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TN Weibull Lomax Distribution and its Applications in Survival and Reliability Analysis

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Abstract: This article proposes the TN family of distribution as a new family of distribution. Several significant characteristics of the TN family are obtained. The shapes of its density and hazard functions are illustrated. Additionally, we propose a new submodel of the TN family of distributions, referred to as the TN Weibull Lomax (TNWL) distribution. For the TN Weibull Lomax model, important statistical characteristics such as moments, skewness, kurtosis and the moment generating function are determined. The method of maximum likelihood (ML), Anderson Darling (AD) method and Cramer-von Mises (CvM) method are used to estimate the parameters of the TN Weibull Lomax distribution. Two real-world datasets are analyzed to show the effectiveness of the TN Weibull Lomax model. The findings of the dataset analysis demonstrate that the TN Weibull Lomax distribution is superior to several of the compared distributions.

Keywords: Hazard Function, Lomax Distribution, Maximum Likelihood Estimation, TN Family, Weibull Lomax Distribution

1 Introduction

In Statistics, the development of statistical distribution is well-established but still significant field. In many applicable fields including engineering, environmental science, economics, finance, reliability theory and medical research, classical distributions are widely used especially for modeling lifetime data. A wide range of approaches for creating new distributions by expanding or modifying existing distributions have been presented and investigated by numerous researchers in recent years.

Newly developed statistical distributions offer more versatility in expressing complex features of data including skewness, kurtosis, heavy tails and different hazard rate shapes in addition to their descriptive ability. Especially in lifetime and survival data, these characteristics are frequently inadequately represented by classical distributions. Researchers aim to provide more accurate inferential results, enhance interpretability and improve model fit by introducing additional parameters or transformation processes. Due to these factors, constant development of new distributional families is still essential to satisfy the evolving requirements of contemporary statistical modeling in a variety of applied fields.

Recent advances in distribution theory have stimulated considerable research interest. Researchers have focused on developing new methods to extend existing families of lifetime distributions. Some of the well-known generators are the exponentiated family proposed by Mudholkar and Srivastava (1993) [22], the Marshall-Olkin generated family (MO-G) introduced by Marshall and Olkin (1997) [20], the beta-G family by Eugene *et al.* (2002) [14], the gamma-G family by Zografos and Balakrishnan (2009) [29], the Kumaraswamy-G family by Cordeiro and De Castro (2011) [11], the McDonald-G (Mc-G) family by Alexander *et al.* (2012) [2], the Transformed-Transformer (T-X) by Alzaatreh *et al.* (2013) [6], the exponentiated generalized class by Cordeiro *et al.* (2013) [12], the exponentiated half-logistic family by Cordeiro *et al.* (2014) [10], the log-gamma-G family by Amini *et al.* (2014) [7], the Lomax generator by Cordeiro *et al.* (2014) [13], the Weibull-G family by Bourguignon *et al.* (2014) [8], the Kumaraswamy Marshall-Olkin family by Alizadeh *et al.* (2015) [4], the Kumaraswamy odd log-logistic-G by Alizadeh *et al.* (2015) [3], the odd generalized exponential-G by Tahir *et al.* (2015) [24], the Logistic-X by Tahir *et al.* (2015) [25] and the beta Marshall-Olkin family by Alizadeh *et al.* (2015) [5].

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The primary motive of this work is to present a family of distributions with a highly flexible, left- or right-skewed shape that improves the characteristics and flexibility of existing distributions. Additionally, an extended baseline distribution with closed forms for the survival, hazard rate and cumulative distribution functions should be presented. Maurya and Nadarajah's (2021) [21] study of Poisson-generated distributions and explanation of different distribution generation techniques served as the inspiration for this work.

The following sections of this article are divided as follows. In Section 2, TN family and its properties are discussed. A special sub-model of the proposed family called TN Weibull Lomax distribution is introduced in Section 3. In Section 4 covers some mathematical properties of TN Weibull Lomax distribution. Section 5 provides various estimation techniques for estimating the parameters of TNWL distribution, while Section 6 offers interpretations based on real data applications. Section 7 presents the conclusion summarizing the key findings of the study and highlighting the practical significance and potential applications of the proposed TN Weibull Lomax model.

2 The TN Family

2.1 Proposed New Family of Distributions

Definition 1.*We have introduced a new family of lifetime distributions called the TN family of distributions. Let X be a random variable with baseline cumulative distribution function (cdf), G(x) and corresponding probability density function (pdf), g(x). The new family is defined by the cdf,*

$$F(x) = \frac{e^{\lambda G(x)} - e^{\theta G(x)}}{e^{\lambda} - e^{\theta}}; \quad \lambda, \theta > 0 \ \textit{and} \ x > 0 \tag{1}$$

The probability density function of TN family is given by,

$$f(x) = \frac{1}{e^{\lambda} - e^{\theta}} \left[\lambda e^{\lambda G(x)} - \theta e^{\theta G(x)}\right] g(x); \quad \lambda, \theta > 0 \ \textit{and} \ x > 0 \tag{2}$$

The new pdf is most tractable when G(x) and g(x) have clear analytical expressions. Henceforth, a random variable X with pdf (2) is denoted by $X \sim TNWL(x, \lambda, \theta)$.

The proposed TN family introduces two additional positive parameters, λ and θ into a baseline distribution with cumulative distribution function $G(x)$. The parameters of the baseline distribution determine its underlying scale and shape, whereas the additional parameters λ and θ modify the allocation of probability across the support through a nonlinear transformation of $G(x)$. Their principal role is to provide greater flexibility in modeling the density, survival function and hazard rate.

By increasing either λ or θ while maintaining $\lambda > \theta$ and keeping the other parameter constant, the transformed cumulative probability at an interior point of the support increases. This is true for fixed baseline parameters. In survival studies, this indicates reduced survival probabilities at a specified follow-up time. However it does not always denote a steadily increasing hazard rate.

