Evolution of Translation Surfaces in Euclidean 3-Space

\( E^3 \)

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Received: 15 Nov. 2013, Revised: 7 Apr. 2014, Accepted: 15 Apr. 2014
Published online: 1 Mar. 2015

Abstract: Geometry and kinematics have been intimately connected in their historical evolution and, although it is currently less fashionable, the further development of such connections is crucial to many computer-aided design and manufacturing. In this paper, the evolution of the translation surfaces and their generating curves in \( E^3 \) are investigated. Integrability conditions of the Gauss-Weingarten equations are obtained. Kinematics of moving frame fields associated to these surfaces are described. The evolution equations of the Christoffel symbols, the second fundamental quantities and Gauss-Codazzi equations for the motion are established. Thus, the evolution equations of the curvatures in terms of their intrinsic geometric formulas are derived. Two examples of translation surfaces and their motions are considered and plotted.

Keywords: Kinematics of surfaces, Evolution Surfaces.

1 Introduction

The evolution of curves and surfaces has significant applications in computer vision and image processing [1], such as scale space by linear and nonlinear diffusions [2] and [3] image enhancement through anisotropic diffusions [4,5] and [6] and image segmentation by active contours [7,8] and [9].

The evolution of curves has been researched and studied extensively in various manifolds and homogeneous spaces. The connection between integrable equations (soliton equations) and the geometric motions of curves in spaces has been known for a long time. In fact, many integrable equations have been shown to describe the evolution invariants associated with certain movements of curves in particular geometric settings.

The dynamics of shapes in physics, chemistry and biology are modelled in terms of motion of surfaces and interfaces, and some dynamics of shapes are reduced to motion of plane curves. These models are specified by velocity fields, which are local or nonlocal functionals of the intrinsic quantities of curves [10].

Evolution of surfaces accompanies manifold physical phenomena: propagation of wave fronts [11], motion of interfaces, growth of crystals [12], geometry [13,14] and [15]. Most often one needs to consider the related three-dimensional problem for the domain the surface encloses to find out how it evolves in time.

Hence, geometry of curves and surfaces evolution is a quite recent field of the differential geometry which deals with curves and surfaces where the time plays the fundamental rule. Geometrically, curves and surfaces evolution means that deforming a curve and surface into another curve and surface in a continuous manner, respectively. This evolution process can be understood through the answer of the following geometrical questions:

- What is the final shape of the evolving curves and surfaces ?
- What is the invariant geometrical property during the evolution process ?

The main goal of this paper is to try to answer the above questions, in order to illustrate the translation surfaces behavior during the evolution process. Consequently, it gives a classification of the geometrical flows according to the resulting geometrical information which characterizes and recognizes each flow in a class of geometric flows.

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2 Geometric preliminaries

Here, and in the sequel, we assume that the indices \( \{i, j, k, l\} \) and \( \{m, n\} \) run over the ranges \( \{1, 2\} \) and \( \{1, 2, 3\} \), respectively.

Let \( C_1 : \alpha = \alpha (s_1) \) and \( C_2 : \beta = \beta (s_2) \) be two curves parametrized by the arc lengths \( s_1 \) in \( E^3 \). Consider the Frenet frame \( \{t_i(s_i), n_i(s_i), b_i(s_i)\} \) associated with the curves \( C_i \). The derivatives of the vectors \( t_i(s_i) \) and \( b_i(s_i) \), when expressed in the basis \( \{t_i, n_i, b_i\} \), yield geometrical entities, the natural curvatures \( \kappa_i(s_i) \) and torsions \( \tau_i(s_i) \), which give us information about the behavior of the curves \( \alpha \) and \( \beta \) in the neighborhood of \( s_i \), respectively. Then the Frenet formulas of the curves \( C_i \) are defined by [10]:

\[
\frac{d}{ds_i} \begin{pmatrix} t_i(s_i) \\ n_i(s_i) \\ b_i(s_i) \end{pmatrix} = \begin{pmatrix} 0 & -\kappa_i(s_i) & \tau_i(s_i) \\ \kappa_i(s_i) & 0 & 0 \\ -\tau_i(s_i) & 0 & 0 \end{pmatrix} \begin{pmatrix} t_i(s_i) \\ n_i(s_i) \\ b_i(s_i) \end{pmatrix}.
\]

We denote a surface \( M \) in \( E^3 \) by

\[
X(s_i) = (x_i(s_i)).
\]

Let \( N \) be the standard unit normal vector field on the surface \( M \) defined by \( N = \frac{\partial X}{\partial s_i} \), where, \( X_1 = \frac{\partial X}{\partial s_i} \).

