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Common Fixed Point Theorem in Fuzzy Metric Spaces with An Application in the Product Space

Manish Jain[1,*], *Satish Kumar*[2] *and Manoj Kumar*[3]

[1] Department of Mathematics, Ahir College, Rewari 123401, India
[2] Department of Mathematics, IGU, Meerpur, Rewari, India
[3] Department of Mathematics, Lovely Professional University, Punjab, India

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Abstract: The purpose of this paper is to investigate the existence and uniqueness of common fixed point for a pair of mappings using implicit functions in the setup of fuzzy metric spaces. The mappings considered need not be commutative (nor minimal commutative) and the results are proved without making an appeal to the continuity of maps. At the end we give an application in the product spaces.

Keywords: Fuzzy metric space; Product space; Common fixed point; Implicit relation.

1 Introduction and Preliminaries

The introduction of the notion of fuzzy sets by Zadeh [1] proved a turning point in the development of mathematics. This notion laid the foundation of fuzzy mathematics. Kramosil and Michalek [2] introduced the notion of fuzzy metric space by generalizing the concept of the probabilistic metric space to the fuzzy situation. George and Veeramani [3] modified the concept of fuzzy metric spaces introduced by Kramosil and Michalek [2]. There are many view points of the notion of the metric space in fuzzy topology; for instance, one can refer to Kaleva and Seikkala [4], Kramosil and Michalek [2] and George and Veeramani [3]. This leads to a milestone in fixed point theory of fuzzy metric space and afterwards a flood of papers appeared for fixed point theorems in fuzzy metric spaces.

An important category in fixed point theory is the class of the problems concerning the computation of common fixed points. An early result in this direction was established by Jungck under commuting maps [23]. Jungcks concept of commuting maps has been enjoyed in various spaces and has been generalized in several ways over the years. One such generalization was made by Mishra et al. [5] by introducing the concept of compatible maps in the setup of fuzzy metric spaces which was further generalized by Singh and Jain [6] with the introduction of the notion of weak compatible maps in fuzzy metric spaces. Popa [10,11] introduced the idea of implicit relations to prove a common xed point theorem in metric spaces. Jain [12] further extended the results of Popa [10,11] in fuzzy metric spaces. Afterwards, implicit relations are used as a tool for nding common xed points of maps under different conditions (see, [13,14,15,16,17, 18,19,20]). We first summarize some basic results that are useful for further study.

Definition 1.1([1]). *A fuzzy set A in X is a function with domain X and values in* $[0,1]$.

Definition 1.2([21]). *A binary operation* $*:[0,1]\times[0,1]\to[0,1]$ *is a continuous t-norm if* $([0,1],*)$ *is a topological abelian monoid with unit* 1 *s.t.* $a*b\le c*d$ *whenever* $a\le c$ *and* $b\le d$*, for all* $a,b,c,d\in[0,1]$.

Definition 1.3([3]). *The 3-tuple* $(X,M,*)$ *is called a fuzzy metric space if X is an arbitrary set,* $*$ *is a continuous t-norm and M is a fuzzy set on* $X^2\times[0,\infty)$ *satisfying the following conditions:*

(FM-1) $M(x,y,0)>0$;
(FM-2) $M(x,y,t)=1$ *iff* $x=y$;
(FM-3) $M(x,y,t)=M(y,x,t)$;
(FM-4) $M(x,y,t)*M(y,z,s)\le M(x,z,t+s)$;
(FM-5) $M(x,y,\cdot):(0,\infty)\to[0,1]$ *is continuous, for all* $x,y,z\in X$ *and* $s,t>0$.

* Corresponding author e-mail: manish_261283@rediffmail.com

Throughout this paper, we consider M to be fuzzy metric space with condition:

(FM-6) $\lim_{t\to\infty} M(x,y,t) = 1$, *for all* $x,y \in X$ *and* $t > 0$.

Definition 1.4([3]). *Let* $(X,M,*)$ *be a fuzzy metric space. A sequence* $\{x_n\}$ *in X is said to be*

(i) *convergent to a point* $x \in X$, *if* $\lim_{n\to\infty} M(x_n,x,t) = 1$ *for all* $t > 0$;
(ii) *Cauchy sequence if* $\lim_{n\to\infty} M(x_{n+p},x_n,t) = 1$ *for all* $t > 0$ *and* $p > 0$.

