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Fixed Point Theorems for Chatterjea Type F-Contraction on Closed Ball

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Abstract: The notions of Chatterjea type F-contraction and Chatterjea type $(\alpha-\eta-AF)$-contraction on closed ball are introduced. Two fixed point theorems in the framework of complete partial metric spaces are obtained. Some comparative examples are constructed to illustrate the novelty of these results. Our results provide substantial generalizations and improvements of several well known results existing in the comparable literature.

Keywords: fixed point; F and $(\alpha-\eta-AF)$ contractions; Closed ball; complete partial metric space

1 Introduction and Preliminaries

Banach Contraction Principle has been extended and generalized in many directions (see [3,7,8,9,10]). One of the most interesting generalization was given by Wardowski [24]. Recently, Abbas et al.[1] further generalized the concept of F-contraction and proved certain fixed point results. Hussain and Salimi [14] introduced an α-GF contraction with respect to a general family of functions G and established Wardowski type fixed point results in ordered metric spaces.
Matthews [22] introduced the concept of partial metric spaces and proved an analogue of Banach's fixed point theorem in partial metric spaces. This remarkable contribution leads many authors to focus on partial metric spaces and its topological properties, see [16] and references therein.
Following Arshad et al. [2,4] and Hussain and Salimi [14], in this article, we shall show existence of fixed points of Chatterjea type F-contraction on closed ball in partial metric spaces. In section 2, we introduce the Chatterjea type F-contraction on closed ball in partial metric spaces, by combining the ideas of Wardowski [24] and Chatterjea [15], and obtain a fixed point theorem. We give an example to illustrate this result. In section 3, we introduce Chatterjea type $(\alpha-\eta-AF)$-contraction on closed ball, by combining the ideas $(\alpha-\eta-AF)$-contraction and Chatterjea [15], and present a fixed point theorem and explain its idea through an example.
Throughout this paper, we denote $(0,\infty)$ by $\mathbb{R}^+$, $[0,\infty)$ by $\mathbb{R}_0^+$, $(-\infty,+\infty)$ by $\mathbb{R}$ and set of natural numbers by $\mathbb{N}$.
Following concepts and results will be required for the proofs of main results

Definition 1.[22] *Let X be a nonempty set and $p: X\times X\to \mathbb{R}_0^+$ satisfies following properties*

$(P_1) x=y \Leftrightarrow p(x,x)=p(x,y)=p(y,y)$,
$(P_2) p(x,x)\leq p(x,y)$,
$(P_3) p(x,y)=p(y,x)$,
$(P_4) p(x,y)\leq p(x,z)+p(z,y)-p(z,z)$,

for all $x,y,z\in X$. Then p is called a partial metric on X and the pair (X,p) is known as partial metric space.

Matthews [22], proved that every partial metric p on X induces a metric $d_p: X\times X\to\mathbb{R}_0^+$ defined by

$$d_p(x,y)=2p(x,y)-p(x,x)-p(y,y), \quad (1)$$

for all $x,y\in X$.
Notice that a metric on a set X is a partial metric p such that $p(x,x)=0$ for all $x\in X$.
Matthews [22] established that each partial metric p on X generates a T_0 topology $\tau(p)$ on X. The base of topology $\tau(p)$ is the family of open p-balls $\{B_p(x,\varepsilon) : x\in X,\ \varepsilon>0\}$, where

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$B_p(x,\varepsilon) = \{y \in X : p(x,y) < p(x,x) + \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$. Corresponding closed ball is defined by

$$\overline{B_p(x,\varepsilon)} = \{y \in X : p(x,y) \leq p(x,x) + \varepsilon\}.$$

A sequence $\{x_n\}_{n\in\mathbb{N}}$ in (X,p) converges to a point $x \in X$ if and only if $p(x,x) = \lim_{n\to\infty} p(x,x_n)$.

Definition 2.*[22] Let (X,p) be a partial metric space, then*

(1)A sequence $\{x_n\}_{n\in\mathbb{N}}$ in (X,p) is called a Cauchy sequence if $\lim_{n,m\to\infty} p(x_n,x_m)$ exists and is finite.
(2)A partial metric space (X,p) is said to be complete if every Cauchy sequence $\{x_n\}_{n\in\mathbb{N}}$ in X converges, with respect to $\tau(p)$, to a point $x \in X$ such that $p(x,x) = \lim_{n,m\to\infty} p(x_n,x_m)$.

Following lemma will be helpful in the sequel.

Lemma 1.[22]

(1)A partial metric space (X,p) is complete if and only if the metric space (X,d_p) is complete.
(2)A sequence $\{x_n\}_{n\in\mathbb{N}}$ in X converges to a point $x \in X$, with respect to $\tau(d_p)$ if and only if $\lim_{n\to\infty} p(x,x_n) = p(x,x) = \lim_{n,m\to\infty} p(x_n,x_m)$.
(3)If $\lim_{n\to\infty} x_n = \upsilon$ such that $p(\upsilon,\upsilon) = 0$ then $\lim_{n\to\infty} p(x_n,y) = p(\upsilon,y)$ for every $y \in X$.

