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Landau Type ψ-Hilfer Fractional Inequalities

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Abstract: Here we present a series of Landau type inequalities related to left and right ψ-Hilfer fractional derivatives.

Keywords: Fractional Landau inequality, right and left ψ-Hilfer fractional derivatives.

1 Introduction

Let $p \in [1,\infty]$, $I = \mathbb{R}_+$ or $I = \mathbb{R}$ and $f : I \to \mathbb{R}$ is twice differentiable with $f, f'' \in L_p(I)$, then $f' \in L_p(I)$. Moreover, there exists a constant $C_p(I) > 0$ independent of f, such that

$$\left\|f'\right\|_{p,I} \leq C_p(I)\left\|f\right\|_{p,I}^{\frac{1}{2}}\left\|f''\right\|_{p,I}^{\frac{1}{2}}, \tag{1}$$

where $\|\cdot\|_{p,I}$ is the p-norm on the interval I, see [1,2].

The research on these inequalities started by E. Landau [3] in 1913. For the case of $p = \infty$ he proved that

$$C_\infty(\mathbb{R}_+) = 2 \text{ and } C_\infty(\mathbb{R}) = \sqrt{2} \tag{2}$$

are the best constants in (1).

In 1932, G. H. Hardy and J. E. Littlewood [4] proved (1) for $p = 2$, with the best constants

$$C_2(\mathbb{R}_+) = \sqrt{2} \text{ and } C_2(\mathbb{R}) = 1. \tag{3}$$

In 1935, G. H. Hardy, E. Landau and J. E. Littlewood [5] showed that the best constants $C_p(\mathbb{R}_+)$ in (1) satisfies the estimate

$$C_p(\mathbb{R}_+) \leq 2, \text{ for } p \in [1,\infty), \tag{4}$$

which yields $C_p(\mathbb{R}) \leq 2$ for $p \in [1,\infty)$.

In fact, in [6] and [7] was shown that $C_p(\mathbb{R}) \leq \sqrt{2}$.

In this article we present Landau type ψ-Hilfer fractional inequalities.

2 Background

Let $-\infty < a < b < \infty$, the left and right Riemann-Liouville fractional integrals of order $\alpha \in \mathbb{C}$ $(\mathscr{R}(\alpha) > 0)$ are defined by

$$\left(I_{a+}^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)\, dt, \tag{5}$$

$x > a$; where Γ stands for the gamma function,

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and

$$\left(I_{b-}^{\alpha}f\right)(x)=\frac{1}{\Gamma(\alpha)}\int_{x}^{b}(t-x)^{\alpha-1}f(t)\,dt, \tag{6}$$

$x<b$.

The Riemann-Liouville left and right fractional derivatives of order $\alpha\in\mathbb{C}$ ($\mathscr{R}(\alpha)\geq 0$) are defined by

$$\left(\Delta_{a+}^{\alpha}y\right)(x)=\left(\frac{d}{dx}\right)^{n}\left(I_{a+}^{n-\alpha}y\right)(x)=\frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dx}\right)^{n}\int_{a}^{x}(x-t)^{n-\alpha-1}y(t)\,dt. \tag{7}$$

($n=\lceil\mathscr{R}(\alpha)\rceil$, $\lceil\cdot\rceil$ means ceiling of the number; $x>a$)

$$\left(\Delta_{b-}^{\alpha}y\right)(x)=(-1)^{n}\left(\frac{d}{dx}\right)^{n}\left(I_{b-}^{n-\alpha}y\right)(x)=$$

$$\frac{(-1)^{n}}{\Gamma(n-\alpha)}\left(\frac{d}{dx}\right)^{n}\int_{x}^{b}(t-x)^{n-\alpha-1}y(t)\,dt \tag{8}$$

($n=\lceil\mathscr{R}(\alpha)\rceil$; $x<b$), respectively, where $\mathscr{R}(\alpha)$ is the real part of α.

In particular, when $\alpha=n\in\mathbb{Z}_{+}$, then

$$\left(\Delta_{a+}^{0}y\right)(x)=\left(\Delta_{b-}^{0}y\right)(x)=y(x);$$

$$\left(\Delta_{a+}^{n}y\right)(x)=y^{(n)}(x),\ \text{and}\ \left(\Delta_{b-}^{n}y\right)(x)=(-1)^{n}y^{(n)}(x),\ n\in\mathbb{N},$$

see [8].

Let $\alpha>0$, $I=[a,b]\subset\mathbb{R}$, f an integrable function defined on I and $\psi\in C^{1}(I)$ an increasing function such that $\psi'(x)\neq 0$, for all $x\in I$. Left fractional integrals and left Riemann-Liouville fractional derivatives of a function f with respect to another function ψ are defined as ([8,9])

$$I_{a+}^{\alpha,\psi}f(x)=\frac{1}{\Gamma(\alpha)}\int_{a}^{x}\psi'(t)(\psi(x)-\psi(t))^{\alpha-1}f(t)\,dt \tag{9}$$

and

$$\Delta_{a+}^{\alpha,\psi}f(x)=\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^{n}I_{a+}^{n-\alpha,\psi}f(x)= \tag{10}$$

$$\frac{1}{\Gamma(n-\alpha)}\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^{n}\int_{a}^{x}\psi'(t)(\psi(x)-\psi(t))^{n-\alpha-1}f(t)\,dt,$$

respectively, where $n=\lceil\alpha\rceil$.

Similarly, we define the right ones:

$$I_{b-}^{\alpha,\psi}f(x)=\frac{1}{\Gamma(\alpha)}\int_{x}^{b}\psi'(t)(\psi(t)-\psi(x))^{\alpha-1}f(t)\,dt \tag{11}$$

and

$$\Delta_{b-}^{\alpha,\psi}f(x)=\left(-\frac{1}{\psi'(x)}\frac{d}{dx}\right)^{n}I_{b-}^{n-\alpha,\psi}f(x)=$$

$$\frac{1}{\Gamma(n-\alpha)}\left(-\frac{1}{\psi'(x)}\frac{d}{dx}\right)^{n}\int_{x}^{b}\psi'(t)(\psi(t)-\psi(x))^{n-\alpha-1}f(t)\,dt. \tag{12}$$

The following semigroup property holds; if $\alpha,\beta>0$, $f\in C(I)$, then

$$I_{a+}^{\alpha,\psi}I_{a+}^{\beta,\psi}f=I_{a+}^{\alpha+\beta,\psi}f\ \ \text{and}\ \ I_{b-}^{\alpha,\psi}I_{b-}^{\beta,\psi}f=I_{b-}^{\alpha+\beta,\psi}f.$$

Next let again $\alpha>0$, $n=\lceil\alpha\rceil$, $I=[a,b]$, $f,\psi\in C^{n}(I)$: ψ is increasing and $\psi'(x)\neq 0$, for all $x\in I$. The left ψ-Caputo fractional derivative of f of order α is given by ([10])

$$^{C}D_{a+}^{\alpha,\psi}f(x)=I_{a+}^{n-\alpha,\psi}\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^{n}f(x), \tag{13}$$

and the right ψ-Caputo fractional derivative ([10])

$$ {}^{C}D_{b-}^{\alpha,\psi}f(x) = I_{b-}^{n-\alpha,\psi}\left(-\frac{1}{\psi'(x)}\frac{d}{dx}\right)^{n} f(x). \tag{14} $$

We set

$$ f_{\psi}^{[n]}(x) := f_{\psi}^{(n)}f(x) := \left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^{n} f(x). \tag{15} $$

Clearly, when $\alpha = m \in \mathbb{N}$ we have

$$ {}^{C}D_{a+}^{\alpha,\psi}f(x) = f_{\psi}^{[m]}(x) \ \text{ and } \ {}^{C}D_{b-}^{\alpha,\psi}f(x) = (-1)^{m} f_{\psi}^{[m]}(x) $$

and if $\alpha \notin \mathbb{N}$, then

$$ {}^{C}D_{a+}^{\alpha,\psi}f(x) = \frac{1}{\Gamma(n-\alpha)}\int_{a}^{x}\psi'(t)(\psi(x)-\psi(t))^{n-\alpha-1} f_{\psi}^{[n]}(t)\,dt \tag{16} $$

and

$$ {}^{C}D_{b-}^{\alpha,\psi}f(x) = \frac{(-1)^{n}}{\Gamma(n-\alpha)}\int_{x}^{b}\psi'(t)(\psi(t)-\psi(x))^{n-\alpha-1} f_{\psi}^{[n]}(t)\,dt. \tag{17} $$

If $\psi(x) = x$, then we get the usual left and right Caputo fractional derivatives

$$ {}^{C}D_{a+}^{m}f(x) = f^{(m)}(x), \ {}^{C}D_{b-}^{m}f(x) = (-1)^{m} f^{(m)}(x), $$

for $m \in \mathbb{N}$, and ($\alpha \notin \mathbb{N}$)

$$ D_{*a}^{\alpha}f(x) = {}^{C}D_{a+}^{\alpha}f(x) = \frac{1}{\Gamma(n-\alpha)}\int_{a}^{x}(x-t)^{n-\alpha-1} f^{(n)}(t)\,dt, \tag{18} $$

$$ D_{b-}^{\alpha}(x) = {}^{C}D_{b-}^{\alpha}f(x) = \frac{(-1)^{n}}{\Gamma(n-\alpha)}\int_{x}^{b}(t-x)^{n-\alpha-1} f^{(n)}(t)\,dt. \tag{19} $$

Also we set

$$ {}^{C}D_{a+}^{0,\psi}f(x) = {}^{C}D_{b-}^{0,\psi}f(x) = f(x). $$

Next we will deal with the ψ-Hilfer fractional derivative.

