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Multi-Team Prey-Predator Model with Delay

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Abstract: The main goal in this paper is to continue the investigations of the important system (see [4]), by considering a delayed multi-team prey-predator model. In the absence of delay, we study the conditions of the existence and stability properties of the equilibrium points. For the full general model with delay, conditions are derived under which there can be no change in stability. Using the discrete time delay as a bifurcation parameter it is found that Hopf bifurcation occurs when the delay passes through a critical value. Results are verified by computer simulation.

Keywords: Delay, predator-prey model, Hopf bifurcation.

1 Introduction

In the natural world, however, species does not exist alone. Also, it competes with the other species for space, food or predated by other species. Therefore, it is more biological significance to consider the effect of interacting species when we study the dynamical behaviors of biological models. Delay-differential equations exhibit much more complicated dynamics than ordinary differential equations since a time delay could cause a stable equilibrium to become unstable and cause the populations to fluctuate. Time delays have been incorporated into biological models by many authors (for example, [2], [3], [5], [7], [8], [9], [12]). In recent years, a delayed prey-predator system is one of the most important fields of interest (see [1], [6], [10], [11] [13]).

In this paper, we formulate a multi-team prey-predator model with time delay $\tau > 0$ denote the gestation of the predator. The results show that if the gestation delay is small enough, their sizes will keep stable in the long run, but if the gestation delay of predator is large enough, their sizes will periodically fluctuate in the long term. By Hurwitz criteria, the local stability of the positive equilibrium of this model is investigated.

The conditions under which the positive equilibrium is locally asymptotically stable are obtained. Under the same conditions, namely, with the same parameters, the stability of the positive equilibrium of predator-prey model would change due to the introduction of gestation time delay for predator. Moreover, with the variation of time delay, the positive equilibrium of the model subjects to Hopf bifurcation. The above theoretical results are validated by numerical simulations with the help of dynamical software MATLAB.

The paper is organized as follows: In Section 2, the model is built. Section 3 is devoted to the investigation of the conditions for local stability. In Section 4, linearization of the delayed model is treated and the conditions for the Andronov-Hopf bifurcation are established (these are the main results of this paper). We show that the increase of delay destabilizes the system and causes the occurrence of periodic oscillations. We consider an example to illustrate what can be expected. Our concluding remarks are presented in section 5.

2 Mathematical model

In this Section, we propose a system consists of two teams of preys with densities $N_1(t)$ and $N_2(t)$, respectively, interact with one team of predator with density $P(t)$. The assumptions of this model are as follow:

(1) In the absence of any predation, each team of preys grows logistically; this is $r_1 N_1 (1-N_1)$ and $r_2 N_2 (1-N_2)$.

(2) The effect of the predation is to reduce the prey growth rate by a term proportional to the prey and predator populations; this is the $(-N_1 P)$, $(-N_2 P)$ terms.

(3) The teams of preys help each other against the predator, that the term $(N_1 N_2 P)$ is exist.

(4) In the absence of any prey for sustenance, the predator's death rate results in inverse decay, that is the term $\left(-\alpha P^2\right)$.

(5) The prey's contribution to the predator growth rate is $(\beta N_1 P.)$, $(\gamma N_2 P)$; that is proportional to the available prey as well as the size of the predator population.

Based on the above discussion, Elettreby (see [4]) have studied the following prey predator model:

$$\begin{aligned}
\frac{dN_1}{dt} &= r_1 N_1 (1-N_1) - N_1 P + N_1 N_2 P, \\
\frac{dN_2}{dt} &= r_2 N_2 (1-N_2) - N_2 P + N_1 N_2 P, \\
\frac{dP}{dt} &= -\alpha P^2 + \beta N_1 P + \gamma N_2 P.
\end{aligned} \tag{1}$$

Time delay is not occurred in the above model. In this paper, our aim is to investigate how the time delay effects the dynamics of system (1). From the above assumptions with time delay τ $(\tau > 0)$ which is the time required for the gestation of predator, the model equations become:

$$\begin{aligned}
\frac{dN_1}{dt} &= r_1 N_1(t) (1-N_1(t)) - N_1(t) P(t) + N_1(t) N_2(t) P(t), \\
\frac{dN_2}{dt} &= r_2 N_2(t) (1-N_2(t)) - N_2(t) P(t) + N_1(t) N_2(t) P(t), \\
\frac{dP}{dt} &= -\alpha P^2(t) + \beta N_1(t-\tau) P(t-\tau) + \gamma N_2(t-\tau) P(t-\tau),
\end{aligned} \tag{2}$$

where the coefficients $r1$, $r2$, α, β and γ are positive constants and $N_1(0)$, $N_2(0)$, $P(0) > 0$. It is clear that the term $N_1 N_2 P$ in the pray equations means that the preys help each other e.g. in foraging and in early warning against predation. Note that this help occurs only in the presence of predator, this is presented by the term $N_1 N_2 P$ in the preys equations.

