Journal of Analysis & Number Theory
An International Journal

http://dx.doi.org/10.18576/jant/050102

New Approach of $(\alpha_0 - \psi)$-Contractive Mappings on a Closed Ball in dislocated metric spaces

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Received: 3 Jun. 2016, Revised: 2 Aug. 2016, Accepted: 4 Aug. 2016
Published online: 1 Jan. 2017

Abstract: The aim of this paper is to introduce an integral type closed ball and establish common fixed point results of integral type $(\alpha_0 - \psi)$-contractive mapping in complete dislocated metric space. We extend the scope of $(\alpha, \psi)-$contractive mappings which are α_0-admissible on a closed ball in complete dislocated metric spaces. Our results provide extensions as well as substantial generalizations and improvements of several well known results in the existing comparable literature.

Keywords: Common fixed point; Complete Dislocated metric Space; α_0-ψ-contractive mappings; α_0-admissible; Closed ball; integral type contraction; Lebesgue integrable.

1 PRELIMINARIES AND SCOPE

Branciari [6] obtained a fixed point theorem for a single valued mapping satisfying an analogue of Banach's contraction principle for an integral type inequality. Rhoades [21] proved two fixed point theorems involving more general contractive condition of integral type. Moradi and Omid [17] established fixed point results for mappings satisfying integral type inequality depending on another function.

Samet et al. [23] introduced a concept of $(\alpha, \psi)-$ contractive type mappings and established fixed point theorems for such mappings in complete metric space. Hussain et al. [11] ,[12], [13] and Salimi et al. [22] obtained fixed point results for single and multi-valued mappings extending the notion of α-admissible mappings. Mohammadi et al. [16] introduced a new notion of $\alpha - \phi-$contractive mappings and show that this is a real generalization for some old results. Recently Arshad et al. [2] established fixed point results of a pair of contractive dominated mappings on a closed ball in an ordered complete dislocated metric space. Over the years, fixed point theory has been generlized in multi-directions by several mathematicians(see [1-22]).

Let Ψ denote the family of all nondecreasing functions $\psi : [0,+\infty) \to [0,+\infty)$ such that $\sum_{n=1}^{+\infty} \psi^n(t) < +\infty$, and $\psi(0) = 0$ for each $t > 0$, where ψ^n is the n^{th} iterate of ψ.

Definition 1. *[10] Let X be a nonempty set and let $d_l : X \times X \to [0,\infty)$ be a function, called a dislocated metric (or simply d_l-metric) if the following conditions hold for any $x,y,z \in X$:*

(i) If $d_l(x,y) = 0$, then $x = y$;
(ii) $d_l(x,y) = d_l(y,x)$;
(iii) $d_l(x,y) \le d_l(x,z) + d_l(z,y)$.

The pair (X,d_l) is then called a dislocated metric space. It is clear that if $d_l(x,y) = 0$, then from (i), $x = y$. But if $x = y$, $d_l(x,y)$ may not be 0.

Definition 2. *[10] A sequence $\{x_n\}$ in a d_l-metric space (X,d_l) is called a Cauchy sequence if given $\varepsilon > 0$, there corresponds $n_0 \in N$ such that for all $n,m \ge n_0$ we have $d_l(x_m,x_n) < \varepsilon$.*

Definition 3. *[10]A sequence $\{x_n\}$ in d_l-metric space converges with respect to d_l if there exists $x \in X$ such that $d_l(x_n,x) \to 0$ as $n \to \infty$. In this case, x is called limit of $\{x_n\}$ and we write $x_n \to x$.*

Definition 4. *[10] A d_l-metric space (X,d_l) is called complete if every Cauchy sequence in X converges to a point in X.*

Definition 5. *[23]. Let (X,d) be a metric space. A mapping $T : X \to X$ is an $(\alpha,\ \psi)-$contractive mapping if*

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there exist two functions $\alpha : X \times X \to [0,+\infty)$ and $\psi \in \Psi$ such that

$$\alpha(x,y)d(Tx,Ty) \le \psi(d(x,y)),$$

for all $x,y \in X$.