Definition 1.*([11]) Let $n-1<\alpha<n$, $n \in \mathbb{N}$, $I = [a,b] \subset \mathbb{R}$ and $f, \psi \in C^{n}([a,b])$, ψ is increasing and $\psi'(x) \neq 0$, for all $x \in I$. The ψ-Hilfer fractional derivative (left-sided and right-sided) ${}^{H}\mathbb{D}_{a+(b-)}^{\alpha,\beta;\psi}f$ of order α and type $0 \leq \beta \leq 1$, respectively, are defined by*

$$ {}^{H}\mathbb{D}_{a+}^{\alpha,\beta;\psi}f(x) = I_{a+}^{\beta(n-\alpha);\psi}\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^{n} I_{a+}^{(1-\beta)(n-\alpha);\psi}f(x) \tag{20} $$

and

$$ {}^{H}\mathbb{D}_{b-}^{\alpha,\beta;\psi}f(x) = I_{b-}^{\beta(n-\alpha);\psi}\left(-\frac{1}{\psi'(x)}\frac{d}{dx}\right)^{n} I_{b-}^{(1-\beta)(n-\alpha);\psi}f(x), \ \ x \in [a,b]. \tag{21} $$

The original Hilfer fractional derivatives ([13]) come from $\psi(x) = x$, and are denoted by ${}^{H}\mathbb{D}_{a+}^{\alpha,\beta}f(x)$ and ${}^{H}\mathbb{D}_{b-}^{\alpha,\beta}f(x)$.

When $\beta = 0$, we get Riemann-Liouville fractional derivatives, while when $\beta = 1$ we have Caputo type fractional derivatives.

We define $\gamma = \alpha+\beta(n-\alpha)$. We notice that $n-1<\alpha \leq \alpha+\beta(n-\alpha) \leq \alpha+n-\alpha = n$, hence $\lceil\gamma\rceil = n$. We can easily write that ([11])

$$ {}^{H}\mathbb{D}_{a+}^{\alpha,\beta;\psi}f(x) = I_{a+}^{\gamma-\alpha;\psi}\Delta_{a+}^{\gamma;\psi}f(x) \tag{22} $$

and

$$ {}^{H}\mathbb{D}_{b-}^{\alpha,\beta;\psi}f(x) = I_{b-}^{\gamma-\alpha;\psi}\Delta_{b-}^{\gamma;\psi}f(x), \ \ x \in [a,b]. \tag{23} $$

We have that ([11])

$$ \Delta_{a+}^{\gamma,\psi}f(x) = \left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^{n} I_{a+}^{(1-\beta)(n-\alpha);\psi}f(x) \tag{24} $$

and

$$\Delta_{b-}^{\gamma,\psi} f(x) = \left(-\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n I_{b-}^{(1-\beta)(n-\alpha);\psi} f(x). \quad (25)$$

In particular, when $0<\alpha<1$ *and* $0\le\beta\le1$; $\gamma=\alpha+\beta(1-\alpha)$, *we have that*

$$^H\mathbb{D}_{a+}^{\alpha,\beta;\psi} f(x) = \frac{1}{\Gamma(\gamma-\alpha)}\int_a^x \psi'(t)(\psi(x)-\psi(t))^{\gamma-\alpha-1}\Delta_{a+}^{\gamma;\psi} f(t)\,dt \quad (26)$$

and

$$^H\mathbb{D}_{b-}^{\alpha,\beta;\psi} f(x) = \frac{1}{\Gamma(\gamma-\alpha)}\int_x^b \psi'(t)(\psi(t)-\psi(x))^{\gamma-\alpha-1}\Delta_{b-}^{\gamma;\psi} f(t)\,dt, \quad (27)$$

$x\in[a,b]$.

Remark.([11,13]) Let $\mu=n(1-\beta)+\beta\alpha$, then $\lceil\mu\rceil=n$.

Assume that $g(x)=I_{a+}^{(1-\beta)(n-\alpha);\psi} f(x)\in C^n([a,b])$, we have that

$$^H\mathbb{D}_{a+}^{\alpha,\beta;\psi} f(x) = I_{a+}^{n-\mu;\psi}\left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n g(x). \quad (28)$$

Thus

$$^H\mathbb{D}_{a+}^{\alpha,\beta;\psi} f = {}^C D_{a+}^{\mu;\psi} g(x) = {}^C D_{a+}^{\mu;\psi}\left[I_{a+}^{(1-\beta)(n-\alpha);\psi} f(x)\right]. \quad (29)$$

Assume that $w(x)=I_{b-}^{(1-\beta)(n-\alpha);\psi} f(x)\in C^n([a,b])$. Hence

$$^H\mathbb{D}_{b-}^{\alpha,\beta;\psi} f(x) = I_{b-}^{\beta(n-\alpha);\psi}\left(-\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n w(x) = I_{b-}^{n-\mu;\psi}\left(-\frac{1}{\psi'(x)}\frac{d}{dx}\right)^n w(x). \quad (30)$$

Thus

$$^H\mathbb{D}_{b-}^{\alpha,\beta;\psi} f = {}^C D_{b-}^{\mu;\psi} w(x) = {}^C D_{b-}^{\mu;\psi}\left(I_{b-}^{(1-\beta)(n-\alpha);\psi} f(x)\right). \quad (31)$$

We need the following ψ-Hilfer fractional Taylor formulae.

Theorem 1.(*[13]*) *Let* $f,\psi\in C^n([a,b])$, *with* ψ *being increasing,* $\psi'(x)\neq0$ *over* $[a,b]\subset\mathbb{R}$, $\alpha>0:\lceil\alpha\rceil=n$, $0\le\beta\le1$, $\mu=n(1-\beta)+\alpha\beta$. *Assume that* $g_a(x)=I_{a+}^{(1-\beta)(n-\alpha);\psi} f(x)$, $w_b(x)=I_{b-}^{(1-\beta)(n-\alpha);\psi} f(x)\in C^n([a,b])$. *Then*

1)

$$I_{a+}^{\mu;\psi}\,{}^H\mathbb{D}_{a+}^{\alpha,\beta;\psi} f(x) = g_a(x)-\sum_{k=0}^{n-1}\frac{g_{a\psi}^{[k]}(a)}{k!}(\psi(x)-\psi(a))^k, \quad (32)$$

where

$$g_{a\psi}^{[k]}(x) = \left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^k g_a(x),\quad k=0,1,\dots,n-1,$$

and

2)

$$I_{b-}^{\mu;\psi}\,{}^H\mathbb{D}_{b-}^{\alpha,\beta;\psi} f(x) = w_b(x)-\sum_{k=0}^{n-1}\frac{w_{b\psi}^{[k]}(b)}{k!}(\psi(x)-\psi(b))^k, \quad (33)$$

where

$$w_{b\psi}^{[k]}(x) = \left(\frac{1}{\psi'(x)}\frac{d}{dx}\right)^k w_b(x),\quad k=0,1,\dots,n-1,$$

$x\in[a,b]$.

