y) \in \mathscr{B}(G,M,I)$ induces $Y = \bigcap_{x \in X} N(x)$.

Proof (1) and (2) are got by Definition 2 and Definition 4.

Let $g \in N(g_j)'$. Since $N(g_j)' = \{x \in G \mid \forall y \in N(g_j), xIy\}$ by Definition 1, we have gIy for $y \in N(g_j)$, and in view of Definition 4, $gy \in E(D_I)$ for every $y \in N(g_j)$. Hence, owing to Definition 2, $N(g_j) \subseteq N(g)$. Say, (3) is true.

By Definition 1 and Definition 2, it is easy to have $Y \subseteq N(x)$, and so $Y \subseteq \bigcap_{x \in X} N(x)$.

If $Y \subset \bigcap_{x \in X} N(x)$, that is, there is $b \in \bigcap_{x \in X} N(x) \setminus Y$. This implies $b \in N(x)$ for $x \in X$, say, $xb \in E(D_I)$ for every $x \in X$. Thus, $b \in X'$. However, $X' = Y$ holds according to $(X,Y) \in \mathscr{B}(G,M,I)$ and Definition 1. This follows $b \in Y$, a contradiction with $b \notin Y$. Hence, $Y = \bigcap_{x \in X} N(x)$, i.e. (4)

is correct.

We are now ready to prove our main results.

Theorem 1 Let $g_j \in G$. Then $(N(g_j)', N(g_j)) \in \mathscr{B}(G,M,I)$, $(j = 1,2,\ldots,k)$.

Proof By Lemma 1, $N(g_j) \subseteq N(g_j)''$ holds. According to Definition 1, it only needs to prove $N(g_j)'' \subseteq N(g_j)$.

In light of Definition 1 and Definition 4, one gets $N(g_j)'' = \{y \in M \mid \forall x \in N(g_j)', xy \in E(D_I)\}$ and $N(g_j)' = \{a \in G \mid \forall b \in N(g_j), ab \in E(D_I)\}$.

By Definition 2 and Definition 4, $g_j b \in E(D_I)$ holds for $\forall b \in N(g_j)$. It follows $g_j \in N(g_j)'$. Hence, $g_j y \in E(D_I)$ for $\forall y \in N(g_j)''$. So $y \in N(g_j)$. Namely, $N(g_j)'' \subseteq N(g_j)$.

That is to say, $(N(g_j)', N(g_j)) \in \mathscr{B}(G,M,I)$.

Theorem 2 Let $g \in G$, $p \in \{1,\ldots,t_s\}$ and $d(g_{s_p}) = max\{d(g_1),\ldots,d(g_k)\}$. Then

(1) $N(g) \cap N(g_{s_p}) = N(g_{s_p}) \Longleftrightarrow g \in N(g_{s_p})'$.

(2) $(N(g_{s_p})', N(g_{s_p}))$ covers $(\emptyset, M)$ in $\mathscr{B}(G,M,I)$.

Proof (1) $(\Rightarrow)$ $N(g) \cap N(g_{s_p}) = N(g_{s_p})$ hints $N(g_{s_p}) \subseteq N(g)$. In light of Lemma 3, this implies $d(g_{s_p}) = |N(g_{s_p})| \leq |N(g)| = d(g)$. However, $d(g_{s_p}) = max\{d(g_1), d(g_2), \ldots, d(g_k)\}$. Thus $d(g) = d(g_{s_p})$ is true, further, $N(g_{s_p}) = N(g)$ holds. Herein, $g \in N(g)' = N(g_{s_p})'$.

$(\Leftarrow)$ $g \in N(g_{s_p})'$ and Lemma 3 together hints $N(g_{s_p}) \subseteq N(g)$, i.e $N(g_{s_p}) = N(g_{s_p}) \cap N(g)$.

(2) In view of Theorem 1, $(N(g_{s_p})', N(g_{s_p})) \in \mathscr{B}(G,M,I)$.

Suppose there is $(X,Y) \in \mathscr{B}(G,M,I)$ satisfying $(\emptyset, M) < (X,Y) < (N(g_{s_p})', N(g_{s_p}))$. It has $\emptyset \neq X \subset N(g_{s_p})'$ and $N(g_{s_p}) \subset Y \subset M$. Let $y_0 \in Y \setminus N(g_{s_p}) \neq \emptyset$.

Because $X = Y' = \{x \in G \mid \forall y \in Y, xIy\} = \{x \in G \mid \forall y \in Y, xy \in E(D_I)\}$. Then, for $y_0 \in Y$ and $\forall x \in Y' = X$, there is $xy_0 \in E(D_I)$. This follows $y_0 \in N(x)$ for $\forall x \in X$. However, $x \in X \subset N(g_{s_p})'$ and the above closed (1) together shows us $N(x) = N(g_{s_p})$. Thus $y_0 \in N(g_{s_p})$, a contradiction to $y_0 \in Y \setminus N(g_{s_p})$.

That is to say, $(N(g_{s_p})', N(g_{s_p}))$ covers $(\emptyset, M)$.

Theorem 2 endows the cover elements of $(\emptyset, M)$ yielded from the members in $\mathscr{T}_s = \{g_{s_1}, g_{s_2}, \ldots, g_{s_{t_s}}\}$. We will now discuss how to find the other cover elements of $(\emptyset, M)$.