Thus, we have the metric \( g_{ij} \) and the coefficients of the second fundamental form \( h_{ij} \),

\[
g_{ij} = <X_i, X_j> \quad , \quad h_{ij} = <X_i, N > ,
\]

where \( (\cdot , \cdot) \) is the Euclidean inner product.

Thus, the Gaussian curvature \( G \) and the mean curvature \( H \) are given by

\[
G = \text{Det}(h_{ij}) / \text{Det}(g_{ij}) \quad \text{and} \quad H = \frac{1}{2} tr (g^{ij} h_{jk}) ,
\]

respectively, where, \( (g^{ij}) \) is the associated contravariant metric tensor field of the covariant metric tensor field \( (g_{ij}) \), i.e., \( g^{ik}g_{jk} = \delta^i_j \).

As one moves along the surface (at a fixed time), the tangent and normal vectors change according to the Gauss-Weingarten equations (GWE),

\[
\frac{\partial}{\partial s_j} \begin{bmatrix} X_1 \\ \cdots \\ N \end{bmatrix} = \begin{bmatrix} \Gamma^k_{ij} & h_{ij} \\ \cdots & \cdots & \cdots \end{bmatrix} \begin{bmatrix} X_1 \\ \cdots \\ N \end{bmatrix} , \quad h^k_j = g^{ik} h_{kj} (4)
\]

where \( \Gamma^k_{ij} \) are called the Christoffel symbols of the 2nd kind, which are given as

\[
\Gamma^k_{ij} = \frac{1}{2} \sum_l g^{lk} \left( \frac{\partial g_{le}}{\partial s_i} \frac{\partial g_{le}}{\partial s_j} - \frac{\partial g_{ij}}{\partial s_l} \right) . (5)
\]

The norm of the 2nd fundamental form is given by:

\[
S^2 = \sum_{i,j} h^i_j h^j_i ,
\]

where \( h^i_j \) are given from (4).

From the compatibility conditions of (4), we get the Gauss-Codazzi equations,

\[
R_{ijk} = h_{ik} h_{je} - h_{ie} h_{jk} , \quad (7)
\]

\[
\nabla_i h_{jk} = \nabla_j h_{ik} , \quad (8)
\]

where \( R_{ijk} \) is the Riemann tensor and \( \nabla_i \) is the covariant derivative.

The Riemann curvature tensor is defined by

\[
R^e_{ijk} = \Gamma^e_{ik,j} - \Gamma^e_{ij,k} + \sum_h \left( \Gamma^e_{jh} \Gamma^h_{ik} - \Gamma^h_{kh} \Gamma^h_{ij} \right) . (9)
\]

where the comma denotes the partial derivative.

Nakayama, et al. [16] and [17] discovered connections between integrable evolution and the motion of curves in a 3-dimensional Euclidean space. They considered that \( \alpha = \alpha (x_1, t) \) denote a point on a space curve at the time \( t \). The conventional geometrical model is specified by velocity fields

\[
\frac{\partial \alpha}{\partial t} = v_1^1 t_1 + v_1^2 n_1 + v_1^3 b_1 , \quad (10)
\]

similarly for a space curve \( \beta = \beta (x_2, t) \), we have

\[
\frac{\partial \beta}{\partial t} = v_2^1 t_2 + v_2^2 n_2 + v_2^3 b_2 , \quad (11)
\]

where \( t_i, n_i \) and \( b_i \) are the unit tangent, normal and binormal vectors along the curves, and \( v_1^1, v_2^1 \) and \( v_1^2, v_2^2 \) are the tangential, normal and binormal velocities, respectively. Velocity fields are functionals of the intrinsic quantities of curves, for example, curvature, \( k_i \), torsion \( \tau_i \) and their \( s_i \) derivatives.