*Remark.*Being $\left(\overline{B_p(x_0,r)},p\right) \subseteq (X,p)$, Lemma 1 holds for $\left(\overline{B_p(x_0,r)},p\right)$.

Definition 3.*[15] Let (X,p) be a partial metric space. A mapping $T : X \to X$ is said to be Chatterjea contraction if it satisfies the following condition*

$$p(T(x),T(y)) \leq \frac{k}{2}[p(x,T(y)) + p(y,T(x))],$$

for all $x,y \in X$ and some $k \in [0,1[$.

Definition 4.*[23] Let T be a self map defined on X and $\alpha : X \times X \to \mathbb{R}_0^+$ be nonnegative function. The mapping T is said to be α-admissible if for all $x,y \in X$, $\alpha(x,y) \geq 1$ implies that $\alpha(T(x),T(y)) \geq 1$.*

Definition 5.*[21] Let $T : X \to X$ be a self mapping and $\alpha,\eta : X \times X \to \mathbb{R}_0^+$ be two functions. The mapping T is said to be α-admissible mapping with respect to η if for all $x,y \in X$, $\alpha(x,y) \geq \eta(x,y)$ implies that $\alpha(T(x),T(y)) \geq \eta(T(x),T(y))$.*

If $\eta(x,y) = 1$, then above definition reduces to definition 4. For the sake of completeness, we recall the concept of F-contraction, which was introduced by Wardowski [24], later we will mention his result.
A mapping $T : X \to X$, is said to be F-contraction if it satisfies following condition

$$(d(T(x),T(y)) > 0 \Rightarrow t + F(d(T(x),T(y)) \leq F(d(x,y))), \tag{2}$$

for all $x,y \in M$ and for some $t > 0$. Where $F : \mathbb{R}^+ \to \mathbb{R}$ is a mapping satisfying following properties

(F_1): F is strictly increasing.
(F_2): For each sequence $\{a_n\}$ of positive numbers $\lim_{n\to\infty} a_n = 0$ if and only if $\lim_{n\to\infty} F(a_n) = -\infty$.
(F_3): There exists $\theta \in (0,1)$ such that $\lim_{\alpha\to 0^+} (\alpha)^\theta F(\alpha) = 0$.

We denote by Δ_F, the set of all functions satisfying the conditions $(F_1)-(F_3)$. Wardowski established the following result using F-contraction.

Theorem 1.[24] Let (X,d) be a complete metric space and let $T : X \to X$ be a F-contraction. Then T has a unique fixed point $\upsilon \in X$ and for every $x_0 \in X$ a sequence $\{T^n(x_0)\}_n \in \mathbb{N}$ converges to υ.

*Remark.*From (F_1) and (2) it is easy to conclude that every F-contraction is necessarily continuous.

In the following example, we shall show that there are mappings which are not F-contractions in metric spaces, nevertheless, such mappings follow the conditions of F-contraction in partial metric spaces.

*Example 1.*Let $M = [0,1]$ and define partial metric by $p(r_1,r_2) = \max\{r_1,r_2\}$ for all $r_1,r_2 \in M$. The metric d induced by partial metric p is given by $d(r_1,r_2) = |r_1 - r_2|$ for all $r_1,r_2 \in M$. Define the mappings $F : \mathbb{R}^+ \to \mathbb{R}$ by $F(r) = \ln(r)$ and T by

$$T(r) = \begin{cases} \dfrac{r}{5} & \text{if } r \in [0,1); \\ 0 & \text{if } r = 1 \end{cases}$$

Then T is not a F- contraction in a metric space (M,d). Indeed, for $r_1 = 1$ and $r_2 = \frac{5}{6}$, $d(T(r_1),T(r_2)) > 0$ and we have

$$\tau + F(d(T(r_1),T(r_2))) \leq F(d(r_1,r_2)),$$
$$\tau + F\left(d(T(1),T(\frac{5}{6}))\right) \leq F\left(d(1,\frac{5}{6})\right),$$
$$\tau + F\left(d(0,\frac{1}{6})\right) \leq F\left(\frac{1}{6}\right),$$
$$\frac{1}{6} < \frac{1}{6},$$

which is a contradiction for all possible values of τ. Now if we work in partial metric space (M,p), we get a positive answer that is

$$\tau + F(p(T(r_1),T(r_2))) \leq F(p(r_1,r_2)) \text{ implies}$$
$$\tau + F\left(\frac{1}{6}\right) \leq F(1),$$

which is true. Similarly, for all other points in M our claim proves true.