Definition 6. *[23]. Let $T : X \to X$ and $\alpha : X \times X \to [0,+\infty)$. We say that T is α-admissible if $x,y \in X$, $\alpha(x,y) \ge 1$ implies that $\alpha(Tx,Ty) \ge 1$.*

Example 1. Let $X = (0,\infty)$ and T an identity mapping on X. Define $\alpha : X \times X \to [0,\infty)$ by

$$\alpha(x,y) = \begin{cases} e^{\frac{y}{x}} \text{ if } x \ge y, \\ 0 \text{ if } x < y. \end{cases}$$

Then T is $\alpha-$admissible.

Definition 7. *([1]). Let $S,T : X \to X$ and $\alpha : X \times X \to [0,+\infty)$. We say that the pair (S,T) is α-admissible if $x,y \in X$ such that $\alpha(x,y) \ge 1$, then we have $\alpha(Sx,Ty) \ge 1$ and $\alpha(Tx,Sy) \ge 1$.*

Definition 8. *([22]). Let $T : X \to X$ and $\alpha,\eta : X \times X \to [0,+\infty)$ two functions. We say that T is α-admissible mapping with respect to η if $x,y \in X$ such that $\alpha(x,y) \ge \eta(x,y)$, then we have $\alpha(Tx,Ty) \ge \eta(Tx,Ty)$. Note that if we take $\eta(x,y) = 1$ then T is called α-admissible mapping [23]. If $\alpha(x,y) = 1$, then T is called an η-subadmissible mapping.*

Definition 9. *Let $S,T : X \to X$ and $\alpha,\eta : X \times X \to [0,+\infty)$ two functions. We say that the pair (S,T) is α-admissible with respect to η if $x,y \in X$ such that $\alpha(x,y) \ge \eta(x,y)$ then we have $\alpha(Sx,Ty) \ge \eta(Sx,Ty)$ and $\alpha(Tx,Sy) \ge \eta(Sx,Ty)$. Also, if we take $\eta(x,y) = 1$, then, the pair (S,T) is called α-admissible, if we take, $\alpha(x,y) = 1$, then we say that the pair (S,T) is η-subadmissible mapping. If we take $S = T$ we obtain Definition 10. Also if we take $\eta(x,y) = 1$, then we obtain the Definition 9 of Abdeljawad [1].*

Definition 10. *([16]). Let $T : X \to X$ and $\alpha_0 : X \times X \to [0,+\infty)$ by*

$$Tx = x+1,\ \alpha_0(x,y) = \left\{ \begin{array}{ll} 1 & \alpha(x,y) \ge \eta(x,y) \\ 0 & otherwise \end{array} \right\}.$$

We say that T is α_0-admissible. If $\alpha_0(x,y) \ge 1$, then $\alpha(x,y) \ge \eta(x,y)$ and so $\alpha(Tx,Ty) \ge \eta(Tx,Ty)$. This implies $\alpha_0(Tx,Ty) = 1$. Also $\alpha_0(x_0,Sx_0) = 1$.

Definition 11. *Let $S,T : X \to X$ and $\alpha_0 : X \times X \to [0,+\infty)$ by*

$$Sx = x^2, Tx = x \text{ and } \alpha_0(x,y) = \left\{ \begin{array}{ll} 1 & \alpha(x,y) \ge \eta(x,y) \\ 0 & otherwise \end{array} \right\}.$$

We say that the pair (S,T) is α_0-admissible. If $\alpha_0(x,y) \ge 1$, then $\alpha(x,y) \ge \eta(x,y)$ and so $\alpha(Sx,Ty) \ge \eta(Sx,Ty)$ and $\alpha(Tx,Sy) \ge \eta(Sx,Ty)$. This implies $\alpha_0(Sx,Ty) = \alpha_0(Tx,Sy) = 1$. Also $\alpha_0(x_0,Sx_0) = 1$. If we take $S = T$ we obtain the Definition 12.