The time evolution equations for \( t_i, n_i \) and \( b_i \) are given by

\[
\frac{\partial}{\partial t} \begin{bmatrix} t_i(s_i) \\ n_i(s_i) \\ b_i(s_i) \end{bmatrix} = \begin{bmatrix} 0 & \alpha_i & \beta_i \\ -\alpha_i & 0 & \gamma_i \\ -\beta_i & -\gamma_i & 0 \end{bmatrix} \begin{bmatrix} t_i(s_i) \\ n_i(s_i) \\ b_i(s_i) \end{bmatrix} , \quad (12)
\]

where,

\[
\alpha_i = \left( \frac{\partial v_i^1}{\partial s_i} - \tau_i v_i^2 + \kappa_i v_i^3 \right) , \quad \beta_i = \left( \frac{\partial v_i^2}{\partial s_i} + \tau_i v_i^1 \right) , \quad \gamma_i = \left( \frac{1}{k_i} \frac{\partial ^2 v_i^1}{\partial s_i^2} + \frac{\tau_i}{k_i} \alpha_i \right) . (13)
\]

Thus, Nakayama, et al. [16] and [17] obtained the time evolution equations for curvature and torsion to motion of space curve.

Also, Nakayama, et al. [18, 19] and [20] introduced the dynamics of the surface. They considered the velocity of the surface is expressed by

\[
\frac{\partial X}{\partial t} = V^i X_i + V^3 N , \quad V^l = V^l (s_i, t) , \quad (14)
\]

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where \( V^t \) and \( V^3 \) are the tangential and the normal velocities, respectively.

Using (4) and (14), one can obtain the time evolution equations for the local frame,

\[
\frac{\partial}{\partial t} \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ N \end{bmatrix} = \begin{bmatrix} -V^3 h^k + \nabla V^k : V^3 + V^j h_{ij} \\ \vdots \\ \vdots \\ -g^{ki} (V^3 + V^j h_{ij}) : 0 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ N \end{bmatrix}
\]

(15)

Thus, and using the compatibility conditions, one can get the time evolution equations for \( g_{ij} \) and \( h_{ij} \). See [18, 19] and [20].

### 3 Intrinsic geometry of the translation surfaces in \( E^3 \)

In this section the translation surface in Euclidean 3-space is considered. The geometric invariants on the translation surfaces are derived.

When a space curve is translated over another space curve, the resulting surface can be considered as the most general appearance of a translation surface. Consequently, this surface can be parameterized as the sum of two space curves. Quite often, the class of translation surfaces is restricted to those that can be parameterized as the sum of two plane curves. So it can be parameterized by a patch [21]:

\[
M : X (s_i) = \alpha (s_1) + \beta (s_2), \quad s_i \in I,
\]

(16)

where \( s_i \) are the parameters of the arc lengths of the curves \( \alpha \) and \( \beta \), respectively.

Thus, the two tangent vectors to the surface (16) are given by

\[
X_i = t_i, \quad X_i = \frac{\partial X}{\partial s_i},
\]

(17)

where \( t_i \) denote the tangents of the curves \( \alpha \) and \( \beta \), respectively.

Using (16) and (17), it is easily checked that the metric of \( M \) is given by

\[
(g_{ij}) = \begin{cases} 
1, & i = j \\
\cos \theta, & i \neq j
\end{cases}, \quad \det (g_{ij}) = \sin^2 \theta,
\]

(18)

\[
\theta \neq n \pi, \quad n = 0, 1, 2, \ldots
\]

The unit normal vector of the surface \( M \) is given by

\[
N(s_i) = (t_1 \times t_2) \csc \theta,
\]

(19)

This leads to the coefficients of the second fundamental form \( h_{ij} \) where

\[
(h_{ij}) = \begin{cases} 
\kappa_i \csc \theta [t_1 t_2 n_j], & i = j \\
0, & i \neq j
\end{cases}, \quad \det (h_{ij}) = \csc^2 \theta \prod_{i \neq j} \kappa_i [t_1 t_2 n_j],
\]

(20)

where \( \kappa_i \) denote the curvatures of the curves \( \alpha \) and \( \beta \), respectively and \([t_1 t_2 n_j]\) denotes the triple scalar product to the vectors \( t_1, t_2 \) and \( n_j \).