3 The model without time delay

In this Section, we will study the system (1) without delay. In particular, we will focus on the existence of equilibria and their local stability. This information will be crucial in the next section where we study the effect of the delay parameters on the stability of the steady states.

The equilibria of the system (1) are:

$$E_0 := (0,0,0),\ E_1 := (1,0,0),\ E_2 := (0,1,0,),\ E_3 := (1,1,0),$$

$$E_4 := \left(0, \frac{\alpha r_2}{\alpha r_2 + \gamma^5}, \frac{\gamma r_2}{\alpha r_2 + \gamma}\right), E_5 := \left(0, \frac{r_1 \alpha}{r_1 \alpha + \beta}, \frac{r_1 \beta}{r_1 \alpha + \beta}\right),$$

and the following two interior solutions which means all teams coexist,

$$E_6 : = \left(1, 1, \frac{\beta+\gamma}{\alpha}\right), E^* := \left(\frac{\alpha r_2 + \gamma\, (1 - \sqrt{\frac{r_2}{r_1}})}{\gamma + \beta \sqrt{\frac{r_2}{r_1}}}, \frac{\alpha \sqrt{r_1 r_2} - \beta (1 - \sqrt{\frac{r_2}{r_1}})}{\gamma + \beta \sqrt{\frac{r_2}{r_1}}}, \sqrt{r_1 r_2}\right)$$

under the existence conditions,

$$\alpha \sqrt{r_1 r_2} \le \beta + \gamma,$$
$$\alpha r_1 + \beta > \beta \sqrt{\tfrac{r_2}{r_1}},$$
$$\alpha r_2 + \gamma > \gamma \sqrt{\tfrac{r_1}{r_2}},$$

A standard stability analysis based on the Jacobian matrix;

$$J = \begin{pmatrix} r_1 (1 - 2N_1) - P(1 - N_2) & N_1 P & -N_1 (1 - N_2) \\ N_2 P & r_2 (1 - 2N_2) - P(1 - N_2) & -N_2 (1 - N_1) \\ \beta P & \gamma P & -2\alpha P + \beta N_1 + \gamma N_2 \end{pmatrix},$$

for the equilibria, we show that: the origin $E_0 := (0,0,0)$ is unstable equilibrium point, which has two positive eigenvalues, $\lambda = r_2, r_1$. Similarly, $E_1 := (1,0,0)$ has two positive eigenvalues, $\lambda = \beta, r_2$. So, it is unstable equilibrium point. By the same way, $E_2 := (0, 1, 0)$ has two positive eigenvalues, $\lambda = \gamma, r_1$. So, it is unstable equilibrium point. Also, the equilibrium point $E_3 := (1, 1, 0)$ has one positive eigenvalue, $\lambda = \beta + \gamma$. So, all of them are unstable equilibrium points. The equilibrium solution $E_4 : = \left(0, \frac{\alpha r_2}{\alpha r_2 + \gamma}, \frac{\gamma r_2}{\alpha r_2 + \gamma}\right)$ is locally asymptotically stable under the condition $r_1 < \frac{r_2 \gamma^2}{(r_2 \alpha + \gamma)^2}$. The equilibrium solution $E_5 : = \left(\frac{\alpha r_1}{\alpha r_1 + \beta}, 0, \frac{\beta r_1}{\alpha r_1 + \beta}\right)$ is locally asymptotically stable if $r_2 < \frac{r_1 \beta^2}{(r_1 \alpha + \beta)^2}$.