2.2 Reliability and Hazard Functions

The reliability function of the TN family of distributions is defined as,

$$\begin{aligned} R(x) &= 1 - F(x), \\ &= \frac{e^{\lambda} - e^{\theta} - e^{\lambda G(x)} + e^{\theta G(x)}}{e^{\lambda} - e^{\theta}}. \end{aligned} \tag{3}$$

The hazard rate function of the TN family of distributions is given by,

$$\begin{aligned} h(x) &= \frac{f(x)}{R(x)}, \\ &= \frac{g(x)\left[\lambda e^{\lambda G(x)} - \theta e^{\theta G(x)}\right]}{e^{\lambda} - e^{\theta} - e^{\lambda G(x)} + e^{\theta G(x)}}. \end{aligned} \tag{4}$$

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Similarly, the reversed hazard rate function of the TN family of distributions is expressed as,

$$\begin{aligned} r_h(x) &= \frac{f(x)}{F(x)}, \\ &= \frac{g(x)\left[\lambda e^{\lambda G(x)} - \theta e^{\theta G(x)}\right]}{e^{\lambda G(x)} - e^{\theta G(x)}}. \end{aligned} \tag{5}$$

Corollary 1.*When $\lambda = 1$ and $\theta = 0$, the TN family reduces to DUS transformation of Kumar et al. (2015) which is given by,*

$$F(x) = \frac{e^{G(x)} - 1}{e - 1}. \tag{6}$$

Corollary 2.*When $\lambda = 0$ and $\theta = -\theta$, the TN family becomes Poisson family of distribution of Chen et al. (1999) which is given by,*

$$F(x) = \frac{1 - e^{-\theta\, G(x)}}{1 - e^{-\theta}}. \tag{7}$$

3 TN Weibull Lomax Distribution

Definition 2.*The TN Weibull Lomax (TNWL) distribution is obtained by using Weibull Lomax (WL) distribution (Tahir et al., 2015) [26] as baseline distribution in the TN family of distribution. Let G(x) be cdf of the Weibull Lomax random variable given by,*

$$F(x;a,b,\alpha,\beta) = 1 - \exp\left\{-a\left[\left(1+\frac{x}{\beta}\right)^{\alpha} - 1\right]^{b}\right\}, \qquad x > 0;\ a,b,\alpha,\beta > 0. \tag{8}$$

Inserting equation (8) into (1), we get the six parameter TNWL cdf which is given by,

$$F(x) = \frac{e^{\left\{\lambda\left[1-\exp\left\{-a\left(\left(1+\frac{x}{\beta}\right)^{\alpha}-1\right)^{b}\right\}\right]\right\}} - e^{\left\{\theta\left[1-\exp\left\{-a\left(\left(1+\frac{x}{\beta}\right)^{\alpha}-1\right)^{b}\right\}\right]\right\}}}{e^{\lambda} - e^{\theta}}, \qquad x > 0;\ \lambda,\theta > 0. \tag{9}$$

And the corresponding pdf is given by,

$$\begin{aligned} f(x) = {} & \frac{\lambda\, e^{\left\{\lambda\left[1-\exp\left\{-a\left(\left(1+\frac{x}{\beta}\right)^{\alpha}-1\right)^{b}\right\}\right]\right\}} - \theta\, e^{\left\{\theta\left[1-\exp\left\{-a\left(\left(1+\frac{x}{\beta}\right)^{\alpha}-1\right)^{b}\right\}\right]\right\}}}{e^{\lambda} - e^{\theta}} \\ & \times \frac{ab\alpha}{\beta}\, e^{\left\{-a\left(\left(1+\frac{x}{\beta}\right)^{\alpha}-1\right)^{b}\right\}} \left(1+\frac{x}{\beta}\right)^{\alpha-1} \left(\left(1+\frac{x}{\beta}\right)^{\alpha} - 1\right)^{b-1}, \end{aligned} \tag{10}$$

where $x > 0$, $\lambda > 0$ and $\theta > 0$ are the additional parameters.

Figure 1 illustrates the shapes of TNWL density function for different values of its parameters. The TNWL density exhibits right skewness with a sharp rise near the origin and a slowly decaying right tail.

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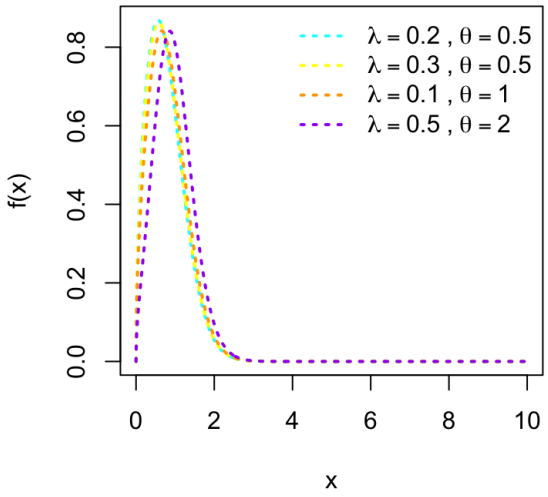

Fig. 1: *Plot of pdf of TNWL distribution for* $a = 1,\ b = 1.5,\ \alpha = 1.2,\ \beta = 1.$

3.1 Reliability and Hazard Function of TNWL Distribution

The reliability function for the TNWL distribution is obtained as follows,

$$\begin{aligned} R(x) &= 1 - F(x), \\ &= \frac{e^{\lambda} - e^{\theta} - e^{\lambda\left[1-e^{-a\left(\left(1+\frac{x}{\beta}\right)^{\alpha}-1\right)^{b}}\right]} + e^{\theta\left[1-e^{-a\left(\left(1+\frac{x}{\beta}\right)^{\alpha}-1\right)^{b}}\right]}}{e^{\lambda} - e^{\theta}}. \end{aligned} \tag{11}$$

The hazard function of the TNWL distribution is defined by,

$$\begin{aligned} h(x) &= \frac{f(x)}{R(x)}, \\ &= \frac{ab\alpha}{\beta}\left(1+\frac{x}{\beta}\right)^{\alpha-1}\left[\left(1+\frac{x}{\beta}\right)^{\alpha}-1\right]^{b-1} e^{-a\left[\left(1+\frac{x}{\beta}\right)^{\alpha}-1\right]^{b}} \\ &\quad \times \frac{\lambda e^{\lambda\left[1-e^{-a\left(\left(1+\frac{x}{\beta}\right)^{\alpha}-1\right)^{b}}\right]} - \theta e^{\theta\left[1-e^{-a\left(\left(1+\frac{x}{\beta}\right)^{\alpha}-1\right)^{b}}\right]}}{e^{\lambda} - e^{\theta} - e^{\lambda\left[1-e^{-a\left(\left(1+\frac{x}{\beta}\right)^{\alpha}-1\right)^{b}}\right]} + e^{\theta\left[1-e^{-a\left(\left(1+\frac{x}{\beta}\right)^{\alpha}-1\right)^{b}}\right]}}. \end{aligned} \tag{12}$$