Definition 1.5([3]). *A fuzzy metric space* $(X,M,*)$ *is said to be complete if and only if every Cauchy sequence in X is convergent.*

Lemma 1.5([22]). $M(x,y,\cdot)$ *is non-decreasing for all* $x,y \in X$.

Lemma 1.6([22]). *Let* $x_n \to x$ *and* $y_n \to y$, *then*

(i) $\lim_{n\to\infty} M(x_n,y_n,t) \geq M(x,y,t)$ *for all* $t > 0$;
(ii) $\lim_{n\to\infty} M(x_n,y_n,t) = M(x,y,t)$ *for all* $t > 0$, *if* $M(x,y,t)$ *is continuous.*

Definition 1.8([23]). *Two maps* $f,g : X \to X$ *are said to be commuting if* $fg(x) = gf(x)$ *for all* $x \in X$.

Definition 1.9([5]). *Let A and B be mappings from a fuzzy metric space* $(X,M,*)$ *into itself. The maps A and B are said to be compatible (or asymptotically commuting), if for all t,* $\lim_{n\to\infty} M(ABx_n,BAx_n,t) = 1$, *whenever* $\{x_n\}$ *is a sequence in X such that* $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = z$ *for some* $z \in X$.

From the above definition it is inferred that A and B are non-compatible maps from a fuzzy metric space $(X,M,*)$ into itself if $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = z$ for some $z \in X$, but either $\lim_{n\to\infty} M(ABx_n,BAx_n,t) \neq 1$ or the limit does not exist.

Definition 1.10([6]). *Let A and B be maps from a Fuzzy metric space* $(X,M,*)$ *into itself. The maps are said to be weakly compatible if they commute at their coincidence points, that is* $Az = Bz$ *implies that* $ABz = BAz$.

Note that compatible mappings are weakly compatible but converse is not true in general.

Lemma 1.11([5]). *If there exists* $k \in (0,1)$ *such that* $M(x,y,kt) \geq M(x,y,t)$, *then* $x = y$.

Lemma 1.12([5]). *If there exists* $k \in (0,1)$ *such that* $M(y_{n+1},y_n+2,kt) \geq M(y_n,y_{n+1},t)$, *for all* $t > 0$, *then* $\{y_n\}$ *is a Cauchy sequence.*

Lemma 1.13([5]). *Let* $(X,M,*)$ *be a complete fuzzy metric space with (FM-6) and* $T : X \to X$ *be a mapping satisfying:*

$$M(Tx,Ty,kt) \geq M(x,y,t), \text{ for all } x,y \in X,$$

where $k \in (0,1)$.

Then T has a unique fixed point in X.

We now introduce our notion.

Definition 1.14.*Let Phi denote the class of all functions* $\emptyset : (R^+)5 \to [0,1]$ *satisfying the following conditions:*

$(\emptyset_i)$ $\emptyset$ *is continuous in all the variables except in the first variable;*
$(\emptyset_{ii})$ $\emptyset$ *is increasing in each of the variables;*
$(\emptyset_{iii})$ $\emptyset(t,1,t,1,1) \geq t$; $\emptyset(1,t,1,t,1) \geq t$; $\emptyset(1,1,t,t,t) \geq t$; $\emptyset(t,t,1,t,t) \geq t$.

2 Main results

In this section, we establish the existence and uniqueness of the common fixed point for the pair of mappings without considering the commutative (or minimal commutative) conditions and continuity hypothesis of the mappings in fuzzy metric spaces.

Our first result is given below:

Theorem 2.1.*Let* $(X,M,*)$ *be a complete Fuzzy metric space. Let A, B be maps from X into itself satisfying the following conditions:*

(2.1) *there exist* $\emptyset \in \Phi$, $k \in (0,1)$ *and* $\mathscr{K} \geq 2$ *such that*

$$M(Ax,By,kt) \geq \emptyset\{M(x,Ax,\mathscr{K}t),M(y,By,\mathscr{K}t), M(y,Ax,\alpha\mathscr{K}t),M(y,Bx,\beta\mathscr{K}t), M(x,y,\mathscr{K}t)\}$$

for all $x,y \in X$, $\alpha,\beta \in (0,2)$ *with* $\alpha+\beta = 2$ *and* $t > 0$.