We also need

Theorem 2.(*[14]*) *Let* $\overline{\alpha},\overline{\beta},A,B>0$ *and the function* $y(h)=\frac{A}{h^{\overline{\alpha}}}+Bh^{\overline{\beta}}$, $h>0$. *It has only one critical number* $h_0=\left(\frac{\overline{\alpha}A}{\overline{\beta}B}\right)^{\frac{1}{(\overline{\alpha}+\overline{\beta})}}$, *such that* $y'(h_0)=0$. *Furthermore* y *has a global minimum which is*

$$y(h_0) = \frac{\left(\overline{\alpha}+\overline{\beta}\right)}{\overline{\alpha}^{\frac{\overline{\alpha}}{\overline{\alpha}+\overline{\beta}}}\,\overline{\beta}^{\frac{\overline{\beta}}{\overline{\alpha}+\overline{\beta}}}}A^{\frac{\overline{\beta}}{\overline{\alpha}+\overline{\beta}}}B^{\frac{\overline{\alpha}}{\overline{\alpha}+\overline{\beta}}}. \quad (34)$$

3 Main results

We make

Convention 3*Let $f:\mathbb{R}\to\mathbb{R}$. Define ${}^{H}\mathbb{D}_{a+}^{\alpha,\beta;\psi}f(x)=0$, for $x<a$, and ${}^{H}\mathbb{D}_{b-}^{\alpha,\beta;\psi}f(x)=0$, for $x>b$, for any $\alpha>0$, $\alpha\notin\mathbb{N}$; $a,b\in\mathbb{R}$.*

Here we prove several Landau type ψ-Hilfer fractional inequalities.
We present the following theorem.

Theorem 4.*Let $f,\psi\in C^2(\mathbb{R}_+)$, with ψ being strictly increasing, $1<\alpha<2$, $0\le\beta\le1$, $\mu=2(1-\beta)+\beta\alpha$. Assume further that $g_a:=I_{a+}^{(1-\beta)(2-\alpha);\psi}f\in C^2([a,+\infty))$, $\forall\ a\in\mathbb{R}_+$; $f\in C_B(\mathbb{R}_+)$ (continuous and bounded functions) and $\sup_{a\in\mathbb{R}_+}\left\|{}^{H}\mathbb{D}_{a+}^{\alpha,\beta;\psi}f\right\|_{\infty,\mathbb{R}_+}<+\infty$.*

Then

$$\sup_{a\in\mathbb{R}_+}\left|g_{a\psi}^{[1]}(a)\right|\le$$

$$\left[\frac{(\mu-(1-\beta)(2-\alpha))}{[\Gamma(\mu+1)(1-(1-\beta)(2-\alpha))]^{\frac{(1-(1-\beta)(2-\alpha))}{(\mu-(1-\beta)(2-\alpha))}}[(\mu-1)\Gamma((1-\beta)(2-\alpha)+1)]^{\frac{(\mu-1)}{(\mu-(1-\beta)(2-\alpha))}}}\right]$$

$$\|f\|_{\infty,\mathbb{R}_+}^{\frac{(\mu-1)}{(\mu-(1-\beta)(2-\alpha))}}\left(\sup_{a\in\mathbb{R}_+}\left\|{}^{H}\mathbb{D}_{a+}^{\alpha,\beta;\psi}f\right\|_{\infty,\mathbb{R}_+}\right)^{\frac{(1-(1-\beta)(2-\alpha))}{(\mu-(1-\beta)(2-\alpha))}}<+\infty. \tag{35}$$

*Proof.*We have that

$$g_a(x)=I_{a+}^{(1-\beta)(2-\alpha);\psi}f(x)=$$

$$\frac{1}{\Gamma((1-\beta)(2-\alpha))}\int_a^x\psi'(t)(\psi(x)-\psi(t))^{(1-\beta)(2-\alpha)-1}f(t)\,dt,$$

for $x\ge a\in\mathbb{R}_+$.

Hence

$$|g_a(x)|\le\frac{1}{\Gamma((1-\beta)(2-\alpha))}\int_a^x\psi'(t)(\psi(x)-\psi(t))^{(1-\beta)(2-\alpha)-1}|f(t)|\,dt\le$$

$$\frac{\|f\|_{\infty,\mathbb{R}_+}}{\Gamma((1-\beta)(2-\alpha)+1)}(\psi(x)-\psi(a))^{(1-\beta)(2-\alpha)},\quad\forall\,x\ge a. \tag{36}$$

Clearly it holds $g_a(a)=0$.

By Theorem 1 (32), we have

$$g_a(x)-g_{a\psi}^{[1]}(a)(\psi(x)-\psi(a))=I_{a+}^{\mu;\psi}\,{}^{H}\mathbb{D}_{a+}^{\alpha,\beta;\psi}f(x),\quad\forall\,x\ge a. \tag{37}$$

That is

$$g_{a\psi}^{[1]}(a)(\psi(x)-\psi(a))=g_a(x)-I_{a+}^{\mu;\psi}\,{}^{H}\mathbb{D}_{a+}^{\alpha,\beta;\psi}f(x),\quad\forall\,x\ge a. \tag{38}$$

Let $\psi(x_1)-\psi(a)=h$ (where $x_1>a$), then

$$g_{a\psi}^{[1]}(a)h=g_a(x_1)-I_{a+}^{\mu;\psi}\,{}^{H}\mathbb{D}_{a+}^{\alpha,\beta;\psi}f(x_1),$$

$$g_{a\psi}^{[1]}(a)=\frac{1}{h}\left(g_a(x_1)-I_{a+}^{\mu;\psi}\,{}^{H}\mathbb{D}_{a+}^{\alpha,\beta;\psi}f(x_1)\right) \tag{39}$$

and

$$\left|g_{a\psi}^{[1]}(a)\right|=\frac{1}{h}\left|g_a(x_1)-I_{a+}^{\mu;\psi}\,{}^{H}\mathbb{D}_{a+}^{\alpha,\beta;\psi}f(x_1)\right|\le$$

$$\frac{1}{h}\left[|g_a(x_1)|+\frac{1}{\Gamma(\mu)}\int_a^{x_1}\psi'(t)(\psi(x_1)-\psi(t))^{\mu-1}\left|{}^{H}\mathbb{D}_{a+}^{\alpha,\beta;\psi}f(t)\right|dt\right]\le$$

$$\frac{1}{h}\left[\frac{1}{\Gamma((1-\beta)(2-\alpha))}\int_a^{x_1}\psi'(t)(\psi(x_1)-\psi(t))^{(1-\beta)(2-\alpha)-1}|f(t)|dt+\right.$$
$$\left.\left\|{}^{H}\mathbb{D}_{a+}^{\alpha,\beta;\psi}f\right\|_{\infty,[a,+\infty)}\frac{(\psi(x_1)-\psi(a))^{\mu}}{\Gamma(\mu+1)}\right]\leq \quad (40)$$
$$\frac{1}{h}\left[\frac{\|f\|_{\infty,\mathbb{R}_+}}{\Gamma((1-\beta)(2-\alpha)+1)}h^{(1-\beta)(2-\alpha)}+\frac{\left\|{}^{H}\mathbb{D}_{a+}^{\alpha,\beta;\psi}f\right\|_{\infty,[a,+\infty)}h^{\mu}}{\Gamma(\mu+1)}\right].$$

Therefore it holds

$$\left|g_{a\psi}^{[1]}(a)\right|\leq\frac{\|f\|_{\infty,\mathbb{R}_+}}{\Gamma((1-\beta)(2-\alpha)+1)}h^{(1-\beta)(2-\alpha)-1}+\frac{\sup_{a\in\mathbb{R}_+}\left\|{}^{H}\mathbb{D}_{a+}^{\alpha,\beta;\psi}f\right\|_{\infty,[a,+\infty)}}{\Gamma(\mu+1)}h^{\mu-1}, \quad (41)$$

$\forall\, h>0$ and $\forall\, a\in\mathbb{R}_+$.

Thus we derive

$$\sup_{a\in\mathbb{R}_+}\left|g_{a\psi}^{[1]}(a)\right|\leq\frac{\|f\|_{\infty,\mathbb{R}_+}}{\Gamma((1-\beta)(2-\alpha)+1)h^{1-(1-\beta)(2-\alpha)}}+ \quad (42)$$
$$\frac{\sup_{a\in\mathbb{R}_+}\left\|{}^{H}\mathbb{D}_{a+}^{\alpha,\beta;\psi}f\right\|_{\infty,\mathbb{R}_+}}{\Gamma(\mu+1)}h^{\mu-1},\quad \forall\, h>0.$$

Here $\mu-1=1-2\beta+\beta\alpha$. Since $\alpha>1$ we have $\frac{1}{2-\alpha}>1$. But $0\leq\beta\leq1<\frac{1}{2-\alpha}$, hence $\beta<\frac{1}{2-\alpha}$, that is $\beta(\alpha-2)+1>0$, giving $\mu-1>0$.

Next we apply Theorem 2.