Define $\digamma = \{\varphi : R^+ \to R^+ : \varphi$ is a Lebesgue integral mapping which is summable, nonnegative and satisfies $\int_0^{\varepsilon} \varphi(t)\,dt > 0$, for each $\varepsilon > 0\}$. The ball $\overline{B(x,r)}$,

$$\text{where } \overline{B(x,r)} = \left\{ y \in X : \int_0^{d_l(x,y)} \varphi(t)\,dt \le r \right\}$$

is a generalized closed ball in dislocated metric space, for some $x \in X$ and $\varepsilon > 0$.

2 Fixed point results

We now prove some fixed point results for $(\alpha_0,\psi)-$ contraction mappings of integral type in complete dislocated metric space.

Theorem 1. *Let (X,d_l) be a complete dislocated metric space and $S,T : X \to X$ be two mappings. Suppose there exist a function $\alpha_0 : X \times X \to [0,+\infty)$ such that the pair (S,T) is α_0-admissible. For $r > 0$, $x_0 \in X$, assume that,*

$$x,y \in \overline{B(x_0,r)},\ \alpha_0(x,y) \ge 1$$

implies

$$\int_0^{d_l(Sx,Ty)} \phi(t)\,dt \le \psi\left(\int_0^{d_l(x,y)} \phi(t)\,dt\right) \tag{1}$$

where $\phi \in \digamma$, $\psi \in \Psi$, and

$$\sum_{i=0}^{j} \psi^i \left(\int_0^{d_l(x_0,Sx_0)} \phi(t)\,dt\right) \le r \tag{2}$$

Suppose that for any sequence $\{x_n\}$ in $\overline{B(x_0,r)}$ such that $\alpha_0(x_n,x_{n+1}) \ge 1$ for all $n \in N \cup \{0\}$ and $x_n \to u \in \overline{B(x_0,r)}$ as $n \to +\infty$ then $\alpha_0(x_n,u) \ge 1$ for all $n \in N \cup \{0\}$.

Then, there exists a point x^* in $\overline{B(x_0,r)}$ such that $x^* = Sx^* = Tx^*$.

Proof. Let x_1 in X be such that $x_1 = Sx_0$ and $x_2 = Tx_1$. Continuing this process, we construct a sequence x_n of points in X such that,

$$x_{2i+1} = Sx_{2i}, \text{ and } x_{2i+2} = Tx_{2i+1}, \text{ where } i = 0,1,2,\ldots.$$

Since $\alpha_0(x_0,x_1) \ge 1$ then $\alpha(x_0,x_1) \ge \eta(x_0,x_1)$ otherwise $\alpha_0(x_0,x_1) = 0$, and the pair (S,T) is α_0-admissible we have, $\alpha(Sx_0,Tx_1) \geq \eta(Sx_0,Tx_1)$ from which we deduce that $\alpha(x_1,x_2) \geq \eta(x_1,x_2)$ which also implies that $\alpha_0(x_1,x_2) = 1$ then $\alpha(x_1,x_2) \geq \eta(x_1,x_2)$ otherwise $\alpha_0(x_1,x_2) = 0$, and the pair (S,T) is α_0-admissible we have $\alpha(Tx_1,Sx_2) \geq \eta(Tx_1,Sx_2)$ implies $\alpha_0(x_2,x_3) = 1$ otherwise $\alpha_0(x_2,x_3) = 0$. Continuing in this way we obtain $\alpha_0(x_n,x_{n+1}) = 1$ then $\alpha(x_n,x_{n+1}) \geq \eta(x_n,x_{n+1})$ otherwise $\alpha_0(x_n,x_{n+1}) = 0$, for all $n \in N \cup \{0\}$. First we show that $x_n \in \overline{B(x_0,r)}$ for all $n \in N$. Using inequality (2.2), we have,

$$\sum_{i=0}^{n} \psi^i \left(\int_0^{d_l(x_0,Sx_0)} \phi(t)\,dt \right) \leq r.$$