Thus, one can get the following:

**Corollary 1.** The Gaussian curvature is given by

\[
G = \csc^2 \theta \prod_{i \neq j} \kappa_i [t_1 t_2 n_j].
\]

(21)

**Corollary 2.** The mean curvature is given by

\[
H = \frac{1}{2} \csc^3 \theta \sum_i \kappa_i [t_1 t_2 n_j].
\]

(22)

**Corollary 3.**

\[
h_j^i = \begin{cases} 
\kappa_i \csc^3 \theta [t_1 t_2 n_j], & i = j \\
-\kappa_j \cot \theta \csc^2 \theta [t_1 t_2 n_j], & i \neq j
\end{cases}
\]

(23)

**Corollary 4.** The norm of the 2nd fundamental form is given by:

\[
S^2 = 4H^2 - 2G
\]

(24)

**Corollary 5.** The Christoffel symbols are given as follows

\[
\Gamma^1_{11} = -\kappa_1 \cot^2 \theta, \quad \Gamma^2_{22} = -\kappa_2 \cot^2 \theta, \quad \Gamma^1_{12} = \kappa_1 \cos \theta \csc^2 \theta, \quad \Gamma^2_{21} = \kappa_2 \cos \theta \csc^2 \theta,
\]

(25)

and the other components are zero.

**Corollary 6.** The Riemann tensor \( R_{ijk} \) and the Riemann curvature tensor \( R^e_{ijk} \) are given by

\[
R_{1212} = R_{2121} = -R_{1212} = -R_{2112} = \csc^2 \theta \prod_i \kappa_i [t_1 t_2 n_j],
\]

(26)

\[
R^2_{122} = R^2_{221} = -R^1_{112} = -R^1_{212} = \kappa_1 \kappa_2 \cos \theta \csc^2 \theta,
\]

(27)

and the other components are zero.

### 4 Evolution of the translation surfaces

In the study of the motion of curves, the author seeks to get the partial differential equations (PDEs), which describe the evolution of the curvature and torsion of the evolving curve. Similarly, the study of the motion of surfaces seeks to get (PDEs), which describe the evolution of the coefficients of the 1st and 2nd fundamental forms because the surface can be derived from them. For this purpose, let a translation surface, moving in 3-dimensional Euclidean space \( E^3 \), be given at the time \( t \) by the position vector

\[
M : X (s_i, t) = \alpha (s_1, t) + \beta (s_2, t), \quad s_i \in I,
\]

(28)

where, \( \alpha (s_1, 0) = \alpha (s_1) \), \( \beta (s_1, 0) = \beta (s_1) \).
4.1 Evolution of the curves $\alpha$ and $\beta$

Nakayama, et al. [16] and [17] studied the evolution and the motion of curves as mentioned in section 2 of this paper. Also, Mukherjee and Balakrishnan [22] studied the motion of curves which have been specified via the evolution of the frame field $\{t, n, b\}$.

In this section we apply the same method of Mukherjee and Balakrishnan [22]. In our case we study the evolution of curves $\alpha$ and $\beta$ which have been specified via the evolution of the frame fields $\{t, n, b\}$.

The Serret-Frenet equations (1) and the equations of evolution (12) can be written concisely in the following form

$$ \frac{\partial F}{\partial s_i} = A_i \cdot F, \quad \frac{\partial F}{\partial t} = B_i \cdot F. $$

(29)

Applying the compatibility conditions, we have

$$ \frac{\partial}{\partial t} \left( \frac{\partial F}{\partial s_i} \right) = \frac{\partial}{\partial s_i} \left( \frac{\partial F}{\partial t} \right), $$

(30)

After some calculation using (1) and (12), we have

$$ \frac{\partial A_i}{\partial t} - \frac{\partial B_i}{\partial s_i} + [A_i, B_i] = O_{3 \times 3} $$

(31)

where $[A_i, B_i] = A_i B_i - B_i A_i$ is the Lie bracket of $A_i$ and $B_i$. Thus, we have

$$ \begin{pmatrix}
0 & \alpha_{ij} - \kappa_j - \tau_i \beta_i & \beta_{ij} - \kappa_i + \tau_j \alpha_i \\
-(\alpha_{ij} - \kappa_j - \tau_i \beta_i) & 0 & \gamma_{ij} - \tau_{ij} - \kappa_i \beta_i \\
-(\beta_{ij} - \kappa_i + \tau_j \alpha_i) & -(\gamma_{ij} - \tau_{ij} - \kappa_i \beta_i) & 0
\end{pmatrix}