The following result play a vital role regarding the existence of the fixed point of the mapping satisfying a contractive condition on the closed ball.

Theorem 2.[17, Theorem 5.1.4] Let (X,d) be a complete metric space, $T: X \to X$ be a mapping, $r > 0$ and x_0 be an arbitrary point in X. Suppose there exists $k \in [0,1)$ with

$$d(T(x),T(y)) \leq kd(x,y), \text{ for all } x,y \in Y = \overline{B(x_0,r)}$$

and $d(x_0,T(x_0)) < (1-k)r$. Then there exists a unique point x^* in $\overline{B(x_0,r)}$ such that $x^* = T(x^*)$.

2 Chatterjea Type F-Contraction on Closed Ball

Definition 6.*Let (X,p) be a partial metric space. The mapping $T: X \to X$ is called Chatterjea type F-contraction on closed ball if for all $x,y \in \overline{B_p(x_0,r)} \subseteq X$, we have*

$$\tau + F(p(T(x),T(y))) \leq F\left(\frac{k}{2}\left[p(x,T(y)) + p(y,T(x))\right]\right), \tag{3}$$

where $0 \leq k < 1$, $F \in \Delta_F$ and $\tau > 0$.

Theorem 3.*Let (X,p) be a complete partial metric space and $T: X \to X$ be a Chatterjea type F-contraction on closed ball $\overline{B_p(x_0,r)}$ in a complete partial metric space. Moreover,*

$$p(x_0,T(x_0)) \leq (1-\lambda)[r + p(x_0,x_0)], \text{ where } \lambda = \frac{k}{2-k}. \tag{4}$$

Then there exists a point x^* in $\overline{B_p(x_0,r)}$ such that $T(x^*) = x^*$ with $p(x^*,x^*) = 0$

*Proof.*Choose a point x_1 in X such that $x_1 = T(x_0)$, $x_2 = T(x_1) = T^2(x_0)$. Continuing in this way, we get $x_{n+1} = T(x_n) = T^n(x_0)$, for all $n \geq 0$. First we show that $x_n \in \overline{B_p(x_0,r)}$ for all $n \in \mathbb{N}$. From (4), we have

$$\begin{aligned} p(x_0,x_1) &= p(x_0,T(x_0)) \leq (1-\lambda)[r + p(x_0,x_0)] \\ &< r + p(x_0,x_0), \end{aligned} \tag{5}$$

which shows that $x_1 \in \overline{B_p(x_0,r)}$. From (3) and (F_1), we get

$$\begin{aligned} F(p(x_1,x_2)) &= F(p(T(x_0),T(x_1))) \\ &\leq F\left(\frac{k}{2}\left[p(x_0,x_2) + p(x_1,x_1)\right]\right) - \tau, \end{aligned}$$

P_4 along with F_1 implies

$$p(x_1,x_2) < \frac{k}{2}\left[p(x_0,x_1) + p(x_1,x_2)\right],$$
$$< \lambda p(x_0,x_1) \leq \lambda[r + p(x_0,x_0)].$$

Now,

$$\begin{aligned} p(x_0,x_2) &\leq p(x_0,x_1) + p(x_1,x_2) - p(x_1,x_1), \\ &< (1-\lambda)[r + p(x_0,x_0)] + \lambda[r + p(x_0,x_0)] \\ &= r + p(x_0,x_0). \end{aligned}$$

This shows that $x_2 \in \overline{B_p(x_0,r)}$. Repeating this process n times we obtain that $x_n \in \overline{B_p(x_0,r)}$, for all $n \in \mathbb{N}$. Now condition (3) implies,

$$\begin{aligned} F(p(x_n,x_{n+1})) &\leq F\left(\frac{k}{2}\left[p(x_{n-1},x_{n+1}) + p(x_n,x_n)\right]\right) - \tau. \\ &\leq F\left(\frac{k}{2}\left[p(x_{n-1},x_n) + p(x_n,x_{n+1})\right]\right) - \tau. \\ &\leq F\left(\frac{k}{2}\left[p(x_{n-1},x_n) + \frac{k}{2-k}p(x_{n-1},x_n)\right]\right) - \tau. \\ &\leq F\left(\frac{k}{2-k}p(x_{n-1},x_n)\right) - \tau. \end{aligned}$$

Rewriting this inequality as

$$\tau + F(p(x_n,x_{n+1})) \leq F(\lambda p(x_{n-1},x_n)) < F(p(x_{n-1},x_n)). \tag{6}$$