It follows that,

$$x_1 \in \overline{B(x_0,r)}.$$

Let $x_2,\cdots,x_j \in \overline{B(x_0,r)}$ for some $j \in N$. If $j = 2i+1$, where $i = 0,1,2,\ldots \frac{j-1}{2}$ so using inequality (1), we obtain,

$$\begin{aligned}\int_0^{d_l(x_{2i+1},x_{2i+2})} \phi(t)\,dt &= \int_0^{d_l(Sx_{2i},Tx_{2i+1})} \phi(t)\,dt \\ &\leq \psi\left(\int_0^{d_l(x_{2i},x_{2i+1})} \phi(t)\,dt \right) \\ &\leq \psi^2\left(\int_0^{d_l(x_{2i-1},x_{2i})} \phi(t)\,dt \right) \\ &\leq \cdots \leq \psi^{2i+1}\left(\int_0^{d_l(x_0,x_1)} \phi(t)\,dt \right).\end{aligned}$$

Thus we have,

$$\int_0^{d_l(x_{2i+1},x_{2i+2})} \phi(t)\,dt \leq \psi^{2i+1}\left(\int_0^{d_l(x_0,x_1)} \phi(t)\,dt \right). \quad (3)$$

If $j = 2i+2$, then as x_1, x_2 ...,$x_j \in \overline{B(x_0,r)}$ where $(i = 0,1,2,\ldots,\frac{j-2}{2})$. We obtain,

$$\int_0^{d_l(x_{2i+2},x_{2i+3})} \phi(t)\,dt \leq \psi^{2(i+1)}\left(\int_0^{d_l(x_0,x_1)} \phi(t)\,dt \right). \quad (4)$$

Thus from inequality (3) and (4), we have

$$\int_0^{d_l(x_j,x_{j+1})} \phi(t)\,dt \leq \psi^{j}\left(\int_0^{d_l(x_0,x_1)} \phi(t)\,dt \right). \quad (5)$$

Now,

$$\begin{aligned}\int_0^{d_l(x_0,x_{j+1})} \phi(t)\,dt &= \int_0^{d_l(x_0,x_1)} \phi(t)\,dt + \int_0^{d_l(x_1,x_2)} \phi(t)\,dt \\ &+ \int_0^{d_l(x_2,x_3)} \phi(t)\,dt + \ldots \\ &+ \int_0^{d_l(x_j,x_{j+1})} \phi(t)\,dt \\ &\leq \sum_{i=0}^{j} \psi^i \left(\int_0^{d_l(x_0,x_1)} \phi(t)\,dt \right) \\ &\leq r.\end{aligned}$$

Thus $x_{j+1} \in \overline{B(x_0,r)}$. Hence $x_n \in \overline{B(x_0,r)}$ for all $n \in N$. Now inequality (5) can be written as

$$\int_0^{d_l(x_n,x_{n+1})} \phi(t)\,dt \leq \psi^n\left(\int_0^{d_l(x_0,x_1)} \phi(t)\,dt \right), \text{ for all } n \in N. \quad (6)$$

Fix $\varepsilon > 0$ and let $n(\varepsilon) \in N$ such that $\sum \psi^n \left(\int_0^{d_l(x_0,x_1} \phi(t)\,dt \right) < \varepsilon$. Let $n,m \in N$ with $m > n > k(\varepsilon)$, using the triangular inequality, we obtain,

$$d(x_n,x_m) \leq \sum_{k=n}^{m-1} d(x_k,x_{k+1}) \quad (7)$$

Now from (6) and (7), we have

$$\begin{aligned}\int_0^{d_l(x_n,x_m)} \phi(t)\,dt &\leq \sum_{k=n}^{m-1} \psi^k \left(\int_0^{d_l(x_k,x_{k+1})} \phi(t)\,dt \right) \\ &\leq \sum_{n \geq n(\varepsilon)} \psi^k \left(\int_0^{d_l(x_0,x_1)} \phi(t)\,dt \right) < \varepsilon.\end{aligned}$$

Hence $\{x_n\}$ is a Cauchy sequence in $(\overline{B(x_0,r)},d_l)$. Since X is complete dislocated metric space, so there exists $x^* \in \overline{B(x_0,r)}$ such that $x_n \to x^*$. Also

$$\lim_{n\to\infty} \left(\int_0^{d_l(x_n,x^*)} \phi(t)\,dt \right) = 0. \quad (8)$$