= O_{3 \times 3} $$

(32)

Thus, the compatibility conditions are given from

$$ \frac{\partial}{\partial t} \frac{\partial \psi}{\partial s_i} = \frac{\partial}{\partial s_i} \frac{\partial \psi}{\partial t} $$

(35)

Therefore, using (4) and (15), we have the following

$$ \frac{\partial C_i}{\partial t} - \frac{\partial D}{\partial s_j} + [C_i, D] = O_{3 \times 3} $$

(36)

Thus, the compatibility conditions (35), split into two equations, which we called the 1st and 2nd compatibility conditions, where in each case we obtain evolution (PDEs) for invariant quantities related to each one.

4.3 The first compatibility conditions

Based on (35), we have

$$ \frac{\partial}{\partial t} \frac{\partial \psi}{\partial s_i} = \frac{\partial}{\partial s_i} \frac{\partial \psi}{\partial t} $$

(37)

Thus, using (4), (15) and (34), one can get

$$ \begin{aligned}
\frac{\partial \Gamma_{12}^2}{\partial t} &= d_{12,1} + d_{12} (\Gamma_{12}^2 - \Gamma_{12}^1) - \Gamma_{11}^2 (d_{22} - d_{11}) \\
&\quad - d_{32} h_{11} - d_{31} h_{12}, \\
\frac{\partial \Gamma_{11}^1}{\partial t} &= d_{21,1} + d_{21} (\Gamma_{11}^1 - \Gamma_{12}^1) + \Gamma_{12}^1 (d_{22} - d_{11}) \\
&\quad - d_{31} h_{12} - d_{32} h_{11}, \\
\frac{\partial \Gamma_{12}^1}{\partial t} &= d_{11,1} + d_{11} \Gamma_{12}^1 - d_{21} \Gamma_{12}^2 - d_{31} h_{11} - d_{32} h_{12}, \\
\frac{\partial \Gamma_{11}^2}{\partial t} &= d_{22,1} + d_{22} \Gamma_{12}^1 - d_{12} \Gamma_{12}^2 - d_{32} h_{11} + d_{31} h_{12}, \\
\frac{\partial \Gamma_{12}^2}{\partial t} &= d_{12,1} + d_{12} \Gamma_{12}^2 - d_{22} \Gamma_{12}^1 - d_{32} h_{12} + d_{31} h_{12}, \\
\frac{\partial \Gamma_{11}^1}{\partial t} &= -d_{31,1} - d_{31} \Gamma_{11}^1 + d_{21} \Gamma_{12}^2 + d_{31} h_{11}, \\
\frac{\partial \Gamma_{12}^1}{\partial t} &= -d_{32,1} - d_{32} \Gamma_{12}^1 - d_{22} \Gamma_{12}^2 - d_{32} h_{12},
\end{aligned} $$

(38)

where, $d_{lm}$ are the elements of the matrix $D$ which are given from (15), (34) and $d_{lm,1} = \frac{\partial d_{lm}}{\partial s_j}$. Thus, using (20), (23) and (25), one can get
4.4 The second compatibility conditions

Adopting (35), we have

\[
\frac{\partial}{\partial t} \frac{\partial \psi}{\partial s_2} = \frac{\partial}{\partial s_2} \frac{\partial \psi}{\partial t}
\]