Suppose we have got the cover elements of $(\emptyset, M)$ yielded from
$\mathscr{T}_{i+1} = \{g_{(i+1)_1}, g_{(i+1)_2}, \ldots, g_{(i+1)_{t_{i+1}}}\} \setminus \Big\{g_j \in G \mid d(g_j) = d(g_{(i+1)_1})$, additionally, there is $g \in G$ satisfying $N(g_j) \subset N(g)\Big\}$. Put
$\mathscr{T}_i = \{g_{i_1}, g_{i_2}, \ldots, g_{i_{t_i}}\} \setminus \Big\{g_j \in G \mid d(g_j) = d(g_{i_1})$, there is $g \in G$ satisfying $N(g_j) \subset N(g)\Big\}$
$= \{g_{i_{\alpha_1}}, g_{i_{\alpha_2}}, \ldots, g_{i_{\alpha_{\beta_i}}}\}$.

Then we get a sequence $\mathscr{T}_1, \mathscr{T}_2, \ldots, \mathscr{T}_s$. Considering with Theorem 1, we will prove that the following Theorem 3 is true.

Theorem 3 $(N(g_{i_{\alpha_h}})', N(g_{i_{\alpha_h}}))$ covers $(\emptyset, M)$, $(h = 1,2,\ldots,\beta_i)$.

Proof Otherwise, there is $(X,Y) \in \mathscr{B}(G,M,I)$ satisfying $(\emptyset, M) < (X,Y) < (N(g_{i_{\alpha_h}})', N(g_{i_{\alpha_h}}))$.

In view of Definition 1, $N(g_{i_{\alpha_h}}) \subset Y$ and $\emptyset \neq X \subset N(g_{i_{\alpha_h}})'$ are correct.

Put $a \in N(g_{i_{\alpha_h}})'$. It has $\{a\}' \supseteq N(g_{i_{\alpha_h}})'' = N(g_{i_{\alpha_h}})$. According to Definition 1 and Section 2, $\{a\}' = \{y \in M \mid \forall x \in \{a\}, xIy\} = \{y \in M \mid aIy\} = \{y \in M \mid ay \in E(D_I)\} = N(a)$. Thus, $N(g_{i_{\alpha_h}}) \subseteq N(a)$.

If $N(g_{i_{\alpha_h}}) \subset N(a)$ for some $a_0 \in N(g_{i_{\alpha_h}})'$. This causes a contradiction to the choice of $g_{i_{\alpha_h}}$. That is to say, $N(a) = N(g_{i_{\alpha_h}})$ for any $a \in N(g_{i_{\alpha_h}})'$. Particularly, $x \in X \subset N(g_{i_{\alpha_h}})'$ leads to $N(x) = N(g_{i_{\alpha_h}})$.

Thus by $(X,Y) \in \mathscr{B}(G,M,I)$ and Lemma 3, it follows $Y = \bigcap_{x \in X} N(x) = \bigcap_{x \in X} N(g_{i_{\alpha_h}}) = N(g_{i_{\alpha_h}})$, a contradiction to $N(g_{i_{\alpha_h}}) \subset Y$.

Therefore, $(N(g_{i_{\alpha_h}})', N(g_{i_{\alpha_h}})) covers (\emptyset, M), (h = 1,2,\ldots,\beta_i)$.

Observing $\mathscr{T}_1, \ldots, \mathscr{T}_s$, Theorem 2 and Theorem 3, by the induction on $s \leq |G| < \infty$, we obtain that the style members $(N(g)', N(g))$ covers $(\emptyset, M)$, where $g \in G$, additionally, for any $x \in G$, $N(g) \subset N(x)$ is wrong.

We will now discuss the converse part of the above closed result as follows.

Theorem 4 Let (X,Y) cover $(\emptyset, M)$ in $\mathscr{B}(G,M,I)$. Then there is $g_Y \in G$ such that $(X,Y) = (N(g_Y)', N(g_Y))$ and for any $g \in G, N(g_Y) \subset N(g)$ is wrong.

Proof That (X,Y) covers $(\emptyset, M)$ hints $\emptyset \neq X \subseteq G$ and $Y \subset M$.

Let $a \in X$. Because $X' = Y = \{y \in M \mid \forall x \in X, xIy\} = \{y \in M \mid \forall x \in X, xy \in E(D_I)\}$, in particular, $ay \in E(D_I)$ for any $y \in Y$. This implies $y \in N(a)$ for $\forall y \in Y$, i.e. $Y \subseteq N(a)$.

Suppose $Y \subset N(a)$ for any $a \in X$.

Since $(N(a)', N(a)) \in \mathscr{B}(G,M,I)$ is correct by Theorem 1. Further, $(N(a)', N(a)) < (X,Y)$ holds by the supposition and Definition 1. These and (X,Y) covers $(\emptyset, M)$ taken together will bring about $(N(a)', N(a)) = (\emptyset, M)$. That is $N(a) = M$.

But $Y = \bigcap_{x \in X} N(x)$ holds according to Lemma 3. Considering the arbitrary of $a \in X$ and the above discussion, we have $Y = M$, a contradiction to $Y \subset M$.

That is to say, $Y = N(g_Y)$ for some $g_Y \in G$. Herein, $(X,Y) = (N(g_Y)', N(g_Y))$.