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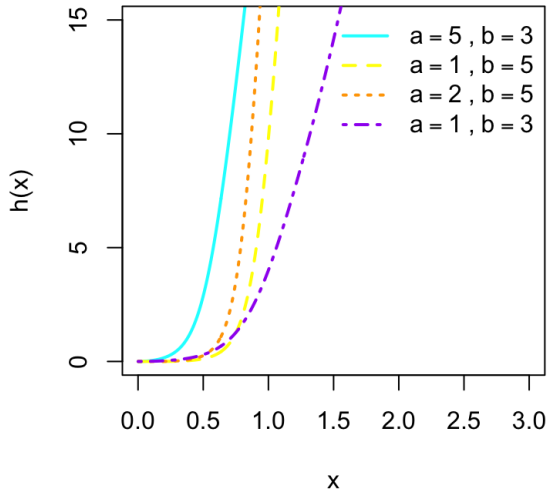

Fig. 2: *Plot of hrf of TNWL distribution for* $\alpha = 1.5,\ \beta = 1.5,\ \lambda = 1.2,\ \theta = 1.8.$

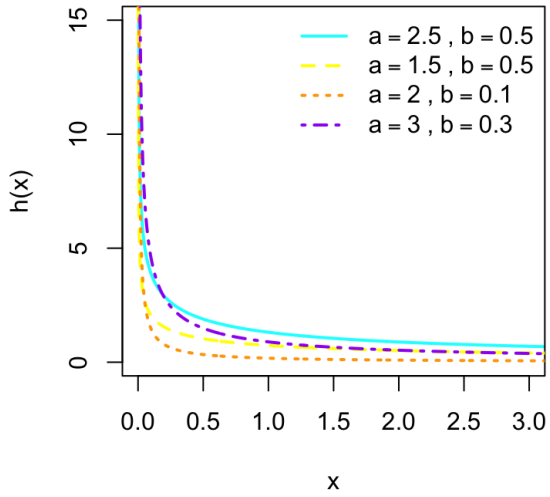

Fig. 3: *Plot of hrf of TNWL distribution for* $\alpha = 0.8,\ \beta = 0.5,\ \lambda = 0.9,\ \theta = 0.5.$

*Remark.*The hazard rate function of the TNWL distribution exhibits flexible behaviour depending on the combined effects of a, b, α, β, λ, and θ. The parameter combinations considered in these figures serve as illustrative examples of the flexibility of the proposed distribution.

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–If $\alpha > 1$, $\beta > 1$, $\lambda > 1$, $\theta > 1$ and $b > 1$, then $h(x)$ is an *increasing function* of x, representing an increasing failure rate (IFR) model. Figure 2 shows increasing hazard rate behaviour, which is commonly associated with aging and wear-out failure mechanisms.
–If $\alpha < 1$, $\beta < 1$, $\lambda < 1$, $\theta < 1$ and $b < 1$, then $h(x)$ is a *decreasing function* of x, indicating a decreasing failure rate (DFR) model. Figure 3 exhibits decreasing hazard rate behaviour, which may represent early-life failures or heterogeneous populations with initially high failure risks.
–The parameter a influences the overall magnitude and progression of the hazard, while b contributes to its local behaviour and rate of change. The parameters α and β determine the shape and scale characteristics of the underlying Weibull–Lomax model, whereas λ and θ introduce additional flexibility through the TN transformation.

These observations highlight the potential of the TNWL distribution for modelling lifetime data with different failure rate characteristics.

Theorem 1(Boundary behaviour of the pdf). *Let X follow the TN Weibull–Lomax (TNWL) distribution with probability density function*

$$f(x) = \frac{\lambda e^{\lambda[1-e^{-ag(x)}]} - \theta e^{\theta[1-e^{-ag(x)}]}}{e^{\lambda} - e^{\theta}} \frac{ab\alpha}{\beta} e^{-ag(x)} \left(1+\frac{x}{\beta}\right)^{\alpha-1} \times \left[\left(1+\frac{x}{\beta}\right)^{\alpha} - 1\right]^{b-1},$$

where

$$g(x) = \left[\left(1+\frac{x}{\beta}\right)^{\alpha} - 1\right]^{b}, \qquad x > 0,$$

and $a, b, \alpha, \beta, \lambda, \theta > 0$, *with* $\lambda \neq \theta$.

Then the following results hold:

1.As $x \to 0^{+}$,

$$\lim_{x\to 0^{+}} f(x) = \begin{cases} 0, & b > 1, \\ \dfrac{\lambda-\theta}{e^{\lambda}-e^{\theta}} \dfrac{a\alpha}{\beta}, & b = 1, \\ +\infty, & 0 < b < 1. \end{cases}$$

2.As $x \to \infty$,

$$\lim_{x\to\infty} f(x) = 0.$$

*Proof.*We first examine the behaviour of the density as $x \to 0^{+}$.

Using the first-order Taylor expansion,

$$\left(1+\frac{x}{\beta}\right)^{\alpha} = 1 + \frac{\alpha x}{\beta} + o(x),$$

$$\left(1+\frac{x}{\beta}\right)^{\alpha} - 1 \sim \frac{\alpha x}{\beta}.$$

Which implies,

$$g(x) \sim \left(\frac{\alpha x}{\beta}\right)^{b} \to 0.$$

Therefore,

$$e^{-ag(x)} \to 1,$$
$$e^{\lambda[1-e^{-ag(x)}]} \to 1,$$
$$e^{\theta[1-e^{-ag(x)}]} \to 1.$$

Moreover,

$$\left(1+\frac{x}{\beta}\right)^{\alpha-1} \to 1.$$

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Substituting these asymptotic relations into the density yields

$$f(x) \sim \frac{\lambda-\theta}{e^{\lambda}-e^{\theta}} \frac{ab\alpha}{\beta} \left(\frac{\alpha x}{\beta}\right)^{b-1}.$$

Equivalently,

$$f(x) \sim \frac{\lambda-\theta}{e^{\lambda}-e^{\theta}} \frac{ab\alpha^{b}}{\beta^{b}} x^{b-1}.$$

Since $\lambda \neq \theta$ and the exponential function is strictly increasing,

$$\frac{\lambda-\theta}{e^{\lambda}-e^{\theta}} > 0.$$

We now evaluate the three cases explicitly.

Case 1: $b > 1$**.**

If $b > 1$, then $b-1 > 0$. Hence,

$$\lim_{x\to 0^{+}} x^{b-1} = 0.$$

Therefore,

$$\begin{aligned}\lim_{x\to 0^{+}} f(x) &= \frac{\lambda-\theta}{e^{\lambda}-e^{\theta}} \frac{ab\alpha^{b}}{\beta^{b}} \lim_{x\to 0^{+}} x^{b-1} \\ &= 0.\end{aligned}$$

Thus, the density approaches zero at the origin when $b > 1$.