The first internal equilibrium solution $E_6 : = \left(1, 1, \frac{\beta + \gamma}{\alpha}\right)$, is locally asymptotically stable if $r_1 r_2 > \left(\frac{\beta+\gamma}{\alpha}\right)^2$. The second internal equilibrium solution:

$$E^* : = \left(\frac{\alpha r_2 + \gamma (1 - \sqrt{\frac{r_2}{r_1}})}{\gamma + \beta \sqrt{\frac{r_2}{r_1}}}, \frac{\alpha \sqrt{r_1 r_2} - \beta\, (1 - \sqrt{\frac{r_2}{r_1}})}{\gamma + \beta \sqrt{\frac{r_2}{r_1}}}, \sqrt{r_1 r_2}\right).$$

To investigate the stability of the positive steady states $E^* := (N_1^*, N_2^*, P^*)$ we consider the linearized system of (1) at E^*. The Jacobian matrix at E^* is given by:

$$\begin{pmatrix} -r_1 N_1^* & N_1^* P^* & -N_1^*(1-N_2^*) \\ N_2^* P^* & -r_2 N_2^* & -N_2^*(1-N_1^*) \\ \beta P^* & \gamma P^* & -\alpha P^* \end{pmatrix} \quad \mathbf{(3)}$$

The characteristic equation of the linearized system is given by:

$$P_3(\lambda) := \lambda^3 + c_2 \lambda^2 + c_1 \lambda + c_0 = 0, \quad \mathbf{(4)}$$

where:

$$\begin{aligned} c_2 &= r_1 N_1^* + r_2 N_2^* + \alpha \sqrt{r_1 r_2}, \\ c_1 &= \sqrt{r_1 r_2} \left[\alpha (r_1 N_1^* + r_2 N_2^*) + (1-N_1^*)(\beta \sqrt{r_1/r_2} N_1^* + \gamma N_2^*)\right], \\ c_0 &= 2 r_1 \sqrt{r_1 r_2} N_1^* N_2^* (1-N_1^*)(\gamma + \beta\sqrt{r_1/r_2}). \end{aligned} \quad \mathbf{(5)}$$

Then according to the Routh-Hurwitz criteria all roots of $P_3(\lambda)$ have negative real parts if:

$$c_2 > 0, \; c_0 > 0, \; c_2 c_1 - c_0 > 0, \quad \mathbf{(6)}$$

hence, the positive equilibrium point E^* is linearly asymptotically stable under the following conditions:

$$\beta + \gamma > \alpha \sqrt{r_1 r_2}, \; r_2 \ge r_1, \; r_1 \alpha \ge 1. \quad \mathbf{(7)}$$

4- The model with time delay

In this Section, we focus on investigate the stability of equilibria and Hopf bifurcation of the positive equilibrium of the system (2). To study the stability of the steady states $E^* := (N_1^*, N_2^*, P^*)$, let us define $u(t) = N_1(t) - N_1^*, v(t) = N_2(t) - N_2^*, w(t) = P(t) - P^*$. Then the linearized system of the system (2) at $E^* := (N_1^*, N_2^*, P^*)$ is given by:

$$\begin{aligned} \frac{du}{dt} &= -r_1 N_1^* u(t) + N_1^* P^* v(t) - N_1^* (1-N_2^*) w(t), \\ \frac{dv}{dt} &= N_2^* P^* u(t) - r_2 N_2^* v(t) - N_2^* (1-N_1^*) w(t), \\ \frac{dw}{dt} &= \beta P^* u(t-\tau) + \gamma P^* v(t-\tau) - 2\alpha P^* w(t) + (\beta N_1^* + \gamma N_2^*) w(t-\tau). \end{aligned} \quad \mathbf{(8)}$$

Then, we express system (8) in matrix form as follows:

$$\frac{d}{dt}\begin{pmatrix} u(t) \\ v(t) \\ w(t) \end{pmatrix} = M_1 \begin{pmatrix} u(t) \\ v(t) \\ w(t) \end{pmatrix} + M_2 \begin{pmatrix} u(t-\tau) \\ v(t-\tau) \\ w(t-\tau) \end{pmatrix} \quad \mathbf{(9)}$$

where M_1 and M_2 are 3×3 matrices given by:

$$M_1 = \begin{pmatrix} -r_1 N_1^* & N_1^* P^* & -N_1^*(1-N_2^*) \\ N_2^* P^* & -r_2 N_2^* & -N_2^*(1-N_1^*) \\ 0 & 0 & -2\alpha P^* \end{pmatrix} \text{ and } M_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \beta P^* & \gamma P^* & \beta N_1^* + \gamma N_2^* \end{pmatrix}.$$