On the other hand, suppose there is $g_0 \in G$ satisfying $N(g_Y) \subset N(g_0)$. By Theorem 1, $(N(g_0)', N(g_0)) \in \mathscr{B}(G,M,I)$. In view of Definition 1, it has $(X,Y) = (N(g_Y)', N(g_Y)) > (N(g_0)', N(g_0))$.

However, that (X,Y) covers $(\emptyset,M)$ compels $(\emptyset,M)=(N(g_0)',N(g_0))$, and so $\emptyset=N(g_0)'$.

Here, we should notice that $g_0\in N(g_0)'$ is right owing to Definition 1. Namely, $N(g_0)'\neq\emptyset$ is correct. This follows a contradiction to $\emptyset=N(g_0)'$.

In one word, there does not exist any $g\in G$ satisfying $N(g_Y)\subset N(g)$.

The following approach is the sketch of an algorithm to obtain the cover elements of $(\emptyset,M)$ yielded from $\mathscr{T}_s=\{g_{s_1},g_{s_2},\ldots,g_{s_{t_s}}\}$.
Step 1. Let $\mathscr{H}=\emptyset,\mathscr{C}_s=\emptyset$ and $\mathscr{T}=\{1,2,\ldots,t_s\}$.
Step 2. If $\mathscr{T}\neq\emptyset$, then $\mathscr{H}=\mathscr{T}$, go to Step 3.
Otherwise, go to Step 7.
Step 3. $\xi=min\mathscr{H},N(g_{s_\xi})'=\emptyset$ and $N'_{s_\xi}=\emptyset$.
Step 4. If $\mathscr{H}\neq\emptyset$, then $\alpha=min\mathscr{H}$, go to Step 5.
Otherwise, go to step 6.
Step 5. If $N(g_{s_\xi})\cap N(g_{s_\alpha})=N(g_{s_\xi})$, then
$N'_{s_\xi}=N'_{s_\xi}\cup g_{s_\alpha}$ and $\mathscr{H}=\mathscr{H}\setminus\alpha$, go to Step 4.
Otherwise, $\mathscr{H}=\mathscr{H}\setminus\alpha$ and $\mathscr{T}=(\mathscr{T}\setminus\xi)\cup\alpha$, go to Step 4.
Step 6. $N(g_{s_\xi})'=N'_{s_\xi},\mathscr{C}_s=\mathscr{C}_s\cup\{(N(g_{s_\xi})',N(g_{s_\xi}))\}$, go to Step 2.
Step 7. Stop.

By the above algorithm, we get that $\mathscr{C}_s$ is the need cover elements.

Suppose we have got the set $\mathscr{C}_{i+1}$ of the cover elements of $(\emptyset,M)$ yielded from $\mathscr{T}_{i+1}=\{g_{(i+1)_1},g_{(i+1)_2},\ldots,g_{(i+1)_{t_{i+1}}}\}\setminus\Big\{g_j\in G\mid d(g_j)=d(g_{(i+1)_1})$, besides, there is $g\in G$ satisfying $N(g_j)\subset N(g)\Big\}$.

Put $\mathscr{T}_i=\{g_{i_1},g_{i_2},\ldots,g_{i_{t_i}}\}\setminus\Big\{g_j\in G\mid d(g_j)=d(g_{i_1})$, besides, there is $g\in G$ satisfying $N(g_j)\subset N(g)\Big\}=\{g_{i_{\alpha_1}},g_{i_{\alpha_2}},\ldots,g_{i_{\alpha_{\beta_i}}}\}$.

If $\mathscr{T}_i\neq\emptyset$, then posit $\mathscr{T}=\{\alpha_1,\alpha_2,\ldots,\alpha_{\beta_i}\}$ and repeat the algorithm above for $\mathscr{T}$, we obtain $\mathscr{C}_i$, the set of the cover elements of $(\emptyset,M)$ yielded from $\mathscr{T}_i$.

If $\mathscr{T}_i=\emptyset$, then $\mathscr{C}_i=\emptyset$. We just consider $\mathscr{T}_\eta$ successively, $\eta=i-1,i-2,\ldots,1$, where $i-1>i-2>\ldots>1$.

According to $|\mathscr{T}_i|\leq|G|<\infty,(i=1,2,\ldots,s\leq|G|)$ and the Theorems from Theorem 1 to Theorem 4, it brings about that the set $\mathscr{C}$ of the cover elements of $(\emptyset,M)$ is $\mathscr{C}=\mathscr{C}_1\cup\mathscr{C}_2\cup\ldots\cup\mathscr{C}_s$.

4 The covers of an element of the lattice

We will use the classical properties of lattice theory and graph theory to construct a one-to-one correspondence between an interval of the lattice and the minimum element in the concept lattice of a new binary relation.

At first, we examine what happens to $(A,B)\in\mathscr{B}(G,M,I)$ when a new underlying graph is established. In Definition 5, we will not suppose that D_I is always connected and Status 3 or Status 4 in Section 2 could not happen.

Definition 5 Let $(A,B)\in\mathscr{B}(G,M,I)$; we will define a new binary relation $I_{AB}\subseteq(G\setminus A)\times B$, as follows:

$$(X_A,Y_A)\in I_{AB}\Leftrightarrow(X_A\cup A,Y_A)=(X,Y)\in I$$
where $X_A=X\setminus A$ and $Y_A=Y$.

For the sake of convenience, before Theorem 5, we just suppose that none of Status 3 and Status 4 in Section 2 will happen here.