Similarly, we can have

$$F(p(x_{n-1},x_n)) < F(p(x_{n-2},x_{n-1})) - \tau.$$

From (6), we obtain

$$F(p(x_n,x_{n+1})) < F(p(x_{n-2},x_{n-1})) - 2\tau.$$

Repeating these steps, we get

$$F(p(x_n,x_{n+1})) < F(p(x_0,x_1)) - n\tau. \tag{7}$$

From (7), we obtain $\lim_{n\to\infty} F(p(x_n,x_{n+1})) = -\infty$. Since $F \in \Delta_F$,

$$\lim_{n\to\infty} p(x_n,x_{n+1}) = 0. \tag{8}$$

From the property (F_3) of F-contraction, there exists $\kappa \in (0,1)$ such that

$$\lim_{n\to\infty} \left((p(x_n,x_{n+1}))^\kappa F(p(x_n,x_{n+1}))\right) = 0. \tag{9}$$

Following (7), for all $n \in \mathbb{N}$, we obtain

$$(p(x_n,x_{n+1}))^\kappa \left(F(p(x_n,x_{n+1})) - F(p(x_0,x_1))\right)$$

$$\leq -(p(x_n,x_{n+1}))^\kappa n\tau \leq 0. \tag{10}$$

Considering (8), (9) and letting $n \to \infty$, in (10), we have

$$\lim_{n\to\infty} \left(n(p(x_n,x_{n+1}))^\kappa\right) = 0. \tag{11}$$

Since (11) holds, there exists $n_1 \in \mathbb{N}$, such that $n(p(x_n,x_{n+1}))^\kappa \leq 1$ for all $n \geq n_1$ or,

$$p(x_n,x_{n+1}) \leq \frac{1}{n^{\frac{1}{\kappa}}} \text{ for all } n \geq n_1. \tag{12}$$

Using (12), we get for $m > n \geq n_1$,

$$\begin{aligned} p(x_n,x_m) &\leq p(x_n,x_{n+1}) + p(x_{n+1},x_{n+2}) + p(x_{n+2},x_{n+3}) \\ &+ \ldots + p(x_{m-1},x_m) - \sum_{j=n+1}^{m-1} p(x_j,x_j), \\ &\leq p(x_n,x_{n+1}) + p(x_{n+1},x_{n+2}) \\ &+ p(x_{n+2},x_{n+3}) + \ldots + p(x_{m-1},x_m), \\ &= \sum_{i=n}^{m-1} p(x_i,x_{i+1}), \\ &\leq \sum_{i=n}^{\infty} p(x_i,x_{i+1}), \\ &\leq \sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{k}}}. \end{aligned}$$

The convergence of the series $\sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{k}}}$ leads to $\lim_{n,m\to\infty} p(x_n,x_m) = 0$. Hence $\{x_n\}$ is a Cauchy sequence in $\left(\overline{B_p(x_0,r)},p\right)$, by Lemma 1, $\{x_n\}$ is also Cauchy sequence in $\left(\overline{B(x_0,r)},d_p\right)$. Moreover, since $\left(\overline{B_p(x_0,r)},p\right)$ is a complete partial metric space, by Lemma 1, $\left(\overline{B(x_0,r)},d_p\right)$ is also a complete metric space. Thus, there exists $x^* \in (\overline{B(x_0,r)},d_p)$ such that $x_n \to x^*$ as $n \to \infty$ and using Lemma 1, we have

$$\lim_{n\to\infty} p(x^*,x_n) = p(x^*,x^*) = \lim_{n,m\to\infty} p(x_n,x_m). \tag{13}$$

Due to $\lim_{n,m\to\infty} p(x_n,x_m) = 0$, we infer from (13) that $p(x^*,x^*) = 0$ and $\{x_n\}$ converges to x^* with respect to $\tau(p)$. In order to show that x^* is a fixed point of T, we assume that $p(x_n,T(x^*) > 0$, otherwise result is obvious. Using contractive condition (3), we obtain

$$F\left(p(x_n,T(x^*))\right) \leq F\left(\frac{k}{2}\left[p\left(x_{n-1},T(x^*)\right)+p\left(x^*,x_n\right)\right]\right) - \tau.$$

Letting $n \to \infty$, we get

$$p(x^*,T(x^*)) < \frac{k}{2} p(x^*,T(x^*)),$$

that is $\left(1-\frac{k}{2}\right) p(x^*,T(x^*)) < 0,$

which implies $p(x^*,T(x^*)) = 0$.

Thus, by using P_1 and P_2, we obtain $x^* = T(x^*)$ which completes the proof. To prove the uniqueness of x^*, assume, on contrary, that $y \in \overline{B_p(x_0,r)}$ is another fixed point of T that is $y = T(y)$, then from (3), we have

$$\begin{aligned} \tau + F\left(p(T(x^*),T(y))\right) &\leq F\left(\frac{k}{2}\left[p(x^*,T(y)) + p(y,T(x^*))\right]\right), \\ \tau + F\left(p(x^*,y)\right) &\leq F\left(\frac{k}{2}\left[p(x^*,y) + p(y,x^*)\right]\right), \\ \tau + F\left(p(x^*,y)\right) &\leq F\left(kp(x^*,y)\right). \end{aligned}$$

This implies that

$$p(x^*,y) \leq kp(x^*,y),$$

which is a contradiction. Hence, $x^* = y$. Therefore, T has a unique fixed point in $\overline{B_p(x_0,r)}$.