On the other hand, from (ii), we have

$$\alpha(x^*,x_n) \geq \eta(x^*,x_n) \text{ for all } n \in N \cup \{0\}.$$

Now using triangle inequality, together with (1), we get

$$\int_0^{d_l(Sx^*,x_{2i+2})} \phi(t)\,dt \leq \psi\left(\int_0^{d_l(x^*,x_{2i+1})} \phi(t)\,dt \right) < \int_0^{d_l(x^*,x_{2i+1})} \phi(t)\,dt.$$

Letting $i \to \infty$ and by using inequality (8), we obtain $d_l(Sx^*,x^*) < 0$. Hence $Sx^* = x^*$. Similarly by using

$$\int_0^{d_l(Tx^*,x_{2i+1})} \phi(t)\,dt \leq \psi\left(\int_0^{d_l(x^*,x_{2i})} \phi(t)\,dt \right) < \int_0^{d_l(x^*,x_{2i})} \phi(t)\,dt,$$

we obtain $d_l(Tx^*,x^*) = 0$, that is, $Tx^* = x^*$. Hence S and T have a common fixed point in $\overline{B(x_0,r)}$.

If $\alpha_0(x,y) = 1$ in Theorem 1 then, we have the following Corollary.

Corollary 1. *Let (X,d_l) be a complete dislocated metric space and $S,T : X \to X$ be two mappings. Suppose there exist a function $\alpha_0 : X \times X \to [0,+\infty)$ such that the pair (S,T) is α_0-admissible. For $r > 0$, $x_0 \in X$, assume that,*

$$\alpha(x,y) \geq \eta(x,y) \textit{ implies}$$

$$\int_0^{d_l(Sx,Ty)} \phi(t)\,dt \leq \psi\left(\int_0^{(d_l(x,y)} \phi(t)\,dt \right) \quad (9)$$

where $\phi \in F$, $\psi \in \Psi$, $x,y \in \overline{B(x_0,r)}$ *and*

$$\sum_{i=0}^{j} \psi^i \left(\int_0^{d_l(x_0,Sx_0)} \phi(t)\,dt \right) \le r \tag{10}$$

Suppose that for any sequence $\{x_n\}$ *in* $\overline{B(x_0,r)}$ *such that* $\alpha_0(x_n,x_{n+1}) \ge 1$ *for all* $n \in N \cup \{0\}$ *and* $x_n \to u \in \overline{B(x_0,r)}$ *as* $n \to +\infty$ *then* $\alpha_0(x_n,u) \ge 1$ *for all* $n \in N \cup \{0\}$.

Then, there exists a point x^* in $\overline{B(x_0,r)}$ such that $x^* = Sx^* = Tx^*$.

If $S=T$, and $\alpha_0(x,y)=1$ in Theorem 1 then, we have the following Corollary.

Corollary 2. *Let* (X,d_l) *be a complete dislocated metric space and* $T : X \to X$ *be two mappings. Suppose there exist two functions,* $\alpha,\eta : X \times X \to [0,+\infty)$ *such that* T *is* α*-admissible with respect to* η*. For* $r>0$, $x_0 \in X$, *assume that,*

$$\alpha(x,y) \ge \eta(x,y) \Rightarrow$$

$$\int_0^{d_l(Tx,Ty)} \phi(t)\,dt \le \psi \left(\int_0^{(d_l(x,y)} \phi(t)\,dt \right) \tag{11}$$

where $\phi \in F$, $\psi \in \Psi$, $x,y \in \overline{B(x_0,r)}$ *and*

$$\sum_{i=0}^{j} \psi^i \left(\int_0^{d_l(x_0,Tx_0)} \phi(t)\,dt \right) \le r \tag{12}$$

Suppose that the following assertions hold:

(i)$\alpha(x_0,Tx_0) \ge \eta(x_0,Tx_0)$;

(ii)for any sequence $\{x_n\}$ in $\overline{B(x_0,r)}$ such that $\alpha(x_n,x_{n+1}) \ge \eta(x_n,x_{n+1})$ for all $n \in N \cup \{0\}$ and $x_n \to u \in \overline{B(x_0,r)}$ as $n \to +\infty$ then $\alpha(x_n,u) \ge \eta(x_n,u)$ for all $n \in N \cup \{0\}$.