Note that $(X,Y)\in[(A,B),(G,\emptyset)]\subseteq\mathscr{B}(G,M,I)$ tells us $A\subseteq X\subseteq G,Y\subseteq B$ and $(X,Y)\in I$. Simultaneously, it also tells us $X\setminus A\subseteq G\setminus A,Y\subseteq B$ and $(X\setminus A,Y)\in I_{AB}$. By Definition 5, it follows that $(X_A,Y_A)\in I_{AB}$ induces $(X,Y)=(X_A\cup A,Y)\in[(A,B),(G,\emptyset)]$.

Analogously to the construction of D_I in Section 2, the associated underlying graph $D_{I_{AB}}$ with $(G\setminus A,B,I_{AB})$ is provided.

We firstly remark that by virtue of Definition 1, $(G\setminus A,B,I_{AB})$ is a new context and $(\emptyset,B)$ is the minimum in $\mathscr{B}(G\setminus A,B,I_{AB})$.

Posit (X_1,Y_1) be a cover element of $(\emptyset,B)$ in $\mathscr{B}(G\setminus A,B,I_{AB})$. Then one gets $\emptyset\neq X_1,Y_1\subset B$ and $(X_1,Y_1)\in I_{AB}$, and further, $(X_2,Y_2)\in[(A,B),(G,\emptyset)]\subseteq\mathscr{B}(G,M,I)$ where $X_2=X_1\cup A$ and $Y_2=Y_1$.

Suppose (X_2,Y_2) does not cover (A,B) in $\mathscr{B}(G,M,I)$. We will get that there is $(X_3,Y_3)\in\mathscr{B}(G,M,I)$ satisfying $(A,B)<(X_3,Y_3)<(X_2,Y_2)$. In view of Definition 5 and the above discussion, $(\emptyset,B)<(X_3\setminus A,Y_3)<(X_2\setminus A,Y_2)=(X_1,Y_1)$ is correct in $\mathscr{B}(G\setminus A,B,I_{AB})$. This brings about a contradiction to the position of (X_1,Y_1) in $\mathscr{B}(G\setminus A,B,I_{AB})$. Thus, (X_2,Y_2) covers (A,B) in $\mathscr{B}(G,M,I)$.

Likewise, if (X,Y) covers (A,B) in $\mathscr{B}(G,M,I)$, then it must have $(X\setminus A,Y)$ covers $(\emptyset,B)$ in $\mathscr{B}(G\setminus A,B,I_{AB})$.

Summary, one will get a result as follows:
$(X,Y)\in\mathscr{B}(G,M,I)$ covers (A,B) in $\mathscr{B}(G,M,I)$ if and only if $(X\setminus A,Y)\in\mathscr{B}(G\setminus A,B,I_{AB})$ covers $(\emptyset,B)$ in $\mathscr{B}(G\setminus A,B,I_{AB})$.

The below in this Section, we will not suppose that both Status 3 and Status 4 will not happen; that D_I is connected will not assumed. Considering the above closed result and the study on the Statuses in Section 2, we pledge that the following Theorem 5 is valid.

Theorem 5 $(X,Y)\in\mathscr{B}(G,M,I)$ covers (A,B) in $\mathscr{B}(G,M,I)$ if and only if $(X\setminus A,Y)\in\mathscr{B}(G\setminus A,B,I_{AB})$ covers $B_\emptyset$ in $\mathscr{B}(G\setminus A,B,I_{AB})$, where $B_\emptyset$ is $min\{C|C\in\mathscr{B}(G\setminus A,B,I_{AB})\}$.

5 Generating the lattice

Computing the cover elements of an element in $\mathscr{B}(G,M,I)$ is an important problem for finding out all the concepts of a given context and the diagram of $\mathscr{B}(G,M,I)$. One may generate all the concepts defined by a binary relation, and at the same time, its diagram.

5.1 The first process

In the lattice $\mathscr{B}(G,M,I)$, if the height function is h, by [3], we will know $h(a) = h(b)+1$ when $a \in \mathscr{B}(G,M,I)$ covers $b \in \mathscr{B}(G,M,I)$. Because the relations we work on are considered as non-empty, one gets $h(G,\emptyset) \geq 1$.

When $h(G,\emptyset) = 1$. $\mathscr{B}(G,M,I)$ has only the two trivial elements.

When $h(G,\emptyset) > 1$. We have $0 = h(\emptyset,M) < h(X,Y) < h(G,\emptyset) < \infty$ for any nontrivial element $(X,Y) \in \mathscr{B}(G,M,I)$.

Applying the method in Section 3, we will have all the cover elements $\mathbf{F}_{(\emptyset,M)}$ of $(\emptyset,M)$. In addition, $h(X,Y) = 1 \Leftrightarrow (X,Y)$ covers $(\emptyset,M)$.

Suppose there are at least two components $D[V_1]$ and $D[V_2]$ of D_I. Let $|V_2| > 1$ and $V_1 \cap G = V_1$ with $|V_1| = 1$. $1 < |V_2|$ and $D[V_2]$ is connected taken together hints $|E(V_2)| \geq 1$. We will prove the following Lemma 4.

Lemma 4 Any concept yielded from $D[V_1]$ will not cover $(\emptyset,M)$ if there is another component $D[V_2]$ satisfying $|V_2| > 1$.