Then A and B have a unique common fixed point in X.

*Proof.*Let $x_0 \in X$. Construct a sequence $\{x_n\}$ in X as follows:

$$Ax_{2n} = x_{2n+1} \text{ and } Bx_{2n+1} = x_{2n+2}, \quad n = 0,1,2,3,\ldots.$$

From (2.1), for $\alpha = \beta = 1$, we have

$$\begin{aligned}
&M(x_{2n+1},x_{2n+2},kt)\\
&= M(Ax_{2n},Bx_{2n+1},kt)\\
&\geq \emptyset\{M(x_{2n},Ax_{2n},\mathscr{K}t),M(x_{2n+1},\\
&\quad Bx_{2n+1},\mathscr{K}t),M(x_{2n+1},Ax_{2n},\mathscr{K}t),\\
&\quad M(x_{2n},Bx_{2n+1},\mathscr{K}t),M(x_{2n},x_{2n+1},\mathscr{K}t))\}\\
&= \emptyset\{M(x_{2n},x_{2n+1},\mathscr{K}t),M(x_{2n+1},x_{2n+2},\mathscr{K}t),\\
&\quad 1,M(x_{2n},x_{2n+2},\mathscr{K}t),M(x_{2n},x_{2n+1},\mathscr{K}t))\}\\
&\geq \emptyset\{M(x_{2n},x_{2n+1},t)*1,M(x_{2n+1},x_{2n+2},t)*1,\\
&\quad 1,M(x_{2n},x_{2n+1},t)*M(x_{2n+1},x_{2n+2},(\mathscr{K}-1)t),\\
&\quad M(x_{2n},x_{2n+1},t)*1))\} \quad (\text{by } (\emptyset_{ii}))\\
&\geq \emptyset\{M(x_{2n},x_{2n+1},t)*M(x_{2n+1},x_{2n+2},(\mathscr{K}-1)t),\\
&\quad M(x_{2n+1},x_{2n+2},t)*M(x_{2n},x_{2n+1},(\mathscr{K}-1)t),\\
&\quad 1,(M(x_{2n},x_{2n+1},t)*M(x_{2n+1},x_{2n+2},(\mathscr{K}-1)t),\\
&\quad M(x_{2n},x_{2n+1},t)*M(x_{2n+1},x_{2n+2},(\mathscr{K}-1)t))))\} \ (\text{by } (\emptyset_{ii}))
\end{aligned}$$

$$\geq M(x_{2n},x_{2n+1},t)*M(x_{2n+1},x_{2n+2},(\mathscr{K}-1)t) \text{ (by } (\emptyset_{\rm iii}))$$
$$\geq M(x_{2n},x_{2n+1},t)*M(x_{2n+1},x_{2n+2},t).$$
$$\geq M(x_{2n},x_{2n+1},t)*M(x_{2n+1},x_{2n+2},t/k^p) \text{ for } p,n\in N,$$

that is,

$$M(x_{2n+1},x_{2n+2},kt)\geq M(x_{2n},x_{2n+1},t)*M\left(x_{2n+1},x_{2n+2},\frac{t}{k^p}\right) \text{ for } p,n\in N.$$

Similarly,

$$M(x_{2n+2},x_{2n+3},kt)\geq M(x_{2n+1},x_{2n+2},t)*M\left(x_{2n+2},x_{2n+3},\frac{t}{k^p}\right) \text{ for } p,n\in N.$$

So, in general

$$M(x_{n+1},x_{n+2},kt)\geq M(x_n,x_{n+1},t)*M\left(x_{n+1},x_{n+2},\frac{t}{k^p}\right) \text{ for } p,n\in N.$$

Since, $M(x_{n+1},x_{n+2},\frac{t}{k^p})\to 1$ as $p\to\infty$, we have

$$M(x_{n+1},x_{n+2},kt)\geq M(x_n,x_{n+1},t) \quad \text{for } n\in N.$$

By Lemma 1.11, $\{x_n\}$ is a Cauchy sequence and has a limit in X, let it be z and hence, $\{Ax_{2n}\}=\{x_{2n+1}\}$ and $\{Bx_{2n+1}\}=\{x_{2n+2}\}$ being sub sequences of $\{x_n\}$ converges to z.