Following example shows that the contractive condition (3) holds on closed ball $\overline{B_p(x_0,r)}$ whereas it does not hold true on the whole space.

Example 2.Let $X = \mathbb{R}^+$ *and* $p(x,y) = \max\{x,y\}$. *Then* (X,p) *is a complete partial metric space. Define the mapping* $T : X \to X$ *by,*

$$T(x) = \begin{cases} \frac{x}{14} & \text{if } x \in [0,1], \\ x - \frac{1}{2} & \text{if } x \in (1,\infty) \end{cases}$$

Set $k = \frac{2}{5}$, $x_0 = \frac{1}{2}$, $r = \frac{1}{2}$ *and* $p(x_0,x_0) = \frac{1}{2}$, *then* $\overline{B_p(x_0,r)} = [0,1]$. *If* $F(\alpha) = \ln(\alpha)$, $\alpha > 0$ *and* $\tau > 0$, *then*

$$p(x_0,T(x_0)) = \max\left\{\frac{1}{2},\frac{1}{28}\right\} = \frac{1}{2} < (1-\lambda)[r + p(x_0,x_0)].$$

For $x,y \in \overline{B_p(x_0,r)}$, *the inequality*

$$\begin{aligned} p(T(x),T(y)) &= \max\left\{\frac{x}{14},\frac{y}{14}\right\} = \frac{1}{14}\max\{x,y\}, \\ &< \frac{1}{5}[x+y] = \frac{1}{5}\left[\max\left\{x,\frac{y}{14}\right\} + \max\left\{y,\frac{x}{14}\right\}\right], \\ &= \frac{k}{2}\left[p(x,T(y)) + p(y,T(x)\right] \end{aligned}$$

holds. Thus,

$$p(T(x),T(y)) < \frac{k}{2}\left[p(x,T(y)) + p(y,T(x)\right],$$

which implies

$$\tau + \ln\left(p(T(x),T(y))\right) \leq \ln\left(\frac{k}{2}\left[p(x,T(y)) + p(y,T(x)\right]\right).$$

So,

$$\tau + F\left(p(T(x),T(y))\right) \leq F\left(\frac{k}{2}\left[p(x,T(y)) + p(y,T(x)\right]\right).$$

Now, if $x = 100, y = 10 \in (1,\infty)$, *then*

$$\begin{aligned} p(T(x),T(y)) &= \max\left\{x-\frac{1}{2},y-\frac{1}{2}\right\}, \\ &\geq \frac{1}{5}[x+y] = \frac{k}{2}\left[p(x,T(y)) + p(y,T(x)\right], \end{aligned}$$

consequently, contractive condition (3) does not hold on X. Hence, hypotheses of Theorem 3 hold on closed ball and $x = 0$ *is a fixed point of* T.

3 Chatterjea Type $(\alpha-\eta-AF)$-Contraction on Closed Ball

We begin by introducing the following family of new functions.
Let Δ_A denotes the set of all functions $A:(\mathbb{R}_0^+)^4\to\mathbb{R}^+$ satisfying the property, if

$$\frac{p_1+p_2+p_3+p_4}{4}\le\frac{p_i+p_{i+1}}{2},\ i=1,2,3,4,$$

then there exists $\tau>0$ such that $A(p_1,p_2,p_3,p_4)=\tau$, for all $p_1,p_2,p_3,p_4\in\mathbb{R}_0^+$.

Definition 7. *Let (X,p) be a partial metric space and T be a self mapping on X. Also suppose that $\alpha,\eta:X\times X\to\mathbb{R}_0^+$ be two functions. T is said to be $(\alpha-\eta-AF)$-contraction if for $x,y\in X$, with $\eta(x,T(x))\le\alpha(x,y)$ and $d(T(x),T(y))>0$, we have*

$$A(p(x,T(x)),p(y,T(y)),p(x,T(y)),p(y,T(x)))$$
$$+F\left(p(T(x),T(y))\right)\le F\left(p(x,y)\right),$$

where $A\in\Delta_A$ and $F\in\Delta_F$.