Proof For any $(A,B) \in \mathscr{B}(G,M,I)$, Lemma 2 says that A,B belong to the same component. Let $V_1 = \{v_1\}$ and $V_2 = \{v_2,v_3,\ldots,v_{\eta_2}\} \cup \{w_2,w_3,\ldots,w_{\varphi_2}\}$, where $\{v_2,v_3,\ldots,v_{\eta_2}\} \subseteq G$ and $\{w_2,w_3,\ldots,w_{\varphi_2}\} \subseteq M$.

It is easy to see that in $D_I, N(v_1) = \emptyset$ is valid. Recalled on Status 2 in Section 2, this leads that there is only one concept $(G,\emptyset)$ yielded from $D[V_1]$.

On the other hand, by the results in Section 3, $(N(v_\pi)', N(v_\pi)) \in \mathscr{B}(G,M,I)$ covers $(\emptyset,M)$, where $d(v_\pi) = max\{d(v_2),d(v_3),\ldots,d(v_{\eta_2})\}$.
By virtue of $|E(V_2)| \geq 1$, one gets that $d(v_\pi) \geq 1$ is effective, and so $|N(v_\pi)| \geq 1$. Herein, $N(v_\pi) \neq \emptyset$ is right. Thus, $(N(v_\pi)', N(v_\pi)) < (G,\emptyset) = (N(v_1)', N(v_1))$.

By the knowledge of lattice theory, here, $(G,\emptyset)$ will not cover $(\emptyset,M)$.

Based on Lemma 4, we get that if all the components $D_1,D_2,\ldots,D_\gamma$ *of* D_I *satisfy* $|V(D_1) \cap G| = |V(D_2) \cap G| = \ldots = |V(D_\gamma) \cap G| = 1$, then there is only $(G,\emptyset)$ belonging to the set of cover elements of $(\emptyset,M)$. In another word to say, for $(A,B) \in \mathscr{B}(G,M,I)$ and $A \neq \emptyset$, the underlying graph $D_{I_{AB}}$ associated to (A,B) satisfies
$E(D_{I_{AB}}) = \emptyset$ if and only if (A,B) is $(G,\emptyset)$.

Considering with Lemma 4, for each component D_t of D_I, we can use the results for Statuses in Section 2 and the manner in Section 3 to obtain the covers $\mathscr{C}_t$ of $(\emptyset,M)$, $(t = 1,2,\ldots,\gamma)$. Calling the results in Section 2 back, we have $\mathbf{F}_{(\emptyset,M)} = \bigcup_{t=1}^{\gamma} \mathscr{C}_t$.

5.2 The second process

Beyond now in this Section, we just consider (G,M,I) with the property that for any $g \in G$, there is $m \in M$ satisfying $(g,m) \notin I$, and at the same time, for any $m_0 \in M$, there is $g_0 \in G$ satisfying $(g_0,m_0) \notin I$. But, below, no such supposition exists.

Using the discussions in Section 2 and 5.1, we have
$(X,Y) \in \mathbf{F}^0_{(\emptyset,M)} \Leftrightarrow (X \cup \{g_{m_1},g_{m_2},\ldots,g_{m_s}\}$,
$Y \cup \{m_{n_1},m_{n_2},\ldots,m_{n_t}\}) \in \mathbf{F}_{M_\emptyset}$, where $\mathbf{F}^0_{(\emptyset,M)}$ is the set of covers of $(\emptyset,M)$ in $\mathscr{B}(G \setminus \{g_{m_1},g_{m_2},\ldots,g_{m_s}\}, M \setminus \{m_{n_1},m_{n_2},\ldots,m_{n_t}\}, I_1)$;
$\mathbf{F}_{M_\emptyset}$ is the set of covers of $M_\emptyset$ in $\mathscr{B}(G,M,I)$;
$(U,V) \in I_1 \Leftrightarrow$
$(U \cup \{g_{m_1},g_{m_2},\ldots,g_{m_s}\}, Y \cup \{m_{n_1},m_{n_2},\ldots,m_{n_t}\}) \in I$;
$\{g_{m_1},g_{m_2},\ldots,g_{m_s}\} \subseteq G, N_{D_I}(g_{m_1}) = N_{D_I}(g_{m_2}) = \ldots = N_{D_I}(g_{m_s}) = M$;
$\{m_{n_1},m_{n_2},\ldots,m_{n_t}\} \subseteq M, N_{D_I}(m_{n_1}) = N_{D_I}(m_{n_2}) = \ldots = N_{D_I}(m_{n_t}) = G$.

As the talk in 5.1, we can get $\mathbf{F}^0_{(\emptyset,M)}$. Considering the above talk, we will obtain $\mathbf{F}_{M_\emptyset}$.

Though by Section 2 especially Lemma 2, for $(A,B) \in \mathscr{B}(G,M,I) \setminus \{M_\emptyset, G_\emptyset\}$, if $(X_t,Y_t) \in \mathscr{B}(G,M,I) \setminus \{M_\emptyset, G_\emptyset\}$ covers (A,B), it causes that all of A,B,X_t,Y_t belong to the same component of D_I, $(t = 1,2)$, we could not say that both the pair of $X_1 \setminus A$ and Y_1 and the pair of $X_2 \setminus A$ and Y_2 belong to the same component of $D_{I_{AB}}$. The reason is that $D_{I_{AB}}$ is perhaps disconnected.