Then, there exists a point x^* in $\overline{B(x_0,r)}$ such that $Tx^* = x^*$.

If $\phi(t)=1$ in Corollary 1, we obtain the following Corollary.

Corollary 3. *Let* (X,d_l) *be a complete dislocated metric space and* $S,T : X \to X$ *be two mappings. Suppose there exist two functions,* $\alpha,\eta : X \times X \to [0,+\infty)$ *such that the pair* (S,T) *is* α*-admissible with respect to* η*. For* $r>0$, $x_0 \in X$, *and* $\psi \in \Psi$ *assume that,*

$$\alpha(x,y) \ge \eta(x,y) \Rightarrow$$

$$d_l(Sx,Ty) \le \psi((d_l(x,y)) \tag{13}$$

$x,y \in \overline{B(x_0,r)}$ *and*

$$\sum_{i=0}^{n} \psi^i (d_l(x_0,Sx_0)) \le r. \tag{14}$$

Suppose that the following assertions hold:

(i)$\alpha(x_0,Sx_0) \ge \eta(x_0,Sx_0)$;

(ii)for any sequence $\{x_n\}$ in $\overline{B(x_0,r)}$ such that $\alpha(x_n,x_{n+1}) \ge \eta(x_n,x_{n+1})$ for all $n \in N \cup \{0\}$ and $x_n \to u \in \overline{B(x_0,r)}$ as $n \to +\infty$, then $\alpha(x_n,u) \ge \eta(x_n,u)$ for all $n \in N \cup \{0\}$.

Then, there exists a point x^* in $\overline{B(x_0,r)}$ such that $x^* = Sx^* = Tx^*$.

If $\alpha(x,y)=1$ in Corollary 3, we obtain the following Corollary.

Corollary 4. *Let* (X,d_l) *be a complete dislocated metric space and* $S,T : X \to X$ *be two mappings. Suppose there exists,* $\eta : X \times X \to [0,+\infty)$ *such that the pair* (S,T) *is* η*-subadmissible. For* $\psi \in \Psi$*, assume that,*

$$\eta(x,y) \le 1 \Rightarrow d_l(Sx,Ty) \le \psi(d_l(x,y)) \tag{15}$$

$x,y \in \overline{B(x_0,r)}$ *and*

$$\sum_{i=0}^{j} \psi^i(d_l(x_0,Sx_0)) \le r. \tag{16}$$

Suppose that the following assertions hold:

(i)$\eta(x_0,Sx_0) \le 1$;

(ii)*for any sequence* $\{x_n\}$ in $\overline{B(x_0,r)}$ such that $\eta(x_n,x_{n+1}) \le 1$ for all $n \in N \cup \{0\}$ and $x_n \to u \in \overline{B(x_0,r)}$ as $n \to +\infty$ then $\eta(x_n,u) \le 1$ for all $n \in N \cup \{0\}$.

Then, there exists a point x^* in $\overline{B(x_0,r)}$ such that $x^* = Sx^* = Tx^*$.

If $\eta(x,y)=1$ in Corollary 1, we obtain the following Corollary.

Corollary 5. *Let* (X,d_l) *be a complete dislocated metric space and* $S,T : X \to X$ *be two mappings. Suppose there exist two functions,* $\alpha,\eta : X \times X \to [0,+\infty)$ *such that the pair* (S,T) *is* α*-admissible with respect to* η*. For* $r>0$, $x_0 \in X$, *assume that,*

$$\alpha(x,y) \ge 1 \Rightarrow$$

$$\int_0^{d_l(Sx,Ty)} \phi(t)\,dt \le \psi \left(\int_0^{(d_l(x,y)} \phi(t)\,dt \right) \tag{17}$$

where $\phi \in F$, $\psi \in \Psi$, $x,y \in \overline{B(x_0,r)}$ *and*

$$\sum_{i=0}^{n} \psi^i \left(\int_0^{d_l(x_0,Sx_0)} \phi(t)\,dt \right) \le r. \tag{18}$$

Suppose that the following assertions hold:

(i)$\alpha(x_0,Sx_0) \ge 1$;

(ii)for any sequence $\{x_n\}$ in $\overline{B(x_0,r)}$ such that $\alpha(x_n,x_{n+1}) \geq 1$ for all $n \in N \cup \{0\}$ and $x_n \to u \in \overline{B(x_0,r)}$ as $n \to +\infty$ then $\alpha(x_n,u) \geq 1$ for all $n \in N \cup \{0\}$.