The characteristic equation of the system (9) is given by:

$$\Delta(\lambda,\tau):=\left|\lambda I - M_1 - \exp(-\lambda\tau) M_2\right| = 0,$$

i.e.

$$\Delta(\lambda,\tau):=\begin{vmatrix} \lambda + r_1 N_1^* & -N_1^* P^* & N_1^*(1-N_2^*) \\ -N_2^* P^* & \lambda + r_2 N_2^* & N_2^*(1-N_1^*) \\ -\beta P^* e^{-\lambda\tau} & -\gamma P^* e^{-\lambda\tau} & \lambda + 2\alpha P^* - (\beta N_1^* + \gamma N_2^*)\, e^{-\lambda\tau} \end{vmatrix} = 0,$$

that is:

$$\Delta(\lambda,\tau):=\lambda^3 + A\lambda^2 + B\lambda - (a\lambda^2 + b\lambda + c)e^{-\lambda\tau} = 0, \tag{10}$$

where:

$$\begin{aligned} A &= r_1 N_1^* + r_2 N_2^* + 2\alpha\sqrt{r_1 r_2}, \\ B &= 2\alpha\sqrt{r_1 r_2}\,(r_1 N_1^* + r_2 N_2^*), \\ a &= (\beta N_1^* + \gamma N_2^*), \\ b &= (r_1 N_1^* + r_2 N_2^*)(\beta N_1^* + \gamma N_2^*) - \sqrt{r_1 r_2}\,(1-N_1^*)(\beta\sqrt{r_1/r_2}\,N_1^* + \gamma N_2^*), \\ c &= -2 r_1\sqrt{r_1 r_2}\, N_1^* N_2^* (1-N_1^*)(\gamma + \beta\sqrt{r_1/r_2}). \end{aligned} \tag{11}$$

It is known that the equilibrium $E^* := (N_1^*, N_2^*, P^*)$ is asymptotically stable if all roots of corresponding characteristic Eq. (10) have negative real parts. We consider Eq. (10) with $\tau = 0$ that is Eq. (4) and assume that all the roots of Eq. (4) have negative real parts. This is equivalent to the assumption (7). Denote $\lambda = \eta(\tau) + i\,w(\tau)$, $(w>0)$, the eigenvalues of the characteristic Eq. (10), where $\eta(\tau)$ and $w(\tau)$ depend on the delay τ. Since the equilibrium $E^* := (N_1^*, N_2^*, P^*)$ of the ODE model is stable, it follows that $\eta(t)<0$ when $\tau=0$. By continuity, if $\tau > 0$ is sufficiently small we still have $\eta(\tau)=0$ and E^* is still stable. If $\eta(\tau_0 = 0)$ for certain value τ_0 (so that $\lambda = i\,w(\tau_0)$) is a purely imaginary root of Eq. (10), then the steady state E^* loses its stability and eventually becomes unstable when $\eta(\tau)$ becomes positive. In other words, if such an $w(\tau_0)$ does not exist, that is, if the characteristic Eq. (10) does not have purely imaginary roots for all delay, then the equilibrium E^* is always stable. We shall show that this indeed is true for the characteristic Eq. (10). Clearly, $\lambda = i\,w(\tau)$, $(w>0)$ is a root of Eq. (10) if and only if:

$$-i w^3 - A w^3 + i B w = (-a w^2 + i b w + c)\, e^{-iwt}.. \tag{12}$$

Separating the real and imaginary parts, we have:

$$\begin{aligned} -A w^3 &= (-a w^3 + c)\cos(wt) + b w \sin(wt), \\ -w^3 + B w &= -(-a w^2 + c)\sin(wt) + b w \cos(wt). \end{aligned} \tag{13}$$

Adding up the squares of both the equations, we obtain:

$$(-A w^2)^2 + w^2 (B - w^2)^2 = b^2 w^2 + (-a w^2 + c)^2 \tag{14}$$

i.e.

$$w^6 + (A^2 - 2B - a^2)\, w^4 + (B^2 - b^2 + 2\,a\,c)\, w^2 - c^2 = 0. \tag{15}$$

Let $z = w^2$, $Q_1 = A^2 - 2B - a^2$, $Q_2 = B^2 - b^2 + 2\,a\,c$, $Q_3 = -c^2$. Then Eq. (15) becomes:

$$h(z) := z^3 + Q_1 z^2 + Q_2 z + Q_3 = 0. \qquad \textbf{(16)}$$

Lemma 1 (a) There is at least one positive root of $h(z)$ if $Q_3 < 0$.