When we examine the cover elements of $(U,W) \in \mathscr{B}(G,M,I)$. If $D_{I_{UW}}$ is disconnected, in virtue of Section 2, we just separately consider the different components of $D_{I_{UW}}$. For each component in $D_{I_{UW}}$, we use the outcomes from Section 2 and that beyond this Section. The main idea is as the following.

Put (X_0,Y_0) be a cover element of $M_\emptyset$ and $\{D_q | q = 1,2,\ldots,n\}$ be the set of components of $D_{I_{X_0Y_0}}$. Now we just want to search out the family $\mathbf{F}_{(X_0,Y_0)}$ of all the cover elements of (X_0,Y_0) in $\mathscr{B}(G,M,I)$. For $(G \setminus X_0, Y_0, I_{X_0Y_0})$, we hope to use the methods appeared in Section 3, Section 4 and that at the above. Unfortunately, we could not pledge that the previous conditions are built on. So we need to consider it under the following two cases to search out the set $\mathbf{F}_{(X_0,Y_0)}$ of covers of (X_0,Y_0) in $\mathscr{B}(G,M,I)$.

Case 1. $|V(D_q)| = 1$ for every $q \in \{1,2,\ldots,n\}$.

By the Statuses in Section 2, we will get all the cover elements of (X_0,Y_0) in $\mathscr{B}(G \setminus X_0, Y_0, I_{X_0Y_0})$. Afterwards, under the instruction of Theorem 5, we obtain the covers of (X_0,Y_0) in $\mathscr{B}(G,M,I)$.

Case 2. If there is D_θ satisfying $|V(D_\theta)| > 1$ for some $\theta \in \{1,2,\ldots,n\}$.

By Lemma 4, we need not to consider the component D_δ where $|V(D_\delta)| = 1, \delta \in \{1,2,\ldots,n\}$. Review Status 1 in Section 2, if $|V(D_\psi)| = 1$ and $V(D_\psi) \cap M \neq \emptyset, (\psi \in \{1,\dots,n\})$, then when we search the covers of (X_0,Y_0), it needs not to consider D_ψ. Therefore, we only put our attention to the components as D_θ where $|V(D_\theta)| > 1$.

Firstly, because D_θ is connected, the Status 1 and Status 2 must not happen for D_θ. If the Status 3 or Status 4 happens, we just use the discussion for the two statuses in Section 2 and Theorem 5 to get the covers of (X_0,Y_0) in $\mathscr{B}(G,M,I)$.

For example, the Status 3 happens, i.e. there are $\{x_1,x_2,\dots,x_\zeta\} \in V(D_\theta) \cap G$ $satisfying$ $N_{D_\theta}(X_t) = V(D_\theta) \cap M$. Then by the result in Section 2, $(\{x_1,x_2,\dots,x_\zeta\}, V(D_\theta) \cap M)$ is the minimum in $\mathscr{B}(G\setminus X_0, Y_0, I_{X_0Y_0})$. Further, in light of Theorem 5, $(X_0 \cup \{x_1,x_2,\dots,x_\zeta\}, V(D_\theta) \cap M)$ is the only one cover of (X_0,Y_0) in $\mathscr{B}(G,M,I)$.

Secondly, if both Status 3 and Status 4 do not happen for D_θ.

According to $|V(D_\theta)| > 1$, Lemma 2 and Section 4, we could use the method in Section 3 to find all the cover elements $\mathscr{C}_\theta(X_0,Y_0)$ of $(\emptyset, Y_0)$ in $\mathscr{B}(G\setminus X_0, Y_0, I_{X_0Y_0})$ born in D_θ.

Under the guide of Theorem 5, we will get the set $\mathbf{H}_\theta(X_0,Y_0)$ of the cover elements of (X_0,Y_0) in $\mathscr{B}(G,M,I)$ associated with D_θ.

Finally, considering Lemma 2 and Section 2, it will have the family $\mathbf{F}_{(X_0,Y_0)}$. In virtue of Section 2, we have $\mathbf{F}_{(X_0,Y_0)} = \bigcup_{\theta\in\mathscr{Q}} \mathbf{H}_\theta(X_0,Y_0)$ where $\mathscr{Q} \subseteq \{1,2,\dots,n\}$ and $|V(D_\theta)| > 1$ for $\theta \in \mathscr{Q}$.

Let $\mathbf{F}_{M_\emptyset}$ be the family of covers of $M_\emptyset$ in $\mathscr{B}(G,M,I)$. Considering Lemma 2 and Section 2, for $(a,b) \in \mathbf{F}_{(X_0,Y_0)}$ and $(X_1,Y_1) \in \mathbf{F}_{M_\emptyset} \setminus (X_0,Y_0)$, (a,b) will not compare with (X_1,Y_1). Therefore, we could say that when $\mathbf{F}_{(X_0,Y_0)}$ is carried out, simultaneously, the relation between (a,b) and (c,d) are represented, besides, $h(a,b) = 2$, where $(c,d) \in \mathbf{F}_{M_\emptyset}$ and h is the height function of $\mathscr{B}(G,M,I)$.

Since $|\mathbf{F}_{M_\emptyset}| < \infty$, repeated application of this above process successively for other members in $\mathbf{F}_{M_\emptyset}$, as the consequence, we will obtain all the members $\mathbf{F}_2$ in $\mathscr{B}(G,M,I)$ with height 2 and the relationships among $(u,v) \in \mathbf{F}_2$ and $(e,f) \in \mathbf{F}_{M_\emptyset}$.