Definition 8. *Let (X,p) be a partial metric space. Let T be a self map defined on X and $\alpha,\eta:X\times X\to\mathbb{R}_0^+$ be two functions. T is said to be (α,η)-continuous mapping on (X,p), if for a given $x\in X$, and the sequence $\{x_n\}_{n\in\mathbb{N}}$ with*

$$x_n\to x \text{ as } n\to\infty,\ \alpha(x_n,x_{n+1})\ge\eta(x_n,x_{n+1})$$
$$\text{implies } T(x_n)\to T(x).$$

Definition 9. *Let (X,p) be a metric space. Suppose that $\alpha,\eta:X\times X\to\mathbb{R}_0^+$ are two functions. The mapping $T:X\to X$ is called Chatterjea type $(\alpha-\eta-AF)$-contraction on closed ball if for all $x,y\in\overline{B_p(x_0,r)}\subseteq X$ with $\eta(x,T(x))\le\alpha(x,y)$ and $p(T(x),T(y))>0$, we have,*

$$\tau(A)+F\left(p(T(x),T(y))\right)\le F\left(\frac{k}{2}\left[p(x,T(y))+p(y,T(x)\right]\right), \tag{14}$$

where $\tau(A)=A(p(x,T(x)),p(y,T(y)),p(x,T(y)),p(y,T(x)))$, $0\le k<1$, $A\in\Delta_A$ and $F\in\Delta_F$.

Theorem 4. *Let (X,p) be a complete metric space. Let $T:X\to X$ be a Chatterjea type $(\alpha-\eta-AF)$-contraction mapping on a closed ball $\overline{B_p(x_0,r)}$ satisfying following assertions:*

(1) T is an α-admissible mapping with respect to η,
(2) there exists $x_0\in X$ such that $\alpha(x_0,T(x_0))\ge\eta(x_0,T(x_0))$,
(3) $p(x_0,T(x_0))\le(1-\lambda)[r+p(x_0,x_0)]$, where $\lambda=\dfrac{k}{2-k}$.

Then there exists a point x^* in $\overline{B_p(x_0,r)}$ such that $T(x^*)=x^*$ with $p(x_0,x_0)=0$

Proof. Suppose x_0 be an initial point of X such that $\alpha(x_0,T(x_0))\ge\eta(x_0,T(x_0))$. Also, for $x_0\in X$, we can construct a sequence $\{x_n\}_{n=1}^{\infty}$ such that $x_1=T(x_0)$, $x_2=T(x_1)=T^2(x_0)$. Continuing this way, $x_{n+1}=T(x_n)=T^{n+1}(x_0)$, for all $n\in\mathbb{N}$. Now since, T is an α-admissible mapping with respect to η, then $\alpha(x_0,x_1)=\alpha(x_0,T(x_0))\ge\eta(x_0,T(x_0))=\eta(x_0,x_1)$. In general we have,

$$\eta(x_{n-1},T(x_{n-1}))=\eta(x_{n-1},x_n)\le\alpha(x_{n-1},x_n),\ \text{for all } n\in\mathbb{N}. \tag{15}$$

If there exists $n\in\mathbb{N}$ such that $p(x_n,T(x_n))=0$, then x_n is a fixed point of T, so we are done. We assume that $p(x_n,T(x_n))>0$, for all $n\in\mathbb{N}$. First we show that $x_n\in\overline{B_p(x_0,r)}$ for all $n\in N$. From hypothesis (3), we obtain

$$p(x_0,x_1)\le(1-\lambda)[r+p(x_0,x_0)]<[r+p(x_0,x_0)]. \tag{16}$$

Thus, $x_1\in\overline{B_p(x_0,r)}$. From (14) and (F_1) we get,

$$\begin{aligned}F\left(p(x_1,x_2)\right)&=F(p(T(x_0),T(x_1)))\\&\le F\left(\frac{k}{2}[p(x_0,x_2)+p(x_1,x_1)]\right)-\tau(A),\\&\le F\left(\frac{k}{2}[p(x_0,x_1)+p(x_1,x_2)]\right)-\tau,\end{aligned}$$

$$\begin{aligned}\text{This implies } p(x_1,x_2)&<\frac{k}{2}[p(x_0,x_1)+p(x_1,x_2)],\\&<\lambda p(x_0,x_1)\le\lambda[r+p(x_0,x_0)].\end{aligned}$$

Where $\tau(A)=\tau$, indeed, $\tau(A)=A(p(x_0,x_1),p(x_1,x_2),p(x_0,x_2),p(x_1,x_1))$ satisfies

$$\frac{p(x_0,x_1)+p(x_1,x_2)+p(x_0,x_2)+p(x_1,x_1)}{4}\le\frac{p(x_0,x_1)+p(x_1,x_2)}{2},$$

so, there exists $\tau>0$ such that

$$A(p(x_0,x_1),p(x_1,x_2),p(x_0,x_2),p(x_1,x_1))=\tau.$$

Now,

$$\begin{aligned}p(x_0,x_2)&\le p(x_0,x_1)+p(x_1,x_2)-p(x_1,x_1),\\&<(1-\lambda)[r+p(x_0,x_0)]+\lambda[r+p(x_0,x_0)]=r+p(x_0,x_0).\end{aligned}$$