(b) There are at least one positive root of $h(z)$ if $Q_1 < 0,\ Q_2 \le \frac{1}{3} Q_1^2$ and $Q_3 > 0$.

Remark: the derivative function is:

$$h'(z) = 3z^2 + 2Q_1 z + Q_2 = 3\left(z + \frac{1}{3} Q_1\right)^2 + Q_2 - \frac{Q_1^2}{3}.$$

We observe that there exist one or two extreme points of $h(z)$ if the discriminate $Q_1^2 - 3Q_2 \ge 0$. Moreover, if $Q_1 < 0$ and $Q_3 > 0$, two roots of $h(z)$ are positive.

The critical value of delay τ_{crit} for which the positive equilibrium point $E^* = (N_1^*, N_2^*, P^*)$ is stable if $\tau < \tau_{crit}$ is given by:

$$T_{crit} = \frac{1}{w} \cos^{-1}\left(\frac{(-A w^2)(-a w^2 + c) + b\, w(-w^3 + m\, w)}{(-a w^2 + c)^2 + (b\, w)^2}\right) + \frac{2 n \pi}{w}, \quad n = 0, 1, 2, 3, \ldots \qquad \textbf{(17)}$$

Theorem 4.0.1 Assume that $Q_1 < 0,\ Q_2 \le \frac{1}{3} Q_1^2$ and $Q3 > 0$, then Hopf bifurcation occurs provided either $2w^6 + w^4 Q_1 - Q_3$ is positive or negative.

Proof: Our assumption gives two positive roots of the function $h(z)$. We consider the sign of the following transversality condition:

$$\frac{d}{dt} \mathrm{Re}\,(\lambda(\tau))\Big|_{\tau = \tau_{crit}} = \frac{d}{dt} \eta(\tau)\Big|_{\tau = \tau_{crit}}. \qquad \textbf{(18)}$$

By continuity, the real part of $\lambda(\tau)$ becomes positive when $\tau > \tau_{crit}$ and the equilibrium $E^* := (N_1^*, N_2^*, P^*)$ becomes unstable. Moreover, a Hopf bifurcation occurs when τ passes through the critical values τ_{crit}. Differentiating characteristic Eq. (10) with respect to τ, we get:

$$(3\lambda^2 + 2A\lambda + B)\frac{d\lambda}{dt} = e^{-\lambda\tau}\left[-\tau(a\lambda^2 + b\lambda + c) + 2a\lambda + b\right]\frac{d\lambda}{dt} - \lambda e^{-\lambda\tau}(a\lambda^2 + b\lambda + c).$$

This gives:

$$\begin{aligned}
\left(\frac{d\lambda}{dt}\right)^{-1} &= \frac{(3\lambda^2 + 2A\lambda + B) + \tau(a\lambda^2 + b\lambda + c)e^{-\lambda\tau} - (2a\lambda + b)e^{-\lambda\tau}}{-\lambda(a\lambda^2 + b\lambda + c)\, e^{-\lambda\tau}} \\
&= \frac{3\lambda^3 + 2A\lambda + B}{-\lambda(a\lambda^2 + b\lambda + c)\, e^{-\lambda\tau}} + \frac{2a\lambda + b}{\lambda(a\lambda^2 + b\lambda + c)} - \frac{\tau}{\lambda} \\
&= \frac{2\lambda^3 + A\lambda^2}{\lambda^2(\lambda^3 + A\lambda^2 + B\lambda)} + \frac{a\lambda^2 - c}{\lambda^2(a\lambda^2 + b\lambda + c)} - \frac{\tau}{\lambda}.
\end{aligned}$$

$$sign\left\{\frac{d\,\mathrm{Re}(\lambda(\tau))}{d\,\tau}\right\}_{\lambda=iw} = sign\left\{\mathrm{Re}\left(\frac{d\,\lambda\,(\tau)}{d\tau}\right)^{-1}\right\}_{\lambda=iw}$$

$$= sign\left\{\mathrm{Re}\left(-\frac{2\lambda^3+A\lambda^2}{\lambda^2(\lambda^3+A\lambda^2+B\lambda)}\right)_{\lambda=iw}+\left(-\frac{a\lambda^2-c}{\lambda^2(a\lambda^2+b\lambda+c)}\right)_{\lambda=iw}\right\}$$

$$= sign\left\{\frac{(Aw^2)(Aw^2)+2w^4(w^2-B)-(aw^2+c)(aw^2-c)}{w^2(aw^2+c)^2+w^4b^2}\right\}$$