Case 2: $b = 1$**.**

If $b = 1$, then $b-1 = 0$, so

$$x^{b-1} = x^{0} = 1.$$

Substituting $b = 1$ gives

$$\begin{aligned}\lim_{x\to 0^{+}} f(x) &= \frac{\lambda-\theta}{e^{\lambda}-e^{\theta}} \frac{a(1)\alpha^{1}}{\beta^{1}} \\ &= \frac{\lambda-\theta}{e^{\lambda}-e^{\theta}} \frac{a\alpha}{\beta}.\end{aligned}$$

Hence, the density approaches a finite positive constant when $b = 1$.

Case 3: $0 < b < 1$**.**

If $0 < b < 1$, then $b-1 < 0$. We can write

$$x^{b-1} = \frac{1}{x^{1-b}}.$$

Since $1-b > 0$,

$$\lim_{x\to 0^{+}} x^{1-b} = 0^{+}.$$

Consequently,

$$\lim_{x\to 0^{+}} x^{b-1} = \lim_{x\to 0^{+}} \frac{1}{x^{1-b}} = +\infty.$$

The multiplicative coefficient $\frac{\lambda-\theta}{e^{\lambda}-e^{\theta}} \frac{ab\alpha^{b}}{\beta^{b}}$ is finite and strictly positive.

Therefore,

$$\lim_{x\to 0^{+}} f(x) = +\infty.$$

Thus, the density diverges at the origin when $0 < b < 1$.

As $x \to \infty$,

$$\left(1+\frac{x}{\beta}\right)^{\alpha} - 1 \sim \left(\frac{x}{\beta}\right)^{\alpha}.$$

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It follows that

$$g(x) \sim \left(\frac{x}{\beta}\right)^{\alpha b} \to \infty.$$

Hence, $e^{-ag(x)} \to 0$, $e^{\lambda[1-e^{-ag(x)}]} \to e^{\lambda}$ and $e^{\theta[1-e^{-ag(x)}]} \to e^{\theta}$.

Furthermore,

$$\left(1+\frac{x}{\beta}\right)^{\alpha-1}\left[\left(1+\frac{x}{\beta}\right)^{\alpha}-1\right]^{b-1} \sim \left(\frac{x}{\beta}\right)^{\alpha b-1}.$$

Therefore,

$$f(x) \sim \frac{\lambda e^{\lambda}-\theta e^{\theta}}{e^{\lambda}-e^{\theta}} \frac{ab\alpha}{\beta}\left(\frac{x}{\beta}\right)^{\alpha b-1} e^{-ag(x)}.$$

Since $\frac{\lambda e^{\lambda}-\theta e^{\theta}}{e^{\lambda}-e^{\theta}} > 0$ for positive λ, θ with $\lambda \neq \theta$, the asymptotic coefficient is finite and positive.

Also, since

$$g(x) \sim \left(\frac{x}{\beta}\right)^{\alpha b},$$

there exist constants $c > 0$ and $x_0 > 0$ such that

$$g(x) \geq cx^{\alpha b}, \qquad x \geq x_0.$$

Consequently,

$$0 < e^{-ag(x)} \leq e^{-acx^{\alpha b}}.$$

For any $\alpha b > 0$,

$$\lim_{x\to\infty} x^{\alpha b-1} e^{-acx^{\alpha b}} = 0.$$

Thus, the exponential decay dominates the polynomial factor, and we conclude that,

$$\lim_{x\to\infty} f(x) = 0.$$

This completes the proof.

*Remark.*The parameter b determines the boundary behaviour of the TNWL density near the origin. Specifically, the density approaches zero for $b > 1$, a finite positive constant for $b = 1$ and diverges to positive infinity for $0 < b < 1$. Furthermore, the density exhibits stretched-exponential-type decay at infinity, ensuring that $f(x) \to 0$ for all admissible parameter values.

Theorem 2(Tail behaviour and absence of regular variation). *Let X follow the TN Weibull–Lomax (TNWL) distribution with cumulative distribution function*

$$F(x) = \frac{e^{\lambda[1-e^{-ag(x)}]} - e^{\theta[1-e^{-ag(x)}]}}{e^{\lambda}-e^{\theta}},$$

where $g(x) = \left[\left(1+\frac{x}{\beta}\right)^{\alpha}-1\right]^{b}$, $x > 0$, *and* $a, b, \alpha, \beta, \lambda, \theta > 0$, *with* $\lambda \neq \theta$.

Then the following properties hold:

1. *The survival function satisfies*

$$\lim_{x\to\infty} \frac{-\log \bar{F}(x)}{x^{\alpha b}} = \frac{a}{\beta^{\alpha b}}.$$

2. *The TNWL distribution is light-tailed if* $\alpha b \geq 1$ *and heavy-tailed if* $0 < \alpha b < 1$, *in the sense of existence or nonexistence of positive exponential moments.*
3. *The survival function* $\bar{F}(x)$ *is not regularly varying for any admissible parameter values.*

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Proof. As $x \to \infty$, we have

$$g(x) \sim \left(\frac{x}{\beta}\right)^{\alpha b}.$$

Using the first-order exponential expansion as $e^{-ag(x)} \to 0$, the survival function satisfies

$$\bar{F}(x) \sim \frac{\lambda e^{\lambda} - \theta e^{\theta}}{e^{\lambda} - e^{\theta}} e^{-ag(x)}.$$

Since the multiplicative constant is finite and strictly positive, taking logarithms yields

$$-\log \bar{F}(x) = ag(x) + O(1).$$

Consequently,

$$\lim_{x\to\infty} \frac{-\log \bar{F}(x)}{x^{\alpha b}} = \frac{a}{\beta^{\alpha b}}.$$

If $\alpha b > 1$, then for every $c > 0$,

$$\log\left[e^{cx}\bar{F}(x)\right] = cx - ag(x) + O(1) \to -\infty.$$

Thus the survival function decays faster than every exponential rate.
If $\alpha b = 1$, then

$$\lim_{x\to\infty} \frac{-\log \bar{F}(x)}{x} = \frac{a}{\beta},$$

so the survival function has an exponential decay rate and admits positive exponential moments for $0 < c < a/\beta$.
If $0 < \alpha b < 1$, then for every $c > 0$,

$$cx - ag(x) \to +\infty.$$

Hence,

$$e^{cx}\bar{F}(x) \to \infty,$$

and the distribution has no finite positive exponential moments.
Finally, for any fixed $t > 0$,

$$\log \frac{\bar{F}(tx)}{\bar{F}(x)} = -a[g(tx) - g(x)] + o(1).$$

Since

$$g(tx) - g(x) \sim \frac{t^{\alpha b} - 1}{\beta^{\alpha b}} x^{\alpha b},$$

we obtain

$$\lim_{x\to\infty} \frac{\bar{F}(tx)}{\bar{F}(x)} = \begin{cases} 0, & t > 1, \\ 1, & t = 1, \\ +\infty, & 0 < t < 1. \end{cases}$$

These limits cannot equal $t^{-\kappa}$ for any finite real κ. Therefore, $\bar{F}(x)$ is not regularly varying.
This completes the proof.