Thus, using (4), (15) and (34), one can get

\[
\begin{align*}
\frac{\partial h_2}{\partial t} &= d_{21.2} + d_{21} (\Gamma_{11}^2 - \Gamma_{22}^2) + \Gamma_{12}^1 (d_{22} - d_{11}) - d_{31} h_{12} - d_{23} h_{12}^2, \\
\frac{\partial h_2}{\partial h_1} &= d_{21.2} + d_{21} (\Gamma_{11}^2 - \Gamma_{22}^2) + \Gamma_{12}^1 (d_{22} - d_{11}) - d_{31} h_{12} - d_{23} h_{12}^2, \\
\frac{\partial h_2}{\partial h_3} &= d_{21.2} + d_{21} (\Gamma_{11}^2 - \Gamma_{22}^2) + \Gamma_{12}^1 (d_{22} - d_{11}) - d_{31} h_{12} - d_{23} h_{12}^2, \\
\frac{\partial h_3}{\partial h_1} &= -d_{31.2} + d_{31} h_{12}^2 + d_{31} \Gamma_{12}^1 + d_{23} h_{12}^2 + d_{11} h_{12}, \\
\frac{\partial h_3}{\partial h_2} &= -d_{31.2} + d_{31} h_{12}^2 + d_{31} \Gamma_{12}^1 + d_{23} h_{12}^2 + d_{11} h_{12}.
\end{align*}
\]

As a consequence, using (20), (23) and (25), one can get

\[
\begin{align*}
\frac{\partial \psi}{\partial t} &= d_{21.1} + d_{21} \Gamma_{12}^1 - \Gamma_{12}^2 - (d_{21} - d_{11}) \frac{\partial \psi}{\partial s_2} \theta + (d_{21} - d_{11}) \frac{\partial \psi}{\partial s_2} \csc^2 \theta \\
\frac{\partial \psi}{\partial h_1} &= \frac{\partial \psi}{\partial h_2} \theta - \frac{\partial \psi}{\partial h_3} \csc^2 \theta \\
\frac{\partial \psi}{\partial h_3} &= \frac{\partial \psi}{\partial h_1} \theta - \frac{\partial \psi}{\partial s} \csc^2 \theta \left. \right|_{t_1 t_2 n_1}.
\end{align*}
\]

4.5 Evolution of the curvatures

Here, (PDEs) which describe the evolution of the curvatures of evolving the translation surface in terms of curvatures and velocities of the surface and the curves are derived.

From the foregoing results, using (3), (6), (7) and (9), we get the proof of the very important theorems

\[
\begin{align*}
\frac{\partial \theta}{\partial t} &= G \left[ \sum d_{ii} - 2 \cot^2 \theta \sum (\alpha_i + \beta_i) \right] + 2H \sum d_{3,i} \\
- \kappa_1 \kappa_2 \cot \theta (d_{13} - d_{23} \cos \theta) \sum t_1 t_2 n_1.
\end{align*}
\]

Theorem 5. Evolution of the mean curvature function is given by

\[
\begin{align*}
\frac{\partial H}{\partial t} &= H \left[ \frac{1}{2} \sum d_{ii} + 2 \cot^2 \theta \sum (\alpha_i + \beta_i) \right] - \frac{\cos \theta \csc^2 \theta (1 + \cos \theta) \sum d_{3,i} \kappa_1}{2 \csc^2 \theta (2 \cot \theta d_{23,1} - \sum d_{3,i})} \\
&- \frac{1}{2} \kappa_1 \kappa_2 \cot \theta \csc \theta \left. \right|_{t_1 t_2 n_1}.
\end{align*}
\]

Theorem 6. Evolution of the norm of the 2nd fundamental form is given by:

\[
\begin{align*}
\frac{\partial S^2}{\partial t} &= S^2 \sum d_{ii} + 4 \left( S^2 + 3G \right) \cot^2 \theta \sum (\alpha_i + \beta_i) \\
&- 4H \cos \theta \csc^2 \theta (1 + \cos \theta) \sum d_{3,i} d_{3,i} \kappa_1 \\
&- 4H \csc^2 \theta (2 \cot \theta d_{23,1} - \sum d_{3,i}) \\
&- 2 \kappa_1 \kappa_2 \cot \theta (d_{23} \cos \theta - d_{13}) \sum t_1 t_2 n_1 \\
&- 8d_{12} H \kappa_1 \kappa_2 \cot \theta \csc \theta \left. \right|_{t_1 t_2 n_1} \\
&- 4H \sum d_{3,i}.
\end{align*}
\]
Theorem 7. Evolution of the Riemann tensor $R_{ijkl}$ are given by

\[
\frac{\partial R_{ijkl}}{\partial t} = -R_{klij} - R_{lkij} = -R_{jkil} = -R_{ijkl}
\]

\[
= R_{2121} \csc \theta \sum d_{ii} + 2H \sin^2 \theta \sum d_{ij} - (d_{13} - d_{23}) k_1 k_2 \cos \theta \csc^2 \theta \sum t_i t_j n_i
\]