This shows that $x_2\in\overline{B_p(x_0,r)}$. Repeating this process n times we obtain that $x_n\in\overline{B_p(x_0,r)}$, for all $n\in\mathbb{N}$. Now contractive condition (14) implies,

$$F\left(p(x_n,x_{n+1})\right)\le F\left(\frac{k}{2}[p(x_{n-1},x_{n+1})+p(x_n,x_n)]\right)-\tau(A). \tag{17}$$

Since $\tau(A)=A(p(x_{n-1},x_n),p(x_n,x_{n+1}),p(x_{n-1},x_{n+1}),p(x_n,x_n))$ satisfies

$$\frac{p(x_{n-1},x_n)+p(x_n,x_{n+1})+p(x_{n-1},x_{n+1})+p(x_n,x_n)}{4}$$
$$\le\frac{p(x_{n-1},x_n)+p(x_n,x_{n+1})}{2},$$

so, there exists $\tau > 0$ such that

$$A(p(x_{n-1},x_n),p(x_n,x_{n+1}),p(x_{n-1},x_{n+1}),p(x_n,x_n)) = \tau.$$

Therefore, from (17), we get

$$\begin{aligned} &F\left(p(x_n,x_{n+1})\right) \\ &\leq F\left(\frac{k}{2}\left[p(x_{n-1},x_{n+1}) + \frac{k}{2-k}p(x_{n-1},x_n)\right]\right) - \tau. \\ &\leq F\left(\frac{k}{2-k}p(x_{n-1},x_n)\right) - \tau. \end{aligned}$$

Rewriting this inequality as

$$\tau + F\left(p(x_n,x_{n+1})\right) \leq F\left(\lambda p(x_{n-1},x_n)\right) < F\left(p(x_{n-1},x_n)\right). \tag{18}$$

Furthermore,

$$F\left(p(x_{n-1},x_n)\right) < F\left(p(x_{n-2},x_{n-1})\right) - \tau.$$

From (18), we obtain

$$F\left(p(x_n,x_{n+1})\right) < F\left(p(x_{n-2},x_{n-1})\right) - 2\tau.$$

Repeating these steps, we get

$$F\left(p(x_n,x_{n+1})\right) < F\left(p(x_0,x_1)\right) - n\tau. \tag{19}$$

Now following the proof of the Theorem 3. We infer that there exists $x^* \in \overline{B_p(x_0,r)}$ such that $p(x^*,x^*) = 0$ and $\{x_n\}$ converges to x^* with respect to $\tau(p)$. In order to show that x^* is a fixed point of T, we assume that $p(x_n,T(x^*)) > 0$, otherwise x^* is a fixed point of T. From contractive condition (14), we obtain

$$\begin{aligned} &F\left(p(x_n,T(x^*))\right) \\ &\leq F\left(\frac{k}{2}\left[p\left(x_{n-1},T(x^*)\right) + p\left(x^*,x_n\right)\right]\right) - \tau(A), \end{aligned}$$

where $\tau(A) =$
$A(p(x_{n-1},x_n),p(x^*,T(x^*)),p(x_{n-1},T(x^*)),p(x^*,x_n))$.
Since $A \in \Delta_A$ and F is continuous, we can have

$$\begin{aligned} &F\left(\lim_{n\to\infty} p(x_n,T(x^*))\right) \\ &\leq F\left(\frac{k}{2}\left[\lim_{n\to\infty} p\left(x_{n-1},T(x^*)\right) + \lim_{n\to\infty} p\left(x^*,x_n\right)\right]\right) - \tau. \end{aligned}$$

Thus,

$$p(x^*,T(x^*)) < \frac{k}{2}p(x^*,T(x^*)),$$

that is $\left(1 - \dfrac{k}{2}\right)p(x^*,T(x^*)) < 0.$

This implies $p(x^*,T(x^*)) = 0$. Consequently, x^* is a fixed point of T. To prove the uniqueness of x^*, assume, on contrary, that $y \in \overline{B_p(x_0,r)}$ is another fixed point of T that is $y = T(y)$, then from (14), we have

$$\begin{aligned} \tau(A) + F\left(p(T(x^*),T(y))\right) & \\ \leq F\left(\frac{k}{2}\left[p(x^*,T(y)) + p(y,T(x^*))\right]\right), & \end{aligned}$$

$$\tau(A) + F\left(p(x^*,y)\right) \leq F\left(\frac{k}{2}\left[p(x^*,y) + p(y,x^*)\right]\right),$$

where $\tau(A) = A(p(x^*,T(x^*)),p(y,T(y)),p(x^*,T(y)),p(y,T(x^*)))$ It is easy to check that $A \in \Delta_A$, therefore, $\tau(A) = \tau$ and we obtain

$$p(x^*,y) < kp(x^*,y),$$

which is a contradiction. Hence, $x^* = y$. Therefore, T has a unique fixed point in $\overline{B_p(x_0,r)}$.