Let

$$A:=\frac{\|f\|_{\infty,\mathbb{R}_+}}{\Gamma((1-\beta)(2-\alpha)+1)},\quad \overline{\alpha}=1-(1-\beta)(2-\alpha),$$

and

$$B:=\frac{\sup_{a\in\mathbb{R}_+}\left\|{}^{H}\mathbb{D}_{a+}^{\alpha,\beta;\psi}f\right\|_{\infty,\mathbb{R}_+}}{\Gamma(\mu+1)},\quad \overline{\beta}=\mu-1.$$

All $A,B,\overline{\alpha},\overline{\beta}>0$.

The critical number is $h_0=\left(\frac{\overline{\alpha}A}{\overline{\beta}B}\right)^{\frac{1}{\overline{\alpha}+\overline{\beta}}}: y'(h_0)=0$, where

$$y(h):=\frac{A}{h^{\overline{\alpha}}}+Bh^{\overline{\beta}},\ h>0. \quad (43)$$

The global minimum of y is given by

$$y(h_0)=\frac{\left(\overline{\alpha}+\overline{\beta}\right)}{\overline{\alpha}^{\frac{\overline{\alpha}}{\overline{\alpha}+\overline{\beta}}}\overline{\beta}^{\frac{\overline{\beta}}{\overline{\alpha}+\overline{\beta}}}}A^{\frac{\overline{\beta}}{\overline{\alpha}+\overline{\beta}}}B^{\frac{\overline{\alpha}}{\overline{\alpha}+\overline{\beta}}}. \quad (44)$$

Consequently it holds

$$\sup_{a\in\mathbb{R}_+}\left|g_{a\psi}^{[1]}(a)\right|\leq\frac{(\mu-(1-\beta)(2-\alpha))}{(1-(1-\beta)(2-\alpha))^{\frac{(1-(1-\beta)(2-\alpha))}{(\mu-(1-\beta)(2-\alpha))}}(\mu-1)^{\frac{(\mu-1)}{(\mu-(1-\beta)(2-\alpha))}}} \quad (45)$$
$$\left(\frac{\|f\|_{\infty,\mathbb{R}_+}}{\Gamma((1-\beta)(2-\alpha)+1)}\right)^{\frac{(\mu-1)}{(\mu-(1-\beta)(2-\alpha))}}\left(\frac{\sup_{a\in\mathbb{R}_+}\left\|{}^{H}\mathbb{D}_{a+}^{\alpha,\beta;\psi}f\right\|_{\infty,\mathbb{R}_+}}{\Gamma(\mu+1)}\right)^{\frac{(1-(1-\beta)(2-\alpha))}{(\mu-(1-\beta)(2-\alpha))}}.$$

The claim is proved.

We also give the following theorem.

Theorem 5. *Let* $f,\psi\in C^{3}(\mathbb{R}_{+})$ *with* ψ *being strictly increasing,* $2<\alpha<3$*,* $0\leq\beta\leq1$*,* $\mu=3(1-\beta)+\beta\alpha$*. Assume further that* $g_{a}:=I_{a+}^{(1-\beta)(3-\alpha);\psi}f\in C^{3}([a,+\infty))$*,* $\forall\, a\in\mathbb{R}_{+}$*;* $f\in C_{B}(\mathbb{R}_{+})$ *and* $\sup_{a\in\mathbb{R}_{+}}\left\|{}^{H}\mathbb{D}_{a+}^{\alpha,\beta;\psi}f\right\|_{\infty,\mathbb{R}_{+}}<+\infty$.

Then
1)

$$\sup_{a\in\mathbb{R}_{+}}\left|g_{a\psi}^{[1]}(a)\right|\leq$$

$$\left[\frac{(\mu-(1-\beta)(3-\alpha))}{\left[2\Gamma(\mu+1)(1-(1-\beta)(3-\alpha))\right]^{\frac{(1-(1-\beta)(3-\alpha))}{(\mu-(1-\beta)(3-\alpha))}}\left[2\Gamma((1-\beta)(3-\alpha)+1)(\mu-1)\right]^{\frac{(\mu-1)}{(\mu-(1-\beta)(3-\alpha))}}}\right]$$

$$\left(4+2^{(1-\beta)(3-\alpha)}\right)^{\frac{(\mu-1)}{(\mu-(1-\beta)(3-\alpha))}}\left(4+2^{\mu}\right)^{\frac{(1-(1-\beta)(3-\alpha))}{(\mu-(1-\beta)(3-\alpha))}}$$

$$\|f\|_{\infty,\mathbb{R}_{+}}^{\frac{(\mu-1)}{(\mu-(1-\beta)(3-\alpha))}}\left(\sup_{a\in\mathbb{R}_{+}}\left\|{}^{H}\mathbb{D}_{a+}^{\alpha,\beta;\psi}f\right\|_{\infty,\mathbb{R}_{+}}\right)^{\frac{(1-(1-\beta)(3-\alpha))}{(\mu-(1-\beta)(3-\alpha))}}<+\infty,\tag{46}$$

and
2)

$$\sup_{a\in\mathbb{R}_{+}}\left|g_{a\psi}^{[2]}(a)\right|\leq$$

$$\left[\frac{(\mu-(1-\beta)(3-\alpha))}{\left[\Gamma(\mu+1)(2-(1-\beta)(3-\alpha))\right]^{\frac{(2-(1-\beta)(3-\alpha))}{(\mu-(1-\beta)(3-\alpha))}}\left[\Gamma((1-\beta)(3-\alpha)+1)(\mu-2)\right]^{\frac{(\mu-2)}{(\mu-(1-\beta)(3-\alpha))}}}\right]$$

$$\left(2+2^{(1-\beta)(3-\alpha)}\right)^{\frac{(\mu-2)}{(\mu-(1-\beta)(3-\alpha))}}\left(2+2^{\mu}\right)^{\frac{(2-(1-\beta)(3-\alpha))}{(\mu-(1-\beta)(3-\alpha))}}$$

$$\|f\|_{\infty,\mathbb{R}_{+}}^{\frac{(\mu-2)}{(\mu-(1-\beta)(3-\alpha))}}\left(\sup_{a\in\mathbb{R}_{+}}\left\|{}^{H}\mathbb{D}_{a+}^{\alpha,\beta;\psi}f\right\|_{\infty,\mathbb{R}_{+}}\right)^{\frac{(2-(1-\beta)(3-\alpha))}{(\mu-(1-\beta)(3-\alpha))}}<+\infty.\tag{47}$$

Proof. Similarly, here it holds $g_{a}(a)=0$. By Theorem 1 (32), we have

$$g_{a}(x)-g_{a\psi}^{[1]}(a)(\psi(x)-\psi(a))-g_{a\psi}^{[2]}(a)\frac{(\psi(x)-\psi(a))^{2}}{2}=I_{a+}^{\mu;\psi}\,{}^{H}\mathbb{D}_{a+}^{\alpha,\beta;\psi}f(x),\tag{48}$$

$\forall\, x\geq a\in\mathbb{R}_{+}$.

That is

$$g_{a\psi}^{[1]}(a)(\psi(x)-\psi(a))+g_{a\psi}^{[2]}(a)\frac{(\psi(x)-\psi(a))^{2}}{2}=g_{a}(x)-I_{a+}^{\mu;\psi}\,{}^{H}\mathbb{D}_{a+}^{\alpha,\beta;\psi}f(x),\tag{49}$$

$\forall\, x\geq a$.

Let $\psi(x_{1})-\psi(a)=h>0$ (where $x_{1}>a$), then

$$g_{a\psi}^{[1]}(a)h+g_{a\psi}^{[2]}(a)\frac{h^{2}}{2}=g_{a}(x_{1})-I_{a+}^{\mu;\psi}\,{}^{H}\mathbb{D}_{a+}^{\alpha,\beta;\psi}f(x_{1})=:A_{1}.\tag{50}$$

Let $\psi(x_{2})-\psi(a)=2h$ (where $x_{2}>a$), then

$$g_{a\psi}^{[1]}(a)2h+g_{a\psi}^{[2]}(a)2h^{2}=g_{a}(x_{2})-I_{a+}^{\mu;\psi}\,{}^{H}\mathbb{D}_{a+}^{\alpha,\beta;\psi}f(x_{2})=:A_{2}.\tag{51}$$

So we solve the system

$$2g_{a\psi}^{[1]}(a)h+g_{a\psi}^{[2]}(a)h^{2}=2A_{1},\tag{52}$$

$$2g_{a\psi}^{[1]}(a)h+2g_{a\psi}^{[2]}(a)h^2=A_2. \tag{53}$$

We get that

$$g_{a\psi}^{[2]}(a)h^2=A_2-2A_1,$$

i.e.