(46)

Theorem 8. Evolution of the Riemann curvature tensor $R^i_{jkl}$ are given by

\[
\frac{\partial R^i_{jkl}}{\partial t} = -R^i_{jkl} - R^i_{klj} = -R^i_{ljk} = -R^i_{jkl}
\]

\[
= R^i_{jkl} d_{ij} \cos \theta - R^i_{jkl} (d_{ij} \csc \theta + d_{jk} \sec \theta \sin^2 \theta) t_i n_j + R^i_{jkl} \csc \theta (\sum d_{ij} (t_i n_j) - d_{ijl} \cos \theta \csc \theta (1 + \cos \theta) + d_{ij} t_i n_j) - 2H d_{ij} \cos \theta (k_1 + k_2 \cos \theta) + (1 + k_2) \cot \theta \csc \theta (\sum d_{ij} (t_i n_j) - d_{ijl} \cos \theta \csc \theta (1 + \cos \theta) + d_{ij} t_i n_j) - \cos \theta \csc \theta (1 + \cos \theta) - d_{ij} (k_2 \cot \theta \csc \theta \sin^2 \theta + d_{ij} t_i n_j)
\]

(47)

The above theorems for the evolution of the curvatures of the translation surfaces describe the change of the shape in time and their solutions have many applications in different areas of science. In geometry, the behavior of evolution equations of the curvatures and their solutions gives us the most geometric information about evolving the surface.

5 Application

In this section, we give some interesting results about how the evolving surfaces will look like after a period of time and the effect of the evolution on the translation surfaces. This is given through some illustrated figures. For this purpose we consider two examples to illustrate our investigation:

(1) Consider the translation surfaces for a circular helix curves $\alpha (s_1, t)$ and $\beta (s_2, t)$ as an example in $E^3$, so that their points can be represented in the form

\[
X (s_1, t) = \sum \left( a_i (t) \cos \left( \frac{s_1}{A} \right), a_i (t) \sin \left( \frac{s_1}{A} \right), 2b_i (t) \frac{s_1}{A} \right)
\]

\[
A = \sqrt{a^2 + b^2}
\]

(48)

(2) Consider the translation surfaces for a circular helix curve $\alpha (s_1, t)$ and the circle curve $\beta (s_2, t)$, where

\[
\beta (s_2, t) = (a_2 (t) \cos \theta, a_2 (t) \sin \theta), \ \theta = \frac{s_2}{a_2}, \ a_2 \neq 0
\]

(49)

Evolution of these surfaces are plotted for different values of the time $t$. In examples (1) and (2) which show that these surfaces are collapsing in length and expanding in width as in the following cases:

Fig. 1: Evolution of the two circular helix $\alpha , \beta$.

Fig. 2: Evolution of the circular helix $\alpha$ and fixed circular helix $\beta$. 
more involved in the case of the motion of translation surfaces.

Acknowledgements

I wish to express my profound thanks and appreciation to our professor, Dr. Nassar H. Abdel All, Department of Mathematics, Assiut University, Egypt, for his strong support, continuous encouragement, revising this paper carefully and providing some important comments on it. I would like also to thank Dr. Samah G. Mohamed, Faculty of Science, Assiut University, for her critical reading of this manuscript and making several useful remarks.

References


6 Conclusion

The study of curves finds applications in various branches of physics. As illustrative examples of a space curve we mention the propagation of light in a twisted optical fiber, the time evolution of the (normalized) classical spin vector field of a one-dimensional Heisenberg chain and the vector field of the H(3) nonlinear sigma model in field theory. This latter model is known to be related to one-dimensional antiferromagnets. In recent years, there has been much interest in the notion of anholonomy in physics. The geometric phases and the corresponding gauge fields have attracted attention in a wide spectrum of problems in classical and quantum systems [26].

As an application of the translation surfaces evolution it is interesting to consider a system, such as fluid and spins, on a moving translation surface. It is also interesting to see what kind of integrable systems is included in this motion. In one dimensional case, it is possible to explain the relation between the motion of the generating curves and integrable system in terms of the inverse scattering method. Situation is, however, much


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