4 Properties of TNWL Distribution

This section presents some important properties of the TN Weibull Lomax distribution such as moments, mean, variance, skewness and kurtosis.

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4.0.1 Moment Function of TNWL Distribution

Moments are essential in statistical analysis, particularly in real-world applications. They contribute to obtaining important distributional features like kurtosis, skewness, central tendency and dispersion.

Theorem 3.*Let X be a random variable following the six-parameter TNWL distribution with parameters $a,b,\alpha,\beta,\lambda,\theta > 0$, where $\lambda \neq \theta$. Then, for any nonnegative integer r, the rth raw moment of X is given by*

$$\begin{aligned}\mu'_r = \frac{a\beta^r}{e^\lambda - e^\theta} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left(\lambda^{k+1} e^\lambda - \theta^{k+1} e^\theta\right) \\ \times \sum_{m=0}^{r} (-1)^{r-m} \binom{r}{m} \\ \times \left[\sum_{n=0}^{\infty} \binom{\frac{m}{\alpha}}{n} \frac{\gamma\left(\frac{n}{b}+1, a(k+1)\right)}{[a(k+1)]^{\frac{n}{b}+1}} \right. \\ \left. + \sum_{n=0}^{\infty} \binom{\frac{m}{\alpha}}{n} \frac{\Gamma\left(\frac{\frac{m}{\alpha}-n}{b}+1, a(k+1)\right)}{[a(k+1)]^{\frac{\frac{m}{\alpha}-n}{b}+1}}\right].\end{aligned} \tag{13}$$

where $\gamma(s,z)$ and $\Gamma(s,z)$ denote the lower and upper incomplete gamma functions, respectively, defined by

$$\gamma(s,z) = \int_0^z u^{s-1} e^{-u}\, du, \qquad s>0,\ z>0, \tag{14}$$

and

$$\Gamma(s,z) = \int_z^\infty u^{s-1} e^{-u}\, du, \qquad z>0. \tag{15}$$

The generalized binomial coefficient is defined as

$$\binom{p}{n} = \frac{p(p-1)\cdots(p-n+1)}{n!}, \qquad \binom{p}{0} = 1. \tag{16}$$

*Proof.*By the definition of the rth raw moment,

$$\mu'_r = E(X^r) = \int_0^\infty x^r f(x)\, dx. \tag{17}$$

Consider the transformation

$$t = \left[\left(1+\frac{x}{\beta}\right)^\alpha - 1\right]^b, \qquad x = \beta\left[(1+t^{1/b})^{1/\alpha} - 1\right].$$

Using the TNWL density and the corresponding Jacobian, we obtain

$$\begin{aligned}\mu'_r = \frac{a\beta^r}{e^\lambda - e^\theta} \int_0^\infty \left[(1+t^{1/b})^{1/\alpha} - 1\right]^r \\ \times e^{-at}\left[\lambda e^{\lambda(1-e^{-at})} - \theta e^{\theta(1-e^{-at})}\right] dt.\end{aligned} \tag{18}$$

Applying the exponential series expansion,

$$e^{-ze^{-at}} = \sum_{k=0}^{\infty} \frac{(-z)^k}{k!} e^{-akt},$$

and the ordinary binomial theorem for $r \in \mathbb{N}_0$, we have

$$\begin{aligned}&\left[(1+t^{1/b})^{1/\alpha} - 1\right]^r \\ &= \sum_{m=0}^{r} (-1)^{r-m} \binom{r}{m} (1+t^{1/b})^{m/\alpha}.\end{aligned}$$

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Consequently,

$$\mu_r' = \frac{a\beta^r}{e^\lambda - e^\theta} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left(\lambda^{k+1} e^\lambda - \theta^{k+1} e^\theta\right) \times \sum_{m=0}^{r} (-1)^{r-m} \binom{r}{m} I_{k,m}, \tag{19}$$

where

$$I_{k,m} = \int_0^\infty (1+t^{1/b})^{m/\alpha} e^{-a(k+1)t}\, dt.$$

To evaluate $I_{k,m}$, the integration domain is divided at $t = 1$, since the generalized binomial expansion is not valid over $(0,\infty)$ in a single power-series form.

For $0 < t < 1$, we have $|t^{1/b}| < 1$, and hence

$$(1+t^{1/b})^{m/\alpha} = \sum_{n=0}^{\infty} \binom{m/\alpha}{n} t^{n/b}.$$

Termwise integration gives

$$I_{k,m}^{(1)} = \int_0^1 (1+t^{1/b})^{m/\alpha} e^{-a(k+1)t}\, dt = \sum_{n=0}^{\infty} \binom{m/\alpha}{n} \frac{\gamma\left(\frac{n}{b}+1, a(k+1)\right)}{[a(k+1)]^{\frac{n}{b}+1}}. \tag{20}$$

For $t > 1$, we rewrite

$$(1+t^{1/b})^{m/\alpha} = t^{m/(\alpha b)} (1+t^{-1/b})^{m/\alpha}.$$

Since $|t^{-1/b}| < 1$, the generalized binomial theorem yields

$$(1+t^{1/b})^{m/\alpha} = \sum_{n=0}^{\infty} \binom{m/\alpha}{n} t^{(m/\alpha - n)/b}.$$

Therefore,

$$I_{k,m}^{(2)} = \int_1^\infty (1+t^{1/b})^{m/\alpha} e^{-a(k+1)t}\, dt = \sum_{n=0}^{\infty} \binom{m/\alpha}{n} \frac{\Gamma\left(\frac{m/\alpha - n}{b}+1, a(k+1)\right)}{[a(k+1)]^{\frac{m/\alpha-n}{b}+1}}. \tag{21}$$

Combining Equations (20) and (21) and substituting into Equation (19), we obtain,

$$\mu_r' = \frac{a\beta^r}{e^\lambda - e^\theta} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left(\lambda^{k+1} e^\lambda - \theta^{k+1} e^\theta\right) \times \sum_{m=0}^{r} (-1)^{r-m} \binom{r}{m} \times \left[\sum_{n=0}^{\infty} \binom{m/\alpha}{n} \frac{\gamma\left(\frac{n}{b}+1, a(k+1)\right)}{[a(k+1)]^{\frac{n}{b}+1}} + \sum_{n=0}^{\infty} \binom{m/\alpha}{n} \frac{\Gamma\left(\frac{m/\alpha-n}{b}+1, a(k+1)\right)}{[a(k+1)]^{\frac{m/\alpha-n}{b}+1}} \right]. \tag{22}$$