Example 3.Let $X = \mathbb{R}_0^+$ and $p(x,y) = \max\{x,y\}$. Then (X,p) is a complete partial metric space. Define $T : X \to X$, $\alpha : X \times X \to \mathbb{R}_0^+$, $\eta : X \times X \to \mathbb{R}_0^+$, $A : (\mathbb{R}_0^+)^4 \to \mathbb{R}^+$ and $F : \mathbb{R}^+ \to \mathbb{R}$ by

$$T(x) = \begin{cases} \frac{5x}{19} & \text{if } x \in [0,1], \\ x - \frac{1}{3} & \text{if } x \in (1,\infty). \end{cases}$$

$$\alpha(x,y) = \begin{cases} e^{x+y} & \text{if } x \in [0,1], \\ \frac{1}{3} & \text{otherwise}. \end{cases}$$

$\eta(x,y) = \frac{1}{2}$ for all $x,y \in X$, $A(t_1,t_2,t_3,t_4) = \tau > 0$ and $F(t) = \ln(t)$ with $t > 0$. Set $k = \frac{4}{5}$ $x_0 = \frac{1}{2}$, $r = \frac{1}{2}$ and $p(x_0,x_0) = \frac{1}{2}$, then $\overline{B(x_0,r)} = [0,1]$. Now

$$p\left(\frac{1}{2},T\left(\frac{1}{2}\right)\right) = \max\left\{\frac{1}{2},\frac{5}{38}\right\} \leq [r + p(x_0,x_0)].$$

For if $x,y \in \overline{B(x_0,r)}$, then $\alpha(x,y) = e^{x+y} \geq \frac{1}{2} = \eta(x,y)$. On the other hand, $T(x) \in [0,1]$ for all $x \in [0,1]$ so $\alpha(T(x),T(y)) \geq \eta(T(x),T(y))$. For $x \neq y$, $p(T(x),T(y)) = \max\left\{\frac{5x}{19},\frac{5y}{19}\right\} > 0$. clearly $\alpha(0,T(0)) \geq \eta(0,T(0))$. Hence, we have

$$p(T(x),T(y)) = \max\left\{\frac{5x}{19},\frac{5y}{19}\right\} = \frac{5}{19}\max\{x,y\}.$$

For $x,y \in \overline{B_p(x_0,r)}$, the inequality

$$\frac{5}{19}\max\{x,y\} < \frac{k}{2}\left[\max\left\{x,\frac{5y}{19}\right\} + \max\left\{y,\frac{5x}{19}\right\}\right] = \frac{k}{2}[x+y],$$

holds. Thus,

$$p(T(x),T(y)) < \frac{k}{2}\left[p(x,T(y)) + p(y,T(x)\right].$$

Consequently,

$$\tau + \ln\left(p(T(x),T(y))\right) \leq \ln\left(\frac{k}{2}\left[p(x,T(y)) + p(y,T(x)\right]\right),$$

which implies

$$\tau + F\left(p(T(x),T(y))\right) \leq F\left(\frac{k}{2}\left[p(x,T(y)) + p(y,T(x)\right]\right).$$

If $x \notin \overline{B_p(x_0,r)}$ or $y \notin \overline{B_p(x_0,r)}$, then $\alpha(x,y) = \frac{1}{3} \ngeq \frac{1}{2} = \eta(x,y)$. Moreover, if $x = 100, y = 10 \in (1,\infty)$, then

$$\begin{aligned} p(T(x),T(y)) &= \max\left\{x - \frac{1}{3}, y - \frac{1}{3}\right\}, \\ &\geq \frac{k}{2}\left[p(x,T(y)) + p(y,T(x)\right], \end{aligned}$$

and consequently, contractive condition (14) does not hold on X. Hence, hypotheses of Theorem 4 hold on closed ball and $x = 0$ *is a fixed point of* T.

4 Conclusion

In this article, following the approach of F-contraction introduced by Wardowski [24], we present some fixed point theorem for F -contraction in partial metric spaces. We observe that there are mappings which are not F-contractions in metric spaces, nevertheless, such mappings follow the conditions of F-contraction in partial metric spaces. This fact makes our results more general and more interesting than that in metric spaces. We observe that there exists some mappings which satisfy given contractive condition only on closed ball. We establish two fixed point theorems and support them with concrete examples. Following Hussain and Salimi [14], we introduce new class of functions Δ_A and establish a fixed point theorem for this family of functions. These new concepts shall lead readers for further investigations and applications. It will also be interesting to apply these concepts in a different metric spaces.

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