$$g_{a\psi}^{[2]}(a)=\frac{A_2-2A_1}{h^2}, \tag{54}$$

and

$$g_{a\psi}^{[1]}(a)=\frac{4A_1-A_2}{2h}. \tag{55}$$

We estimate

$$\left|g_{a\psi}^{[1]}(a)\right|=\frac{1}{2h}|4A_1-A_2|\leq\frac{1}{2h}(4|A_1|+|A_2|)=$$

$$\frac{1}{2h}\left[4\left|g_a(x_1)-I_{a+}^{\mu;\psi}\,{}^H\mathbb{D}_{a+}^{\alpha,\beta;\psi}f(x_1)\right|+\left|g_a(x_2)-I_{a+}^{\mu;\psi}\,{}^H\mathbb{D}_{a+}^{\alpha,\beta;\psi}f(x_2)\right|\right]\leq \tag{56}$$

$$\frac{1}{2h}\left[4\left[|g_a(x_1)|+I_{a+}^{\mu;\psi}\left|{}^H\mathbb{D}_{a+}^{\alpha,\beta;\psi}f\right|(x_1)\right]+\left[|g_a(x_2)|+I_{a+}^{\mu;\psi}\left|{}^H\mathbb{D}_{a+}^{\alpha,\beta;\psi}f\right|(x_2)\right]\right]\leq$$

$$\frac{1}{2h}\left[4\left[\frac{1}{\Gamma((1-\beta)(3-\alpha))}\int_a^{x_1}\psi'(t)(\psi(x_1)-\psi(a))^{(1-\beta)(3-\alpha)-1}|f(t)|dt+\right.\right.$$

$$\left.\frac{1}{\Gamma(\mu)}\int_a^{x_1}\psi'(t)(\psi(x_1)-\psi(a))^{\mu-1}\left|{}^H\mathbb{D}_{a+}^{\alpha,\beta;\psi}f(t)\right|dt\right]+$$

$$\left[\frac{1}{\Gamma((1-\beta)(3-\alpha))}\int_a^{x_2}\psi'(t)(\psi(x_2)-\psi(a))^{(1-\beta)(3-\alpha)-1}|f(t)|dt+\right.$$

$$\left.\left.\frac{1}{\Gamma(\mu)}\int_a^{x_2}\psi'(t)(\psi(x_2)-\psi(a))^{\mu-1}\left|{}^H\mathbb{D}_{a+}^{\alpha,\beta;\psi}f(t)\right|dt\right]\right]\leq$$

$$\frac{1}{2h}\left[4\left[\frac{\|f\|_{\infty,\mathbb{R}_+}}{\Gamma((1-\beta)(3-\alpha)+1)}h^{(1-\beta)(3-\alpha)}+\frac{\left\|{}^H\mathbb{D}_{a+}^{\alpha,\beta;\psi}f\right\|_{\infty,[a,+\infty)}}{\Gamma(\mu+1)}h^{\mu}\right]+\right. \tag{57}$$

$$\left.\left[\frac{\|f\|_{\infty,\mathbb{R}_+}}{\Gamma((1-\beta)(3-\alpha)+1)}(2h)^{(1-\beta)(3-\alpha)}+\frac{\left\|{}^H\mathbb{D}_{a+}^{\alpha,\beta;\psi}f\right\|_{\infty,[a,+\infty)}}{\Gamma(\mu+1)}(2h)^{\mu}\right]\right]=$$

$$\frac{1}{2h}\left[\frac{\|f\|_{\infty,\mathbb{R}_+}}{\Gamma((1-\beta)(3-\alpha)+1)}\left(4+2^{(1-\beta)(3-\alpha)}\right)h^{(1-\beta)(3-\alpha)}+\right. \tag{58}$$

$$\left.\frac{\left\|{}^H\mathbb{D}_{a+}^{\alpha,\beta;\psi}f\right\|_{\infty,[a,+\infty)}}{\Gamma(\mu+1)}(4+2^{\mu})h^{\mu}\right]=$$

$$\frac{\|f\|_{\infty,\mathbb{R}_+}\left(4+2^{(1-\beta)(3-\alpha)}\right)}{2\Gamma((1-\beta)(3-\alpha)+1)h^{1-(1-\beta)(3-\alpha)}}+\left(\frac{\left\|{}^H\mathbb{D}_{a+}^{\alpha,\beta;\psi}f\right\|_{\infty,[a,+\infty)}(4+2^{\mu})}{2\Gamma(\mu+1)}\right)h^{\mu-1},$$

$\forall\, a\in\mathbb{R}_+$, $\forall\, h>0$.

Consequently, we get

$$\sup_{a\in\mathbb{R}_+}\left|g_{a\psi}^{[1]}(a)\right|\leq\left(\frac{\|f\|_{\infty,\mathbb{R}_+}\left(4+2^{(1-\beta)(3-\alpha)}\right)}{2\Gamma((1-\beta)(3-\alpha)+1)}\right)\frac{1}{h^{1-(1-\beta)(3-\alpha)}} \tag{59}$$

$$+\left(\frac{\sup_{a\in\mathbb{R}_+}\left\|{}^{H}\mathbb{D}_{a+}^{\alpha,\beta;\psi}f\right\|_{\infty,[a,+\infty)}\left(4+2^{\mu}\right)}{2\Gamma\left(\mu+1\right)}\right)h^{\mu-1},\ \ \forall\, h>0.$$

Notice here that $1-(1-\beta)(3-\alpha)>0$, and since $\alpha>1$, then $2>3-\alpha$, $\frac{2}{3-\alpha}>1$, $\beta<\frac{2}{3-\alpha}$, $2>\beta(3-\alpha)$, and finally we have $\mu-1>0$.

Next we apply Theorem 2 to (59). Here it is

$$A:=\left(\frac{\|f\|_{\infty,\mathbb{R}_+}\left(4+2^{(1-\beta)(3-\alpha)}\right)}{2\Gamma\left((1-\beta)(3-\alpha)+1\right)}\right),\ \ \overline{\alpha}=1-(1-\beta)(3-\alpha),$$
and
$$B:=\left(\frac{\sup_{a\in\mathbb{R}_+}\left\|{}^{H}\mathbb{D}_{a+}^{\alpha,\beta;\psi}f\right\|_{\infty,\mathbb{R}_+}\left(4+2^{\mu}\right)}{2\Gamma(\mu+1)}\right),\ \ \overline{\beta}=\mu-1. \tag{60}$$

It is all $A,B,\overline{\alpha},\overline{\beta}>0$.

Therefore we get

$$\sup_{a\in\mathbb{R}_+}\left|g_{a\psi}^{[1]}(a)\right|\leq\frac{(\mu-(1-\beta)(3-\alpha))}{(1-(1-\beta)(3-\alpha))^{\frac{(1-(1-\beta)(3-\alpha))}{(\mu-(1-\beta)(3-\alpha))}}(\mu-1)^{\frac{(\mu-1)}{(\mu-(1-\beta)(3-\alpha))}}} \tag{61}$$
$$\left(\frac{\|f\|_{\infty,\mathbb{R}_+}\left(4+2^{(1-\beta)(3-\alpha)}\right)}{2\Gamma\left((1-\beta)(3-\alpha)+1\right)}\right)^{\frac{(\mu-1)}{(\mu-(1-\beta)(3-\alpha))}}$$
$$\left(\frac{\sup_{a\in\mathbb{R}_+}\left\|{}^{H}\mathbb{D}_{a+}^{\alpha,\beta;\psi}f\right\|_{\infty,\mathbb{R}_+}\left(4+2^{\mu}\right)}{2\Gamma\left(\mu+1\right)}\right)^{\frac{(1-(1-\beta)(3-\alpha))}{(\mu-(1-\beta)(3-\alpha))}},$$

proving (46).