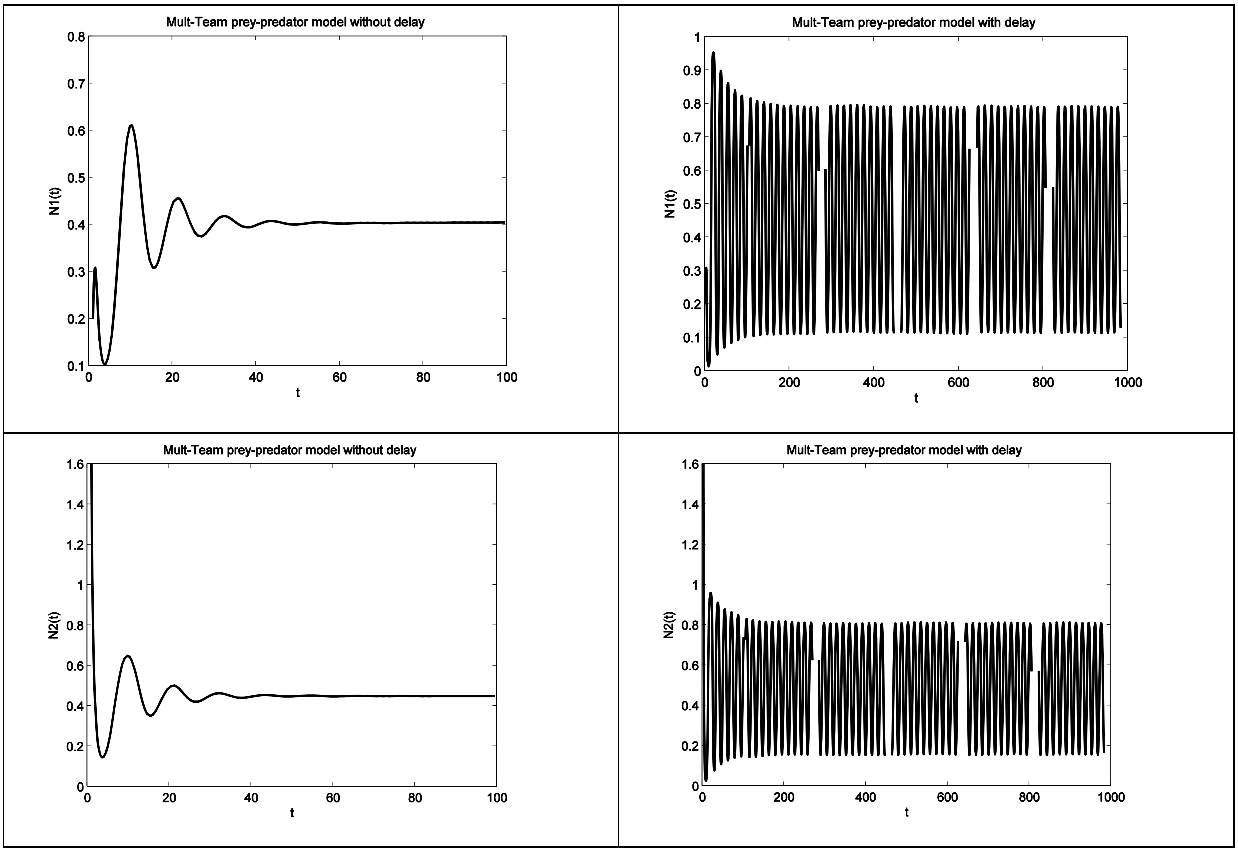
$$= sign\left\{\frac{(A^2w^4)+2w^4(w^2-B)+c^2-a^2w^4)}{w^2(aw^2+c)^2+w^4b^2}\right\}$$

$$= sign\left\{w^4(A^2-2B-a^2+2w^2)+c^2\right\}$$

$$= sign\left\{2w^6+w^4Q_1-Q_3\right\}.$$

If $2w^6+w^4Q_1-Q_3$ is either positive or negative, then the transversality condition holds and hence Hopf bifurcation occurs at $\tau=\tau_{crit}$. This will be clear from the following numerical example.