Then, there exists a point x^* in $\overline{B(x_0,r)}$ such that $x^* = Sx^* = Tx^*$.

If $\eta(x,y) = 1$ in Corollary 3, we have the following Corollary.

Corollary 6. *Let (X,d_l) be a complete dislocated metric space and $S,T : X \to X$ be two mappings. Suppose there exist two functions, $\alpha : X \times X \to [0,+\infty)$ such that the pair (S,T) is α-admissible. For $r > 0$, $x_0 \in X$, and $\psi \in \Psi$ assume that,*

$$\alpha(x,y) \geq 1 \Rightarrow$$

$$d_l(Sx,Ty) \leq \psi((d_l(x,y)) \tag{19}$$

$x,y \in \overline{B(x_0,r)}$ and

$$\sum_{i=0}^{j} \psi^i(d_l(x_0,Sx_0)) \leq r \tag{20}$$

Suppose that the following assertions hold:

(i)$\alpha(x_0,Sx_0) \geq 1$;
(ii)for any sequence $\{x_n\}$ in $\overline{B(x_0,r)}$ such that $\alpha(x_n,x_{n+1}) \geq 1$ for all $n \in N \cup \{0\}$ and $x_n \to u \in \overline{B(x_0,r)}$ as $n \to +\infty$ then $\alpha(x_n,u) \geq 1$ for all $n \in N \cup \{0\}$.

Then, there exists a point x^* in $\overline{B(x_0,r)}$ such that $x^* = Sx^* = Tx^*$.

If $S = T$ in Corollary 6, we obtain the following Corollary.

Corollary 7. *Let (X,d_l) be a complete dislocated metric space and $T : X \to X$ be two mappings. Suppose there exist two functions, $\alpha : X \times X \to [0,+\infty)$ such that T is α-admissible. For $r > 0$, $x_0 \in X$, and $\psi \in \Psi$, assume that,*

$$\alpha(x,y) \geq 1 \Rightarrow$$

$$d_l(Tx,Ty) \leq \psi((d_l(x,y)) \tag{21}$$

$x,y \in \overline{B(x_0,r)}$ and

$$\sum_{i=0}^{j} \psi^i(d_l(x_0,Tx_0)) \leq r. \tag{22}$$

Suppose that the following assertions hold:

(i)$\alpha(x_0,Tx_0) \geq 1$;
(ii)for any sequence $\{x_n\}$ in $\overline{B(x_0,r)}$ such that $\alpha(x_n,x_{n+1}) \geq 1$ for all $n \in N \cup \{0\}$ and $x_n \to u \in \overline{B(x_0,r)}$ as $n \to +\infty$ then $\alpha(x_n,u) \geq 1$ for all $n \in N \cup \{0\}$.

Then, there exists a point x^* in $\overline{B(x_0,r)}$ such that $Tx^* = x^*$.

Example 2. Let $X = R^+ \cup \{0\}$ and be endowed with usual order and let $d_l : X \times X \to X$ be the complete ordered dislocated metric on X defined by $d_l(x,y) = x + y$. Let $S,T : X \to X$ be defined by,

$$Sx = \begin{cases} \frac{x}{5} & \text{if } x \in [0,1] \\ x - \frac{1}{2} & \text{if } x \in (1,\infty) \end{cases}$$

and

$$Tx = \begin{cases} \frac{2x}{5} & \text{if } x \in [0,1] \\ x - \frac{1}{4} & \text{if } x \in (1,\infty). \end{cases}$$

Considering, $x_0 = 1$, $r = 2$, then $\overline{B(x_0,r)} = [0,1]$, and

$$\alpha(x,y) = \begin{cases} 1 \text{if } x,y \in X; \\ 0 \text{ otherwise.} \end{cases}$$