We claim that $Az=z$.

Using (2.1), for $\alpha=1=\beta$,

$$\begin{aligned}M(Az,Bx_{2n+1},kt)\geq \emptyset\{&M(z,Az,\mathscr{K}t),M(x_{2n+1},Bx_{2n+1},\mathscr{K}t),\\&M(x_{2n+1},Az,\mathscr{K}t),M(z,Bx_{2n+1},\mathscr{K}t),\\&M(z,x_{2n+1},\mathscr{K}t))\}.\end{aligned}$$

Proceeding limit as $n\to\infty$, and using $(\emptyset_i)$ we have

$$\begin{aligned}M(Az,z,kt)&\geq \emptyset\{M(z,Az,\mathscr{K}t),1,M(z,Az,\mathscr{K}t),1,1\}\\&\geq M(z,Az,\mathscr{K}t) \quad (\text{by } (\emptyset_{\rm iii}))\\&\geq M(z,Az,t).\end{aligned}$$

By Lemma 1.10, $Az=z$.

Next we show $Bz=z$.

Using (2.1), for $\alpha=1=\beta$,

$$\begin{aligned}M(z,Bz,kt)&=M(Az,Bz,kt)\\&\geq \emptyset\{1,M(z,Bz,\mathscr{K}t),1,M(z,Bz,\mathscr{K}t),1\}\\&\geq M(z,Bz,\mathscr{K}t) \quad (\text{by } (\emptyset_{\rm iii}))\\&\geq M(z,Bz,t).\end{aligned}$$

By Lemma 1.10, $Bz=z$. Subsequently, $Az=z=Bz$.

Uniqueness follows immediately by (2.1).

Theorem 2.2. *Let $(X,M,*)$ be a complete Fuzzy metric space. Let A, B be maps from X into itself satisfying the following conditions:*

(2.2) *there exist $\emptyset\in\Phi$, $k\in(0,1)$ and $K\geq 2$ such that*

$$\begin{aligned}M(Ax,By,kt)\geq \theta^n\{\emptyset\{&M(x,Ax,\mathscr{K}t),M(y,By,\mathscr{K}t),\\&M(y,Ax,\alpha\mathscr{K}t),M(y,Bx,\beta\mathscr{K}t),\\&M(x,y,\mathscr{K}t)\}\}\end{aligned}$$

for all $x,y\in X$, $\alpha,\beta\in(0,2)$ with $\alpha+\beta=2$ and $t>0$, where $\theta:[0,1]\to[0,1]$ being a continuous, non-decreasing function with $\theta(t)\geq t$, for all $t\in[0,1]$ and $n\in N$.

Then A and B have a unique common fixed point in X.

Proof. Proof follows immediately by Theorem 2.1.

3 An application

In this section, we provide an application of our main result in the product spaces.

Theorem 3.1. *Let $(X,M,*)$ be a complete Fuzzy metric space with $t*t\geq t$, for all $t\in[0,1]$. Let A and B be two maps on the product space $X\times X$ with values in X satisfying the following condition:*

(3.1) *there exist $\emptyset\in\Phi$, a constant $k\in(0,1)$ and $\mathscr{K}\geq 2$ such that*

$$\begin{aligned}&M(A(x,y),B(u,v),kt)\\&\geq \emptyset\{M(A(x,y),x,\mathscr{K}t),M(B(u,v),u,\mathscr{K}t),\\&\quad M(A(x,y),u,\alpha\mathscr{K}t),M(B(u,v),x,\beta\mathscr{K}t)\\&\quad M(x,u,\mathscr{K}t)*M(y,v,\mathscr{K}t))\}\end{aligned}$$

for all x,y,u,v in X, $t>0$, $\alpha,\beta\in(0,2)$ such that $\alpha+\beta=2$, then there exists a unique point w in X such that $A(w,w)=w=B(w,w)$.