4.1 Numerical simulation



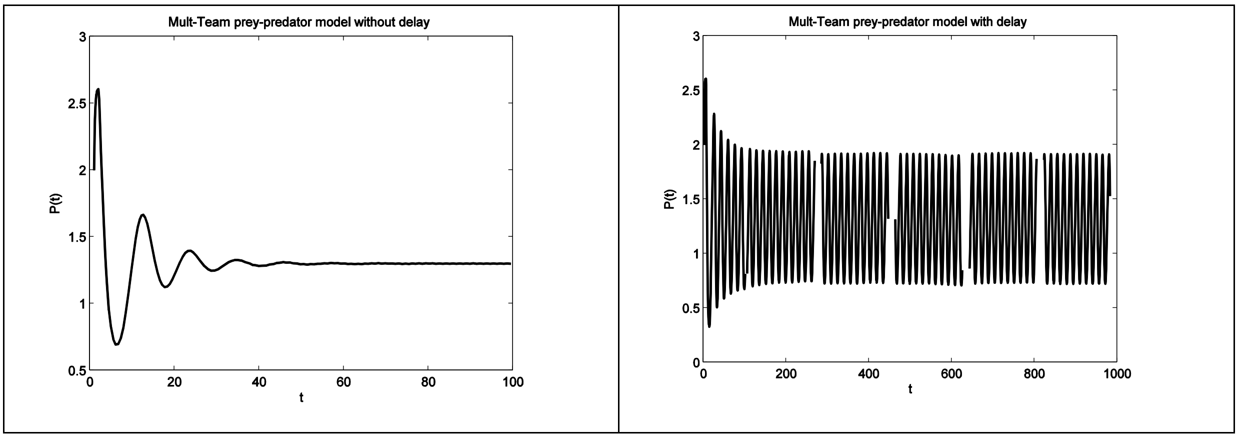


Figure: Left figures: The case of delay $\tau_{crit} = 2$: The solutions N_1 , N_2 and P shows convergence to the equilibrium solution $E*$ as time increases. Right figures: The case of delay τ_{crit} = 2.5: The solution N_1 , N_2 and P at the equilibrium point $E*$ loses its stability and subcritical Hopf bifurcation occurs, that is a family of periodic solution bifurcates from equilibrium $E*$ as time increases (Figure produced by applying MATLAB).

Example: Let $r_1 = 1.2,\ r_2 = 1.4, \alpha = 1.0,\ \beta = 1.0,\ \gamma = 2.0.$ For the set of parameter values, the unique positive stable equilibrium is given by $E^* = (0.40, 0.45, 1.29)$. Substituting these parameter values in Eq. (16), we obtain:

$$h\ (z) : = z^3\ - 1.64481\ z^2\ + 5.51471z\ - .95655 = 0. \qquad (19)$$

Solving Eq. (19), we get one positive value of z, that is z = 0.1822644829; one positive value of w, $w = 0.4269244464$. There exist a critical value of delay, $\tau_{crit} = 2:453264850 + \dfrac{2n\pi}{w}$, such that the positive equilibrium E^* bifurcates to periodic solutions when τ lies near τ_{crit} $= 2.453265; 17.163127; 31.872990,\ \ldots$.

We treat the two cases of delay, near to τ_{crit} with the same initial values are taken such that $N_1(0) = 0.2,\ N_2(0) = 0.4,\ P(0) = 2$. In the former case, depicted as in the Figures, if the delay is slightly shorter than the critical one $\tau_{crit} = 2.453264850$, the solution coverage to the equilibrium solution $E* := (N_1^*\ ,\ N_2^*\ ,\ P)$. In the latter case, if the delay is slightly longer than the critical delay, we observer Hopf bifurcation which oscillate, as shown in the Figures.

5 Conclusions

Many animals live in group. Different groups share one habitat hence these groups may cooperate, compete with each other or form predator-prey system. In this work we present a new model for predator-prey model with delay. Equilibrium solutions are derived, their local stability are studied. A biological realization of our model is two cooperating teams of gazelles and zebras attacked by one predator. For the full general model with delay, conditions are derived under which there can be no change in stability. Using the discrete time delay as a bifurcation parameter it is found that Hopf bifurcation occurs when the delay passes through a critical value. Results are verified by computer simulation.

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