We also have

$$\left|g_{a\psi}^{[2]}(a)\right|\leq\frac{1}{h^2}\left(|A_2|+2|A_1|\right)=$$
$$\frac{1}{h^2}\left[\left[|g_a(x_2)|+I_{a+}^{\mu;\psi}\left|{}^{H}\mathbb{D}_{a+}^{\alpha,\beta;\psi}f\right|(x_2)\right]+2\left[|g_a(x_1)|+I_{a+}^{\mu;\psi}\left|{}^{H}\mathbb{D}_{a+}^{\alpha,\beta;\psi}f\right|(x_1)\right]\right]\leq$$
$$\frac{1}{h^2}\left[\left[\frac{\|f\|_{\infty,\mathbb{R}_+}}{\Gamma\left((1-\beta)(3-\alpha)+1\right)}(2h)^{(1-\beta)(3-\alpha)}+\frac{\left\|{}^{H}\mathbb{D}_{a+}^{\alpha,\beta;\psi}f\right\|_{\infty,[a,+\infty)}}{\Gamma(\mu+1)}(2h)^{\mu}\right]+\right. \tag{62}$$
$$\left.2\left[\frac{\|f\|_{\infty,\mathbb{R}_+}}{\Gamma\left((1-\beta)(3-\alpha)+1\right)}h^{(1-\beta)(3-\alpha)}+\frac{\left\|{}^{H}\mathbb{D}_{a+}^{\alpha,\beta;\psi}f\right\|_{\infty,[a,+\infty)}}{\Gamma(\mu+1)}h^{\mu}\right]\right]=$$
$$\frac{1}{h^2}\left[\frac{\|f\|_{\infty,\mathbb{R}_+}}{\Gamma\left((1-\beta)(3-\alpha)+1\right)}\left(2+2^{(1-\beta)(3-\alpha)}\right)h^{(1-\beta)(3-\alpha)}+\right.$$
$$\left.\frac{\left\|{}^{H}\mathbb{D}_{a+}^{\alpha,\beta;\psi}f\right\|_{\infty,[a,+\infty)}}{\Gamma(\mu+1)}\left(2+2^{\mu}\right)h^{\mu}\right]\leq$$
$$\left(\frac{\|f\|_{\infty,\mathbb{R}_+}\left(2+2^{(1-\beta)(3-\alpha)}\right)}{\Gamma\left((1-\beta)(3-\alpha)+1\right)}\right)\frac{1}{h^{2-(1-\beta)(3-\alpha)}}+ \tag{63}$$

$$\left(\frac{\sup_{a\in\mathbb{R}_+}\left\|{}^{H}\mathbb{D}_{a+}^{\alpha,\beta;\psi}f\right\|_{\infty,[a,+\infty)}\left(2+2^{\mu}\right)}{2\Gamma(\mu+1)}\right)h^{\mu-2},$$

$\forall\, a\in\mathbb{R}_+$, $\forall\, h>0$.

That is

$$\sup_{a\in\mathbb{R}_+}\left|g_{a\psi}^{[2]}(a)\right|\leq\left(\frac{\|f\|_{\infty,\mathbb{R}_+}\left(2+2^{(1-\beta)(3-\alpha)}\right)}{\Gamma((1-\beta)(3-\alpha)+1)}\right)\frac{1}{h^{2-(1-\beta)(3-\alpha)}}$$
$$+\left(\frac{\sup_{a\in\mathbb{R}_+}\left\|{}^{H}\mathbb{D}_{a+}^{\alpha,\beta;\psi}f\right\|_{\infty,\mathbb{R}_+}\left(2+2^{\mu}\right)}{\Gamma(\mu+1)}\right)h^{\mu-2},\quad\forall\, h>0. \tag{64}$$

Notice here that $2-(1-\beta)(3-\alpha)>0$, and for $\beta\neq 0$ we have $\mu=3(1-\beta)+\beta\alpha>3(1-\beta)+2\beta=3-\beta$, and $\mu-2>3-\beta-2=1-\beta\geq 0$, hence $\mu-2>0$; if $\beta=0$, $\mu-2=1>0$.

So we can apply Theorem 2 to (64). Call

$$A^{*}:=\left(\frac{\|f\|_{\infty,\mathbb{R}_+}\left(2+2^{(1-\beta)(3-\alpha)}\right)}{\Gamma((1-\beta)(3-\alpha)+1)}\right),\quad \alpha^{*}=2-(1-\beta)(3-\alpha),$$

and

$$B^{*}:=\left(\frac{\sup_{a\in\mathbb{R}_+}\left\|{}^{H}\mathbb{D}_{a+}^{\alpha,\beta;\psi}f\right\|_{\infty,\mathbb{R}_+}\left(2+2^{\mu}\right)}{\Gamma(\mu+1)}\right),\quad \beta^{*}=\mu-2. \tag{65}$$

All $A^{*},B^{*},\alpha^{*},\beta^{*}>0$.

We will have that

$$\sup_{a\in\mathbb{R}_+}\left|g_{a\psi}^{[2]}(a)\right|\leq\frac{(\mu-(1-\beta)(3-\alpha))}{(2-(1-\beta)(3-\alpha))^{\frac{(2-(1-\beta)(3-\alpha))}{(\mu-(1-\beta)(3-\alpha))}}(\mu-2)^{\frac{(\mu-2)}{(\mu-(1-\beta)(3-\alpha))}}} \tag{66}$$
$$\left(\frac{\|f\|_{\infty,\mathbb{R}_+}\left(2+2^{(1-\beta)(3-\alpha)}\right)}{\Gamma((1-\beta)(3-\alpha)+1)}\right)^{\frac{(\mu-2)}{(\mu-(1-\beta)(3-\alpha))}}$$
$$\left(\frac{\sup_{a\in\mathbb{R}_+}\left\|{}^{H}\mathbb{D}_{a+}^{\alpha,\beta;\psi}f\right\|_{\infty,\mathbb{R}_+}\left(2+2^{\mu}\right)}{\Gamma(\mu+1)}\right)^{\frac{(2-(1-\beta)(3-\alpha))}{(\mu-(1-\beta)(3-\alpha))}},$$

proving (47).

We present

Theorem 6. *Let $f,\psi\in C^{2}(\mathbb{R}_-)$, with ψ being strictly increasing, $1<\alpha<2$, $0\leq\beta\leq 1$, $\mu=2(1-\beta)+\beta\alpha$. Assume further that $w_b:=I_{b-}^{(1-\beta)(2-\alpha);\psi}f\in C^{2}((-\infty,b])$, $\forall\, b\in\mathbb{R}_-$; $f\in C_B(\mathbb{R}_-)$ and $\sup_{b\in\mathbb{R}_-}\left\|{}^{H}\mathbb{D}_{b-}^{\alpha,\beta;\psi}f\right\|_{\infty,\mathbb{R}_-}<+\infty$.*

Then

$$\sup_{b\in\mathbb{R}_-}\left|w_{b\psi}^{[1]}(b)\right|\leq$$
$$\left[\frac{(\mu-(1-\beta)(2-\alpha))}{\left[\Gamma(\mu+1)(1-(1-\beta)(2-\alpha))\right]^{\frac{(1-(1-\beta)(2-\alpha))}{(\mu-(1-\beta)(2-\alpha))}}\left[(\mu-1)\Gamma((1-\beta)(2-\alpha)+1)\right]^{\frac{(\mu-1)}{(\mu-(1-\beta)(2-\alpha))}}}\right]$$
$$\|f\|_{\infty,\mathbb{R}_-}^{\frac{(\mu-1)}{(\mu-(1-\beta)(2-\alpha))}}\left(\sup_{b\in\mathbb{R}_-}\left\|{}^{H}\mathbb{D}_{b-}^{\alpha,\beta;\psi}f\right\|_{\infty,\mathbb{R}_-}\right)^{\frac{(1-(1-\beta)(2-\alpha))}{(\mu-(1-\beta)(2-\alpha))}}<+\infty. \tag{67}$$

Proof. We have that

$$w_b(x) = I_{b-}^{(1-\beta)(2-\alpha);\psi} f(x) =$$

$$\frac{1}{\Gamma((1-\beta)(2-\alpha))} \int_x^b \psi'(t)(\psi(t)-\psi(x))^{(1-\beta)(2-\alpha)-1} f(t)\,dt,$$

for $x \leq b \in \mathbb{R}_-$.

Hence

$$|w_b(x)| \leq \frac{1}{\Gamma((1-\beta)(2-\alpha))} \int_x^b \psi'(t)(\psi(t)-\psi(x))^{(1-\beta)(2-\alpha)-1} |f(t)|\,dt \leq$$

$$\frac{\|f\|_{\infty,\mathbb{R}_-}}{\Gamma((1-\beta)(2-\alpha)+1)} (\psi(b)-\psi(x))^{(1-\beta)(2-\alpha)}, \quad \forall\, x \leq b. \tag{68}$$

Clearly it holds $w_b(b) = 0$.