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Proposition 1.*The mean of the TNWL distribution is obtained by setting $r = 1$ in the expression for the rth raw moment. Therefore,*

$$\begin{aligned} E(X) = \frac{a\beta}{e^{\lambda} - e^{\theta}} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left(\lambda^{k+1} e^{\lambda} - \theta^{k+1} e^{\theta}\right) \\ \times \left[\sum_{n=0}^{\infty} \binom{1/\alpha}{n} \frac{\gamma\left(\frac{n}{b}+1, a(k+1)\right)}{[a(k+1)]^{\frac{n}{b}+1}} \right. \\ + \sum_{n=0}^{\infty} \binom{1/\alpha}{n} \frac{\Gamma\left(\frac{1/\alpha - n}{b}+1, a(k+1)\right)}{[a(k+1)]^{\frac{1/\alpha-n}{b}+1}} \\ \left. - \frac{1}{a(k+1)} \right]. \end{aligned} \tag{23}$$

Proposition 2.*The variance of the TNWL distribution is obtained from the first two raw moments as,*

$$\mathrm{Var}(X) = \mu_2' - (\mu_1')^2. \tag{24}$$

where $\mu_1' =$ mean and

$$\begin{aligned} \mu_2' = \frac{a\beta^2}{e^{\lambda} - e^{\theta}} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left(\lambda^{k+1} e^{\lambda} - \theta^{k+1} e^{\theta}\right) \\ \times \left[\sum_{n=0}^{\infty} \binom{2/\alpha}{n} \frac{\gamma\left(\frac{n}{b}+1, a(k+1)\right)}{[a(k+1)]^{\frac{n}{b}+1}} \right. \\ + \sum_{n=0}^{\infty} \binom{2/\alpha}{n} \frac{\Gamma\left(\frac{2/\alpha - n}{b}+1, a(k+1)\right)}{[a(k+1)]^{\frac{2/\alpha-n}{b}+1}} \\ - 2\sum_{n=0}^{\infty} \binom{1/\alpha}{n} \frac{\gamma\left(\frac{n}{b}+1, a(k+1)\right)}{[a(k+1)]^{\frac{n}{b}+1}} \\ - 2\sum_{n=0}^{\infty} \binom{1/\alpha}{n} \frac{\Gamma\left(\frac{1/\alpha - n}{b}+1, a(k+1)\right)}{[a(k+1)]^{\frac{1/\alpha-n}{b}+1}} \\ \left. + \frac{1}{a(k+1)} \right]. \end{aligned} \tag{25}$$

Proposition 3.*The skewness (γ_1) of TNWL distribution is given by,*

$$\gamma_1 = \frac{\mu_3' - 3\mu_2'\mu_1' + 2(\mu_1')^3}{\left(\mu_2' - (\mu_1')^2\right)^{3/2}}. \tag{26}$$

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Where $\mu_3^{'}$ is calculated as follows,

$$\begin{aligned}\mu_3' = {} & \frac{a\beta^3}{e^{\lambda}-e^{\theta}}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\left(\lambda^{k+1}e^{\lambda}-\theta^{k+1}e^{\theta}\right)\\ & \times\left[\sum_{n=0}^{\infty}\binom{3/\alpha}{n}\frac{\gamma\left(\frac{n}{b}+1,a(k+1)\right)}{[a(k+1)]^{\frac{n}{b}+1}}+\sum_{n=0}^{\infty}\binom{3/\alpha}{n}\frac{\Gamma\left(\frac{3/\alpha-n}{b}+1,a(k+1)\right)}{[a(k+1)]^{\frac{3/\alpha-n}{b}+1}}\right.\\ & -3\sum_{n=0}^{\infty}\binom{2/\alpha}{n}\frac{\gamma\left(\frac{n}{b}+1,a(k+1)\right)}{[a(k+1)]^{\frac{n}{b}+1}}-3\sum_{n=0}^{\infty}\binom{2/\alpha}{n}\frac{\Gamma\left(\frac{2/\alpha-n}{b}+1,a(k+1)\right)}{[a(k+1)]^{\frac{2/\alpha-n}{b}+1}}\\ & +3\sum_{n=0}^{\infty}\binom{1/\alpha}{n}\frac{\gamma\left(\frac{n}{b}+1,a(k+1)\right)}{[a(k+1)]^{\frac{n}{b}+1}}+3\sum_{n=0}^{\infty}\binom{1/\alpha}{n}\frac{\Gamma\left(\frac{1/\alpha-n}{b}+1,a(k+1)\right)}{[a(k+1)]^{\frac{1/\alpha-n}{b}+1}}\\ & \left.-\frac{1}{a(k+1)}\right].\end{aligned} \tag{27}$$

Proposition 4.*The kurtosis (γ_2) of TNWL distribution is given by,*

$$\gamma_2 = \frac{\mu_4' - 4\mu_3'\mu_1' + 6\mu_2'(\mu_1')^2 - 3(\mu_1')^4}{(\mu_2' - (\mu_1')^2)^2}. \tag{28}$$

Where μ_1', μ_2', μ_3' and μ_4' are the first, second, third and fourth raw moments respectively. The fourth raw moment μ_4' is given by,

$$\begin{aligned}\mu_4' = {} & \frac{a\beta^4}{e^{\lambda}-e^{\theta}}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\left(\lambda^{k+1}e^{\lambda}-\theta^{k+1}e^{\theta}\right)\\ & \times\left[\sum_{n=0}^{\infty}\binom{4/\alpha}{n}\frac{\gamma\left(\frac{n}{b}+1,a(k+1)\right)}{[a(k+1)]^{\frac{n}{b}+1}}+\sum_{n=0}^{\infty}\binom{4/\alpha}{n}\frac{\Gamma\left(\frac{4/\alpha-n}{b}+1,a(k+1)\right)}{[a(k+1)]^{\frac{4/\alpha-n}{b}+1}}\right.\\ & -4\sum_{n=0}^{\infty}\binom{3/\alpha}{n}\frac{\gamma\left(\frac{n}{b}+1,a(k+1)\right)}{[a(k+1)]^{\frac{n}{b}+1}}-4\sum_{n=0}^{\infty}\binom{3/\alpha}{n}\frac{\Gamma\left(\frac{3/\alpha-n}{b}+1,a(k+1)\right)}{[a(k+1)]^{\frac{3/\alpha-n}{b}+1}}\\ & +6\sum_{n=0}^{\infty}\binom{2/\alpha}{n}\frac{\gamma\left(\frac{n}{b}+1,a(k+1)\right)}{[a(k+1)]^{\frac{n}{b}+1}}+6\sum_{n=0}^{\infty}\binom{2/\alpha}{n}\frac{\Gamma\left(\frac{2/\alpha-n}{b}+1,a(k+1)\right)}{[a(k+1)]^{\frac{2/\alpha-n}{b}+1}}\\ & -4\sum_{n=0}^{\infty}\binom{1/\alpha}{n}\frac{\gamma\left(\frac{n}{b}+1,a(k+1)\right)}{[a(k+1)]^{\frac{n}{b}+1}}-4\sum_{n=0}^{\infty}\binom{1/\alpha}{n}\frac{\Gamma\left(\frac{1/\alpha-n}{b}+1,a(k+1)\right)}{[a(k+1)]^{\frac{1/\alpha-n}{b}+1}}\\ & \left.+\frac{1}{a(k+1)}\right].\end{aligned} \tag{29}$$