Finally, by the principle of induction and recursively compute the covers of each element in a breath-first fashion, we will get all the concepts for a given context (G,M,I) and the diagram $\mathscr{B}(G,M,I)$.

By the definition of $D_{I_{AB}}$, for $x \in G\setminus A$, it must have $d_{D_{I_{AB}}}(x) \leq d_{D_I}(x) \leq |M| < \infty$. Thus, after finite steps, the above process must be stopped and get $G_\emptyset$ as the last obtained element in $\mathscr{B}(G,M,I)$. Namely, it is a practicable approach provided above to get the members in $\mathscr{B}(G,M,I)$ and the diagram of $\mathscr{B}(G,M,I)$.

6 Example

We give an example to show how to use the manners presented from Section 2 to Section 5 to find out the concept lattice $\mathscr{B}(G,M,I)$ and its diagram for a given context (G,M,I).

Example Let $M = \{1,2,3,4,5,6\}$ and $G = \{a,b,c,d,e,f\}$. The table below describes binary relation I.

Table 1 The binary relation I for the given context

	1	2	3	4	5	6
a		×	×			×
b	×	×	×			
c	×	×			×	
d	×			×	×	
e	×			×		
f			×			

By the following steps to find out $\mathscr{B}(G,M,I)$.

The underlying graph D_I associated to (G,M,I) is shown as Figure 1.

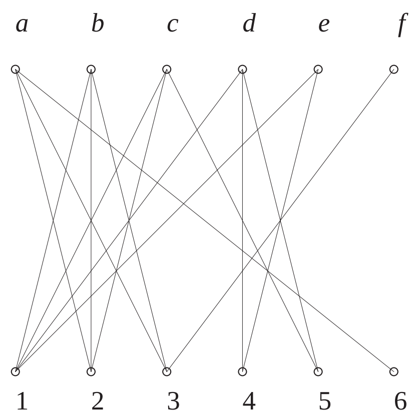

Figure 1 Underlying graph D_I

Since $f(f3)3(3b)b(b1)1(1e)e(e4)4(4d)d(d1)1(1c)c(c2)2(2a)a(a6)6(6a)a(a2)2(2b)b(b1)1(1c)c(c5)5$ is a $(f,5)$-path throughout all the vertices in D_I, where (xy) is the edge xy, for $x,y \in V(D_I)$, it follows that D_I is connected.

By D_I, it is easier to get $d(a) = d(b) = d(c) = d(d) = 3, N(a) = \{2,3,6\}, N(b) = \{1,2,3\}, N(c) = \{1,2,5\}, N(d) = \{1,4,5\}; d(e) = 2, N(e) = \{1,4\}$; $d(f) = 1, N(f) = \{3\}$. Thus, we have $d(f) = 1 = min\{d(a),\dots,d(f)\} < d(e) < d(a) = d(b) = d(c) = d(d) = max\{d(a),\dots,d(f)\}$.

Because $N(a) \cap N(g) \neq N(a)$ where $g = b,c,d$. This hints $N(a)' = \{a\}$. By the consequence in Section 3,

$(N(a)', N(a)) = (\{a\}, \{2,3,6\}) \in \mathscr{B}(G,M,I)$ is sound, and simultaneously, it covers $(\emptyset, M)$.

Similarly, $(N(b)', N(b)) = (\{b\}, \{1,2,3\})$, $(N(c)', N(c)) = (\{c\}, \{1,2,5\})$ and $(N(d)', N(d)) = (\{d\}, \{1,4,5\})$, besides, all of the three cover $(\emptyset, M)$.

Because $\{x \in G \mid d(x) = 2\} = \{e\}$, and additionally, $N(e) \subset N(d)$ hints $\{x \in G \mid d(x) = 2$, there is $g \in G$ satisfying $N(x) \subset N(g)\} = \{e\}$. Hence, $\mathscr{T}_2 = \emptyset$. According to $\{x \in G \mid d(x) = 1\} = \{f\}$ and $N(f) \subset N(d)$ hints $\{x \in G \mid d(x) = 2$, there is $g \in G$ satisfying $N(x) \subset N(g)\} = \{f\}$. Thus $\mathscr{T}_1 = \emptyset$.

Therefore, all the cover elements of $(\emptyset, M)$ is $\mathbf{F}_{(\emptyset,M)} = \{(N(a)', N(a)), (N(b)', N(b)), (N(c)', N(c)), (N(d)', N(d))\} = \{(\{a\}, \{2,3,6\}), (\{b\}, \{1,2,3\}), (\{c\}, \{1,2,5\}), (\{d\}, \{1,4,5\})\}$.

We see $h(N(a)', N(a)) = 1$. A new context associated with $(N(a)', N(a))$ is $(G \setminus N(a)', N(a), I_{N(a)'N(a)}) = (\{b,c,d,e,f\}, \{2,3,6\}, I_{\{a\}\{2,3,6\}})$. In $\mathscr{B}(\{b,c,d,e,f\}, \{2,3,6\}, I_{\{a\}\{2,3,6\}})$, the minimum is $(\emptyset, \{2,3,6\})$.

The underlying graph $D_{I_{\{a\}\{2,3,6\}}}$ is as Figure 2.