By Theorem 1 (33), we have

$$w_b(x) - w_{b\psi}^{[1]}(b)(\psi(x)-\psi(b)) = I_{b-}^{\mu;\psi}\, {}^H\mathbb{D}_{b-}^{\alpha,\beta;\psi} f(x), \quad \forall\, x \leq b. \tag{69}$$

That is

$$w_{b\psi}^{[1]}(b)(\psi(x)-\psi(b)) = w_b(x) - I_{b-}^{\mu;\psi}\, {}^H\mathbb{D}_{b-}^{\alpha,\beta;\psi} f(x), \quad \forall\, x \leq b. \tag{70}$$

Let $\psi(b) - \psi(x_1) = h > 0$ $(x_1 < b)$, then

$$-w_{b\psi}^{[1]}(b)\, h = w_b(x_1) - I_{b-}^{\mu;\psi}\, {}^H\mathbb{D}_{b-}^{\alpha,\beta;\psi} f(x_1), \tag{71}$$

and

$$\left| w_{b\psi}^{[1]}(b) \right| = \frac{1}{h} \left| w_b(x_1) - I_{b-}^{\mu;\psi}\, {}^H\mathbb{D}_{b-}^{\alpha,\beta;\psi} f(x_1) \right| \leq$$

$$\frac{1}{h}\left[|w_b(x_1)| + I_{b-}^{\mu;\psi} \left| {}^H\mathbb{D}_{b-}^{\alpha,\beta;\psi} f \right| (x_1) \right] \leq$$

$$\frac{1}{h}\left[\frac{1}{\Gamma((1-\beta)(2-\alpha))} \int_{x_1}^b \psi'(t)(\psi(t)-\psi(x_1))^{(1-\beta)(2-\alpha)-1} |f(t)|\,dt + \right.$$

$$\left. \frac{1}{\Gamma(\mu)} \int_{x_1}^b \psi'(t)(\psi(t)-\psi(x_1))^{\mu-1} \left| {}^H\mathbb{D}_{b-}^{\alpha,\beta;\psi} f(t) \right| dt \right] \leq$$

$$\frac{1}{h}\left[\frac{\|f\|_{\infty,\mathbb{R}_-}}{\Gamma((1-\beta)(2-\alpha)+1)} (\psi(b)-\psi(x_1))^{(1-\beta)(2-\alpha)} + \right.$$

$$\left. \frac{\left\| {}^H\mathbb{D}_{b-}^{\alpha,\beta;\psi} f \right\|_{\infty,(-\infty,b]}}{\Gamma(\mu+1)} (\psi(b)-\psi(x_1))^{\mu} \right] = \tag{72}$$

$$\frac{1}{h}\left[\frac{\|f\|_{\infty,\mathbb{R}_-}}{\Gamma((1-\beta)(2-\alpha)+1)} h^{(1-\beta)(2-\alpha)} + \frac{\left\| {}^H\mathbb{D}_{b-}^{\alpha,\beta;\psi} f \right\|_{\infty,(-\infty,b]} h^{\mu}}{\Gamma(\mu+1)} \right].$$

Therefore it holds

$$\left| w_{b\psi}^{[1]}(b) \right| \leq \frac{\|f\|_{\infty,\mathbb{R}_-}}{\Gamma((1-\beta)(2-\alpha)+1)} h^{(1-\beta)(2-\alpha)-1} + \frac{\sup_{b\in\mathbb{R}_-} \left\| {}^H\mathbb{D}_{b-}^{\alpha,\beta;\psi} f \right\|_{\infty,(-\infty,b]}}{\Gamma(\mu+1)} h^{\mu-1}, \tag{73}$$

$\forall\, h > 0$ and $\forall\, b \in \mathbb{R}_-$.

Thus we derive

$$\sup_{b\in\mathbb{R}_-} \left| w_{b\psi}^{[1]}(b) \right| \leq \left(\frac{\|f\|_{\infty,\mathbb{R}_-}}{\Gamma((1-\beta)(2-\alpha)+1)} \right) \frac{1}{h^{1-(1-\beta)(2-\alpha)}} +$$

$$\left(\frac{\sup_{b\in\mathbb{R}_-}\left\|{}^{H}\mathbb{D}_{b-}^{\alpha,\beta;\psi}f\right\|_{\infty,\mathbb{R}_-}}{\Gamma(\mu+1)}\right)h^{\mu-1},\quad \forall\, h>0. \tag{74}$$

The rest of the proof comes by application of Theorem 2, as in the proof of Theorem 4.

We finish with

Theorem 7. *Let $f,\psi\in C^3(\mathbb{R}_-)$, with ψ being strictly increasing, $2<\alpha<3$, $0\le\beta\le 1$, $\mu=3(1-\beta)+\beta\alpha$. Assume further that $w_b:=I_{b-}^{(1-\beta)(3-\alpha);\psi}f\in C^3((-\infty,b])$, $\forall\, b\in\mathbb{R}_-$; $f\in C_B(\mathbb{R}_-)$ and $\sup_{b\in\mathbb{R}_-}\left\|{}^{H}\mathbb{D}_{b-}^{\alpha,\beta;\psi}f\right\|_{\infty,\mathbb{R}_-}<+\infty$.*

Then

1)

$$\sup_{b\in\mathbb{R}_-}\left|w_{b\psi}^{[1]}(b)\right|\le$$

$$\left[\frac{(\mu-(1-\beta)(3-\alpha))}{[2\Gamma(\mu+1)(1-(1-\beta)(3-\alpha))]^{\frac{(1-(1-\beta)(3-\alpha))}{(\mu-(1-\beta)(3-\alpha))}}[2\Gamma((1-\beta)(3-\alpha)+1)(\mu-1)]^{\frac{(\mu-1)}{(\mu-(1-\beta)(3-\alpha))}}}\right]$$

$$\left(4+2^{(1-\beta)(3-\alpha)}\right)^{\frac{(\mu-1)}{(\mu-(1-\beta)(3-\alpha))}}\left(4+2^{\mu}\right)^{\frac{(1-(1-\beta)(3-\alpha))}{(\mu-(1-\beta)(3-\alpha))}}$$

$$\|f\|_{\infty,\mathbb{R}_-}^{\frac{(\mu-1)}{(\mu-(1-\beta)(3-\alpha))}}\left(\sup_{b\in\mathbb{R}_-}\left\|{}^{H}\mathbb{D}_{b-}^{\alpha,\beta;\psi}f\right\|_{\infty,\mathbb{R}_-}\right)^{\frac{(1-(1-\beta)(3-\alpha))}{(\mu-(1-\beta)(3-\alpha))}}<+\infty, \tag{75}$$

and

2)

$$\sup_{b\in\mathbb{R}_-}\left|w_{b\psi}^{[2]}(b)\right|\le$$

$$\left[\frac{(\mu-(1-\beta)(3-\alpha))}{[\Gamma(\mu+1)(2-(1-\beta)(3-\alpha))]^{\frac{(2-(1-\beta)(3-\alpha))}{(\mu-(1-\beta)(3-\alpha))}}[\Gamma((1-\beta)(3-\alpha)+1)(\mu-2)]^{\frac{(\mu-2)}{(\mu-(1-\beta)(3-\alpha))}}}\right]$$

$$\left(2+2^{(1-\beta)(3-\alpha)}\right)^{\frac{(\mu-2)}{(\mu-(1-\beta)(3-\alpha))}}\left(2+2^{\mu}\right)^{\frac{(2-(1-\beta)(3-\alpha))}{(\mu-(1-\beta)(3-\alpha))}}$$

$$\|f\|_{\infty,\mathbb{R}_-}^{\frac{(\mu-2)}{(\mu-(1-\beta)(3-\alpha))}}\left(\sup_{b\in\mathbb{R}_-}\left\|{}^{H}\mathbb{D}_{b-}^{\alpha,\beta;\psi}f\right\|_{\infty,\mathbb{R}_-}\right)^{\frac{(2-(1-\beta)(3-\alpha))}{(\mu-(1-\beta)(3-\alpha))}}<+\infty. \tag{76}$$

Proof. Similarly, here it holds $w_b(b)=0$. By Theorem 1 (33), we have

$$w_b(x)-w_{b\psi}^{[1]}(b)(\psi(x)-\psi(b))-w_{b\psi}^{[2]}(b)\frac{(\psi(x)-\psi(b))^2}{2}=I_{b-}^{\mu;\psi}\,{}^{H}\mathbb{D}_{b-}^{\alpha,\beta;\psi}f(x), \tag{77}$$

$\forall\, x\le b\in\mathbb{R}_-$.

That is

$$w_{b\psi}^{[1]}(b)(\psi(x)-\psi(b))+w_{b\psi}^{[2]}(b)\frac{(\psi(x)-\psi(b))^2}{2}=w_b(x)-I_{b-}^{\mu;\psi}\,{}^{H}\mathbb{D}_{b-}^{\alpha,\beta;\psi}f(x), \tag{78}$$

$\forall\, x\le b$.