Proof. Taking $v=y$ in (3.1),

$$\begin{aligned}&M(A(x,y),B(u,y),kt)\\&\geq \emptyset\{M(A(x,y),x,\mathscr{K}t),M(B(u,y),u,\mathscr{K}t),\\&\quad M(A(x,y),u,\alpha\mathscr{K}t),M(B(u,y),x,\beta\mathscr{K}t),\\&\quad M(x,u,\mathscr{K}t)*1)\}\end{aligned}$$

for all x,y,u in X. Therefore, by Theorem 2.1, for each $y\in X$, there exists one and only one $z(y)$ in X such that

(3.2) $A(z(y),y,t)=z(y)=B(z(y),y,t)$.

Now, for any y,y' in X, using (3.1) with $\alpha=1=\beta$, we have

$$\begin{aligned}&M(z(y),z(y'),kt)\\&=M(A(z(y),y),B(z(y'),y'),kt)\\&\geq \emptyset\{M(A(z(y),y),z(y),\mathscr{K}t),M(B(z(y'),y'),z(y'),\mathscr{K}t),\\&\quad M(A(z(y),y),z(y'),\mathscr{K}t),M(B(z(y'),y'),z(y),\mathscr{K}t),\\&\quad M(z(y),z(y'),\mathscr{K}t)*M(y,y',\mathscr{K}t))\}\end{aligned}$$

$$
\begin{aligned}
&= \emptyset\{1,1,M(z(y),z(y'),\mathscr{K}t),M(z(y),z(y'),\mathscr{K}t),\\
&\quad M(z(y),z(y'),\mathscr{K}t)*M(y,y',\mathscr{K}t))\}\\
&= \emptyset\{1,1,M(z(y),z(y'),\mathscr{K}t)*1,M(z(y),z(y'),\mathscr{K}t)*1,\\
&\quad M(z(y),z(y'),\mathscr{K}t)*M(y,y',\mathscr{K}t))\}\\
&\geq \emptyset\{1,1,M(z(y),z(y'),\mathscr{K}t)*M(y,y',\mathscr{K}t),\\
&\quad M(z(y),z(y'),\mathscr{K}t)*M(y,y',\mathscr{K}t),\\
&\quad M(z(y),z(y'),\mathscr{K}t)*M(y,y',\mathscr{K}t))\} \ \text{(by } (\emptyset_{ii}))\\
&\geq M(z(y),z(y'),\mathscr{K}t)*M(y,y',\mathscr{K}t) \quad \text{(by } (\emptyset_{iii}))\\
&\geq M(z(y),z(y'),t)*M(y,y',t)\\
&\geq M\left(z(y),z(y'),\frac{t}{k^n}\right)*M(y,y',t) \to 1*M(y,y',t),
\end{aligned}
$$

so that, we have

$$M(z(y),z(y'),kt) \geq M(y,y',t).$$

Therefore, Lemma 1.12 implies that the map $z(\cdot)$ of X into itself has exactly one fixed point w in X, i.e., $z(w) = w$. Thus, by (3.2), $w = z(w) = A(w,w) = w = B(w,w)$. Uniqueness follows immediately using (3.1) and $t*t \geq t$.

4 Conclusion

Without making an appeal to the continuity and commutative (or minimal commutative) conditions of maps, we have established the existence and uniqueness of common fixed point for a pair of mappings in the framework of fuzzy metric spaces using implicit functions. Further, an application of the main result in the product spaces illustrates the usability of the proved results.

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Manish Jain is presently working as Assistant Professor in the Department of Mathematics at Ahir College, Rewari (Haryana) India and has nearly 10 years of teaching and research experience. He has been reviewer to many International Journals and has contributed numerous research papers in various reputed International Journals. His area of interest includes functional analysis, real analysis, nonlinear analysis, fixed point theory and applications.

Satish Kumar is working as Associate Professor in the Department of Mathematics, Indira Gandhi University, Meerpur, Rewari (Haryana) India. He has been teaching real analysis, topology, functional analysis at postgraduate level since last 25 years. His areas of interest are fixed point theory and inventory models.

Manoj Kumar is working as an Assistant Professor in Department of Mathematics at Lovely Professional University, Punjab. He did his Ph.D. from Guru Jambheshwar University of Science and Technology. He has published a lot of papers in the field of fixed point theory. He has authored a book on "Elements of Engineering Mathematics". Functional analysis, Topology and Fixed point theory are his research interests.