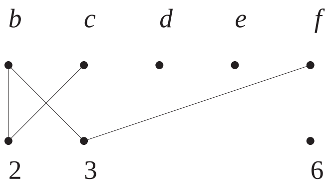

Figure 2 Underlying graph $D_{I_{\{a\}\{2,3,6\}}}$

We find that $D_{I_{\{a\}\{2,3,6\}}}$ has four components $D[V_1] = D(\{b,c,f\} \cup \{2,3\}), D[d], D[e]$ and $D[6]$. According to Lemma 4 and the other results in Section 5, it only needs to consider $D[V_1]$ to search out the covers of $(\{a\}, \{2,3,6\})$.

Because $N_{D[V_1]}(b) = \{2,3\} = V(D[V_1]) \cap M$, considering Status 3 in Section 2 with the discussion in Section 5, we obtain that $(\{b\}, \{2,3\})$ is the minimum in $\mathscr{B}(\{b,c,f\}, \{2,3\}, I_1)$, where $(X,Y) \in I_1 \iff (X,Y) \in I_{\{a\}\{2,3,6\}}$ for any $(X,Y) \subseteq \{b,c,f\} \times \{2,3\}$.

Moreover, by Theorem 5, $(\{b\} \cup \{a\}, \{2,3\}) = (\{a,b\}, \{2,3\})$ is a cover of $(\{a\}, \{2,3,6\})$ in $\mathscr{B}(G,M,I)$. Therefore, by the minimum property of $(\{b\}, \{2,3\})$ in $\mathscr{B}(\{b,c,f\}, \{2,3\}, I_1)$, one gets that in $\mathscr{B}(G,M,I)$, the set of cover elements of $(\{a\}, \{2,3,6\})$ is consisted by only $(\{b\} \cup \{a\}, \{2,3\}) = (\{a,b\}, \{2,3\})$.

For any of the other cover elements of a member in $\mathbf{F}_{(\emptyset,M)} \setminus (\{a\}, \{2,3,6\})$, by the similar way as the above, we will get their covers. That is to say, we get $\mathbf{F}_2$, all the members in (G,M,I) with height 2, and the relationships between $\mathbf{F}_2$ and $\mathbf{F}_{(\emptyset,M)}$. The diagram of $\mathbf{F}_2 \cup \mathbf{F}_{(\emptyset,M)} \cup (\emptyset, M)$ in $\mathscr{B}(G,M,I)$ is shown as Figure 3.

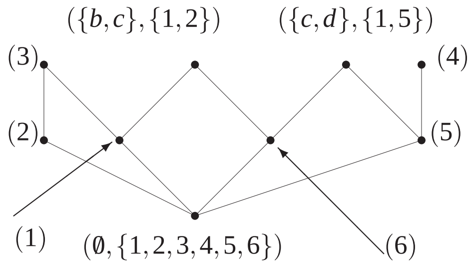

Figure 3 Diagram of $\mathbf{F}_2 \cup \mathbf{F}_{(\emptyset,M)} \cup (\emptyset, M)$ in the concept lattice

where (1)=$(\{b\}, \{1,2,3\})$, (2)=$(\{a\}, \{2,3,6\})$, (3)=$(\{a,b\}, \{2,3\})$, (4)=$(\{d,e\}, \{1,4\})$ (5)=$(\{d\}, \{1,4,5\})$, (6)=$(\{c\}, \{1,2,5\})$

Analogously, we will obtain all the members in $\mathscr{B}(G,M,I)$ and at the last, the diagram of $\mathscr{B}(G,M,I)$ is produced at the same time. The diagram of $\mathscr{B}(G,M,I)$ is shown as Figure 4.

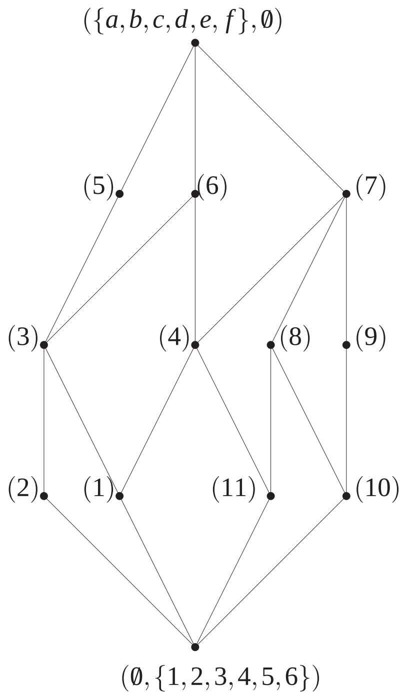

Figure 4 Diagram of the concept lattice

where (1)=$(\{b\}, \{1,2,3\})$, (2)=$(\{a\}, \{2,3,6\})$, (3)=$(\{a,b\}, \{2,3\})$, (4)=$(\{b,c\}, \{1,2\})$, (5)=$(\{a,b,f\}, \{3\})$,(6)=$(\{a,b,c\}, \{2\})$, (7)=$(\{b,c,d,e\}, \{1\})$,(8)=$(\{c,d\}, \{1,5\})$, (9)=$(\{d,e\}, \{1,4\})$,(10)=$(\{d\}, \{1,4,5\})$, (11)=$(\{c\}, \{1,2,5\})$.

At the final part, we say that this Example above is just the running example in [4]. [4] solves this example by its way which is quite different from ours born in this paper. Here, using the approach provided in this paper, we obtain the same result about the running example as that in [4]. This also illustrates that the approach here is an efficient computing method and would be a better generating algorithm for concepts.

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