Let $\psi(b)-\psi(x_1)=h>0$ $(x_1<b)$, then

$$\left(-w_{b\psi}^{[1]}(b)\right)h+w_{b\psi}^{[2]}(b)\frac{h^2}{2}=w_b(x_1)-I_{b-}^{\mu;\psi}\,{}^{H}\mathbb{D}_{b-}^{\alpha,\beta;\psi}f(x_1)=:B_1. \tag{79}$$

Let also $\psi(b)-\psi(x_2)=2h$ $(x_2<b)$, then

$$2\left(-w_{b\psi}^{[1]}(b)\right)h+2w_{b\psi}^{[2]}(b)h^2=w_b(x_2)-I_{b-}^{\mu;\psi}\,{}^{H}\mathbb{D}_{b-}^{\alpha,\beta;\psi}f(x_2)=:B_2. \tag{80}$$

So we solve the system

$$\begin{aligned}2\left(-w_{b\psi}^{[1]}(b)\right)h+w_{b\psi}^{[2]}(b)h^2&=2B_1,\\ 2\left(-w_{b\psi}^{[1]}(b)\right)h+2w_{b\psi}^{[2]}(b)h^2&=B_2.\end{aligned} \tag{81}$$

We get that

$$w_{b\psi}^{[2]}(b)h^2=B_2-2B_1,$$

i.e.

$$w_{b\psi}^{[2]}(b)=\frac{B_2-2B_1}{h^2}, \tag{82}$$

and

$$-w_{b\psi}^{[1]}(b)=\frac{4B_1-B_2}{2h}. \tag{83}$$

Consequently, we get that

$$\left|w_{b\psi}^{[1]}(b)\right|\leq\frac{1}{2h}\left(4|B_1|+|B_2|\right), \tag{84}$$

and

$$\left|w_{b\psi}^{[2]}(b)\right|\leq\frac{1}{h^2}\left(|B_2|+2|B_1|\right). \tag{85}$$

Therefore we have

$$\begin{aligned}&\left|w_{b\psi}^{[1]}(b)\right|\leq\\ &\frac{1}{2h}\Bigg[4\Bigg[\frac{1}{\Gamma((1-\beta)(3-\alpha))}\int_{x_1}^{b}\psi'(t)(\psi(t)-\psi(x_1))^{(1-\beta)(3-\alpha)-1}|f(t)|dt+\\ &\frac{1}{\Gamma(\mu)}\int_{x_1}^{b}\psi'(t)(\psi(t)-\psi(x_1))^{\mu-1}\left|{}^{H}\mathbb{D}_{b-}^{\alpha,\beta;\psi}f(t)\right|dt\Bigg]+\\ &\Bigg[\frac{1}{\Gamma((1-\beta)(3-\alpha))}\int_{x_2}^{b}\psi'(t)(\psi(t)-\psi(x_2))^{(1-\beta)(3-\alpha)-1}|f(t)|dt+\\ &\frac{1}{\Gamma(\mu)}\int_{x_2}^{b}\psi'(t)(\psi(t)-\psi(x_2))^{\mu-1}\left|{}^{H}\mathbb{D}_{b-}^{\alpha,\beta;\psi}f(t)\right|dt\Bigg]\Bigg]\leq\\ &\frac{1}{2h}\Bigg[4\Bigg[\frac{\|f\|_{\infty,\mathbb{R}_-}}{\Gamma((1-\beta)(3-\alpha)+1)}h^{(1-\beta)(3-\alpha)}+\frac{\left\|{}^{H}\mathbb{D}_{b-}^{\alpha,\beta;\psi}f\right\|_{\infty,\mathbb{R}_-}}{\Gamma(\mu+1)}h^{\mu}\Bigg]+\\ &\Bigg[\frac{\|f\|_{\infty,\mathbb{R}_-}}{\Gamma((1-\beta)(3-\alpha)+1)}(2h)^{(1-\beta)(3-\alpha)}+\frac{\left\|{}^{H}\mathbb{D}_{b-}^{\alpha,\beta;\psi}f\right\|_{\infty,\mathbb{R}_-}}{\Gamma(\mu+1)}(2h)^{\mu}\Bigg]\Bigg]=\\ &\frac{1}{2h}\Bigg[\frac{\|f\|_{\infty,\mathbb{R}_-}}{\Gamma((1-\beta)(3-\alpha)+1)}\left(4+2^{(1-\beta)(3-\alpha)}\right)h^{(1-\beta)(3-\alpha)}+\\ &\frac{\left\|{}^{H}\mathbb{D}_{b-}^{\alpha,\beta;\psi}f\right\|_{\infty,\mathbb{R}_-}}{\Gamma(\mu+1)}(4+2^{\mu})h^{\mu}\Bigg]\leq\\ &\frac{\|f\|_{\infty,\mathbb{R}_-}\left(4+2^{(1-\beta)(3-\alpha)}\right)}{2\Gamma((1-\beta)(3-\alpha)+1)}\frac{1}{h^{1-(1-\beta)(3-\alpha)}}+\frac{\sup\limits_{b\in\mathbb{R}_-}\left\|{}^{H}\mathbb{D}_{b-}^{\alpha,\beta;\psi}f\right\|_{\infty,\mathbb{R}_-}(4+2^{\mu})}{2\Gamma(\mu+1)}h^{\mu-1},\end{aligned} \tag{86}$$

$\forall\, b \in \mathbb{R}_-, \forall\, h > 0.$

Therefore it holds

$$\sup_{b\in\mathbb{R}_-}\left|w_{b\psi}^{[1]}(b)\right| \le \left(\frac{\|f\|_{\infty,\mathbb{R}_-}\left(4+2^{(1-\beta)(3-\alpha)}\right)}{2\Gamma((1-\beta)(3-\alpha)+1)}\right)\frac{1}{h^{1-(1-\beta)(3-\alpha)}} + \left(\frac{\sup\limits_{b\in\mathbb{R}_-}\left\|{}^{H}\mathbb{D}_{b-}^{\alpha,\beta;\psi}f\right\|_{\infty,\mathbb{R}_-}(4+2^{\mu})}{2\Gamma(\mu+1)}\right)h^{\mu-1},\ \ \forall\, h>0. \tag{87}$$

Next we apply Theorem 2 to (87), to derive (75).

Finally, we estimate

$$\left|w_{b\psi}^{[2]}(b)\right| \le \frac{1}{h^2}\left[\left[\frac{\|f\|_{\infty,\mathbb{R}_-}}{\Gamma((1-\beta)(3-\alpha)+1)}(2h)^{(1-\beta)(3-\alpha)} + \frac{\left\|{}^{H}\mathbb{D}_{b-}^{\alpha,\beta;\psi}f\right\|_{\infty,\mathbb{R}_-}}{\Gamma(\mu+1)}(2h)^{\mu}\right] + 2\left[\frac{\|f\|_{\infty,\mathbb{R}_-}}{\Gamma((1-\beta)(3-\alpha)+1)}h^{(1-\beta)(3-\alpha)} + \frac{\left\|{}^{H}\mathbb{D}_{b-}^{\alpha,\beta;\psi}f\right\|_{\infty,\mathbb{R}_-}}{\Gamma(\mu+1)}h^{\mu}\right]\right] =$$

$$\left(\frac{\|f\|_{\infty,\mathbb{R}_-}\left(2+2^{(1-\beta)(3-\alpha)}\right)}{\Gamma((1-\beta)(3-\alpha)+1)}\right)\frac{1}{h^{2-(1-\beta)(3-\alpha)}} + \left(\frac{\sup\limits_{b\in\mathbb{R}_-}\left\|{}^{H}\mathbb{D}_{b-}^{\alpha,\beta;\psi}f\right\|_{\infty,\mathbb{R}_-}(2+2^{\mu})}{\Gamma(\mu+1)}\right)h^{\mu-2}, \tag{88}$$

$\forall\, b \in \mathbb{R}_-, \forall\, h > 0.$

Therefore it holds

$$\sup_{b\in\mathbb{R}_-}\left|w_{b\psi}^{[2]}(b)\right| \le \left(\frac{\|f\|_{\infty,\mathbb{R}_-}\left(2+2^{(1-\beta)(3-\alpha)}\right)}{\Gamma((1-\beta)(3-\alpha)+1)}\right)\frac{1}{h^{2-(1-\beta)(3-\alpha)}} + \left(\frac{\sup\limits_{b\in\mathbb{R}_-}\left\|{}^{H}\mathbb{D}_{b-}^{\alpha,\beta;\psi}f\right\|_{\infty,\mathbb{R}_-}(2+2^{\mu})}{\Gamma(\mu+1)}\right)h^{\mu-2},\ \ \forall\, h>0. \tag{89}$$

Next by applying Theorem 2 to (89), we obtain (76).

The theorem is proved.

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