Clearly, the pair (S,T) is an α-ψ-contractive mapping with $\psi(t) = \frac{t}{2}$. Now,

$$d_l(x_0,Sx_0) = d_l(1,S1) = d_l(1,\frac{1}{5}) = 1 + \frac{1}{5} = \frac{6}{5}$$

$$\sum_{i=0}^{n} \psi^n(d_l(x_0,Sx_0)) = \frac{6}{5}\sum_{i=0}^{n}\frac{1}{2^n} < \frac{3}{2}(\frac{6}{5}) = \frac{9}{5} < 2$$

Also if $x,y \in (1,\infty)$, then

$$\begin{aligned} 2x + 2y - \frac{3}{2} &> x + y \\ (x + y - \frac{3}{4}) &> \frac{x+y}{2} \\ x + y - \frac{3}{4} &> \psi(x+y) \\ d_l(Sx,Ty) &> \psi(d_l(x,y)) \end{aligned}$$

Then the contractive condition does not hold on X. Also if, $x,y \in \overline{B(x_0,r)}$, then

$$\begin{aligned} \frac{2x}{5} + \frac{4y}{5} &\leq x + y \\ \frac{x}{5} + \frac{2y}{5} &\leq \frac{x+y}{2} \\ \frac{x}{5} + \frac{2y}{5} &\leq \psi(x+y) \\ d_l(Sx,Ty) &\leq \psi(d_l(x,y)). \end{aligned}$$

$$\int_0^{d_l(Sx,Ty)} \phi(t)\,dt \leq \psi\left(\int_0^{d_l(x,y)} \phi(t)\,dt\right).$$

Theorem 2. *Adding condition "if p is any common fixed point in $\overline{B(x_0,r)}$ of S and T, x be any fixed point of S or T in $\overline{B(x_0,r)}$, then $\alpha_0(x,p)\geq 1$" to the hypotheses of Theorem 15. Then S and T have a unique common fixed point p and $d_l(x,p)=0$.*

Proof. Assume that q be another fixed point of S and T in $\overline{B(x_0,r)}$, then, by assumption, $\alpha_0(p,q)\geq 1$ implies $\alpha(p,q)\geq\eta(p,q)$, otherwise $\alpha_0(p,q)=0$.

$$\int_0^{d_l(p,q)}\phi(t)\,dt<\int_0^{d_l(p,q)}\phi(t)\,dt.$$

Which contradiction to the fact that for each $t>0$, $\psi(t)<t$. So $\int_0^{d_l(p,q)}\phi(t)\,dt=0$ then $d_l(p,q)$ implies $p=q$. Hence S and T have no fixed point other than p. Now, $\alpha_0(p,p)\geq 1$ implies $\alpha(p,p)\geq\eta(p,p)$, otherwise $\alpha_0(p,p)=0$, then,

$$\int_0^{d_l(p,p)}\phi(t)\,dt=\int_0^{d_l(Sp,Tp)}\phi(t)\,dt\leq\psi\left(\int_0^{d_l(p,p)}\phi(t)\,dt\right).$$

This implies that,

$$d_l(p,p)=0.$$

Remark. (i) Every modified closed ball is a closed ball by setting $\varphi(t)=1$ over R^+.

(ii) Every contractive condition of integral type automatically includes a corresponding contractive condition, not involving integrals, by setting $\varphi(t)=1$ over R^+.

3 Conclusion

In this approach, the main aim of our paper is to introduce new concepts of an integral type closed ball and establish common fixed point results of integral type contractive mapping in complete dislocated metric space. Existennce of fixed point results of such type of contraction on closed ball in complete metric space are established. In this article we observe that the integral type contraction automatically includes a corresponding contractive condition, not involving integral, by setting $\varphi(t)=1$. The study of results is very useful in the sense that it requires the integral type contraction mapping only on the closed ball instead on the whole space. The new concepts lead to further investigations and applications. It will be also interesting to apply these concepts in a different metric spaces.

Acknowledgment

The authors are grateful to the anonymous referee for a careful checking of the details and for helpful comments that improved this paper.

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