4.0.2 Moment Generating Function of TNWL Distribution

Theorem 4.*Existence of Uniqueness Theorem of TNWL Distribution*
If $M_X(t)$ exists in an open interval around $t=0$, then it uniquely determines the probability distribution of X.

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*Proof.*We obtain the mgf of a TNWL random variable X as

$$\begin{aligned} M_X(t) &= \sum_{r=0}^{\infty} \frac{t^r}{r!}\mu_r' \\ &= \sum_{r=0}^{\infty} \frac{t^r}{r!} \frac{a\beta^r}{e^{\lambda}-e^{\theta}} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left(\lambda^{k+1}e^{\lambda} - \theta^{k+1}e^{\theta}\right) \\ &\quad \times \sum_{m=0}^{r} (-1)^{r-m} \binom{r}{m} \\ &\quad \times \left[\sum_{n=0}^{\infty} \binom{m/\alpha}{n} \frac{\gamma\left(\frac{n}{b}+1, a(k+1)\right)}{[a(k+1)]^{\frac{n}{b}+1}} \right. \\ &\quad \left. + \sum_{n=0}^{\infty} \binom{m/\alpha}{n} \frac{\Gamma\left(\frac{m/\alpha-n}{b}+1, a(k+1)\right)}{[a(k+1)]^{\frac{m/\alpha-n}{b}+1}} \right]. \end{aligned} \tag{30}$$

5 Parameter Estimation

In this section, we employ the Maximum Likelihood (ML), Anderson-Darling (AD) and Cramer-von Mises (CVM) methods to estimate the unknown parameters $\alpha, \beta, a, b, \lambda$ and θ of the TNWL distributions. We assume that $x_1, x_2, \ldots, x_n$ form a random sample of size n drawn from TNWL distributions with the parameters $\alpha, \beta, a, b, \lambda$ and θ being unknown.

5.1 Method of Maximum Likelihood

We emphasize using a technique called maximum likelihood estimation (MLE) to estimate the unknown parameters of the TNWL distribution. The maximum likelihood approach is the most commonly used of the different parameter estimate methods developed in the literature. Let $x_1, x_2, x_3, ..., x_n$ be a sample of size n from the TNWL distribution. The log likelihood (ℓ) function is given by,

$$\begin{aligned} \ell &= n\log a + n\log b + n\log\alpha - n\log\beta - n\log\left(e^{\lambda}-e^{\theta}\right) + (\alpha-1)\sum_{i=1}^{n}\log\left(1+\frac{x_i}{\beta}\right) \\ &\quad + (b-1)\sum_{i=1}^{n}\log\left[\left(1+\frac{x_i}{\beta}\right)^{\alpha}-1\right] - a\sum_{i=1}^{n}\left[\left(1+\frac{x_i}{\beta}\right)^{\alpha}-1\right]^{b} \\ &\quad + \sum_{i=1}^{n}\log\left[\lambda\exp\left\{\lambda\left(1-\exp\left\{-a\left[\left(1+\frac{x_i}{\beta}\right)^{\alpha}-1\right]^{b}\right\}\right)\right\}\right. \\ &\quad \left. - \theta\exp\left\{\theta\left(1-\exp\left\{-a\left[\left(1+\frac{x_i}{\beta}\right)^{\alpha}-1\right]^{b}\right\}\right)\right\}\right]. \end{aligned} \tag{31}$$

Differentiating ℓ with respect to each parameter $\alpha, \beta, a, b, \lambda$ and θ and setting the result equals to zero, we obtain maximum likelihood estimates (MLEs) of the parameters.

$$\begin{aligned} \frac{\partial\ell}{\partial a} &= \frac{n}{a} - \sum_{i=1}^{n}\left[\left(1+\frac{x_i}{\beta}\right)^{\alpha}-1\right]^{b} \\ &\quad + \sum_{i=1}^{n} \frac{\lambda^2\exp\left\{\lambda\left(1-e^{-a\left[\left(1+\frac{x_i}{\beta}\right)^{\alpha}-1\right]^{b}}\right)\right\} - \theta^2\exp\left\{\theta\left(1-e^{-a\left[\left(1+\frac{x_i}{\beta}\right)^{\alpha}-1\right]^{b}}\right)\right\}}{\lambda\exp\left\{\lambda\left(1-e^{-a\left[\left(1+\frac{x_i}{\beta}\right)^{\alpha}-1\right]^{b}}\right)\right\} - \theta\exp\left\{\theta\left(1-e^{-a\left[\left(1+\frac{x_i}{\beta}\right)^{\alpha}-1\right]^{b}}\right)\right\}} \\ &\quad \times e^{-a\left[\left(1+\frac{x_i}{\beta}\right)^{\alpha}-1\right]^{b}}\left[\left(1+\frac{x_i}{\beta}\right)^{\alpha}-1\right]^{b}. \end{aligned} \tag{32}$$

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$$\begin{aligned}\frac{\partial \ell}{\partial b} &= \frac{n}{b}+\sum_{i=1}^{n}\log\left[\left(1+\frac{x_i}{\beta}\right)^{\alpha}-1\right]\\ &\quad -a\sum_{i=1}^{n}\left[\left(1+\frac{x_i}{\beta}\right)^{\alpha}-1\right]^{b}\log\left[\left(1+\frac{x_i}{\beta}\right)^{\alpha}-1\right]\\ &\quad +\sum_{i=1}^{n}\frac{\lambda^2 e^{\lambda(1-e^{-a[\cdot]^b})}-\theta^2 e^{\theta(1-e^{-a[\cdot]^b})}}{\lambda e^{\lambda(1-e^{-a[\cdot]^b})}-\theta e^{\theta(1-e^{-a[\cdot]^b})}}\\ &\quad \times e^{-a\left[\left(1+\frac{x_i}{\beta}\right)^{\alpha}-1\right]^{b}}\cdot a\left[\left(1+\frac{x_i}{\beta}\right)^{\alpha}-1\right]^{b}\log\left[\left(1+\frac{x_i}{\beta}\right)^{\alpha}-1\right].\end{aligned} \tag{33}$$

$$\begin{aligned}\frac{\partial \ell}{\partial \alpha} &= \frac{n}{\alpha}+\sum_{i=1}^{n}\log\left(1+\frac{x_i}{\beta}\right)\\ &\quad +(b-1)\sum_{i=1}^{n}\frac{\left(1+\frac{x_i}{\beta}\right)^{\alpha}\log\left(1+\frac{x_i}{\beta}\right)}{\left(1+\frac{x_i}{\beta}\right)^{\alpha}-1}\\ &\quad -ab\sum_{i=1}^{n}\left[\left(1+\frac{x_i}{\beta}\right)^{\alpha}-1\right]^{b-1}\left(1+\frac{x_i}{\beta}\right)^{\alpha}\log\left(1+\frac{x_i}{\beta}\right)\\ &\quad +\sum_{i=1}^{n}\frac{\lambda^2 e^{\lambda(1-e^{-a[\cdot]^b})}-\theta^2 e^{\theta(1-e^{-a[\cdot]^b})}}{\lambda e^{\lambda(1-e^{-a[\cdot]^b})}-\theta e^{\theta(1-e^{-a[\cdot]^b})}}\\ &\quad \times e^{-a[\cdot]^b}\cdot ab\left[\left(1+\frac{x_i}{\beta}\right)^{\alpha}-1\right]^{b-1}\left(1+\frac{x_i}{\beta}\right)^{\alpha}\log\left(1+\frac{x_i}{\beta}\right).\end{aligned} \tag{34}$$

$$\begin{aligned}\frac{\partial \ell}{\partial \beta} &= -\frac{n}{\beta}-(\alpha-1)\sum_{i=1}^{n}\frac{x_i}{\beta(\beta+x_i)}\\ &\quad +(b-1)\sum_{i=1}^{n}\frac{-\alpha x_i(1+\frac{x_i}{\beta})^{\alpha-1}}{\beta^2\left[\left(1+\frac{x_i}{\beta}\right)^{\alpha}-1\right]}\\ &\quad -ab\sum_{i=1}^{n}\left[\left(1+\frac{x_i}{\beta}\right)^{\alpha}-1\right]^{b-1}\left[-\frac{\alpha x_i}{\beta^2}\left(1+\frac{x_i}{\beta}\right)^{\alpha-1}\right]\\ &\quad +\sum_{i=1}^{n}\frac{\lambda^2 e^{\lambda(1-e^{-a[\cdot]^b})}-\theta^2 e^{\theta(1-e^{-a[\cdot]^b})}}{\lambda e^{\lambda(1-e^{-a[\cdot]^b})}-\theta e^{\theta(1-e^{-a[\cdot]^b})}}\\ &\quad \times e^{-a[\cdot]^b}\cdot ab\left[\left(1+\frac{x_i}{\beta}\right)^{\alpha}-1\right]^{b-1}\left[-\frac{\alpha x_i}{\beta^2}\left(1+\frac{x_i}{\beta}\right)^{\alpha-1}\right].\end{aligned} \tag{35}$$

$$\begin{aligned}\frac{\partial \ell}{\partial \lambda} &= -\frac{ne^{\lambda}}{e^{\lambda}-e^{\theta}}\\ &\quad +\sum_{i=1}^{n}\frac{\exp\left\{\lambda\left(1-e^{-a[\cdot]^b}\right)\right\}\left[1+\lambda\left(1-e^{-a[\cdot]^b}\right)\right]}{\lambda\exp\left\{\lambda\left(1-e^{-a[\cdot]^b}\right)\right\}-\theta\exp\left\{\theta\left(1-e^{-a[\cdot]^b}\right)\right\}}.\end{aligned} \tag{36}$$

$$\begin{aligned}\frac{\partial \ell}{\partial \theta} &= \frac{ne^{\theta}}{e^{\lambda}-e^{\theta}}\\ &\quad +\sum_{i=1}^{n}\frac{-\exp\left\{\theta\left(1-e^{-a[\cdot]^b}\right)\right\}\left[1+\theta\left(1-e^{-a[\cdot]^b}\right)\right]}{\lambda\exp\left\{\lambda\left(1-e^{-a[\cdot]^b}\right)\right\}-\theta\exp\left\{\theta\left(1-e^{-a[\cdot]^b}\right)\right\}}.\end{aligned} \tag{37}$$

Solving equation (32), (33), (34), (35), (36) and (37) by equating to zero, we can find the maximum likelihood estimates of α,β,a,b,λ and θ.

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5.2 Method of Anderson-Darling

In order to determine whether a sample deviates from a given distribution, Anderson and Darling (1952) developed the Anderson-Darling (AD) test as an alternative statistical technique. This test is particularly useful for identifying departures from normality which is particularly important for reliability and lifetime data.

For a sample $x_1, x_2, \ldots, x_n$ of size n, the AD statistic is defined as,

$$A^2 = -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[\ln F(x_{(i)}) + \ln\left(1 - F(x_{(n-i+1)})\right)\right], \tag{38}$$

where $x_{(i)}$ denotes the ith order statistic and $F(x)$ is the theoretical cdf.

A smaller value of A^2 indicates a closer agreement between the sample data and the assumed distribution.

For the TNWL distribution, define

$$z_i = a\left[\left(1 + \frac{x_{(i)}}{\beta}\right)^{\alpha} - 1\right]^{b}. \tag{39}$$

The AD statistic for TNWL distribution is defined as,

$$\begin{aligned} A^2 = & -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[\ln\left(\frac{\exp\{\lambda(1-e^{-z_i})\} - \exp\{\theta(1-e^{-z_i})\}}{e^{\lambda} - e^{\theta}}\right)\right. \\ & \left. + \ln\left(1 - \frac{\exp\{\lambda(1-e^{-z_{n+1-i}})\} - \exp\{\theta(1-e^{-z_{n+1-i}})\}}{e^{\lambda} - e^{\theta}}\right)\right]. \end{aligned} \tag{40}$$

Minimizing A^2 with respect to the parameters $\alpha, \beta, a, b, \lambda$ and θ yields the AD estimates of the TNWL distribution, giving extra weight to the tails of the distribution which is important in reliability and lifetime data analysis.

5.3 Method of Cramer-von Mises

The Cramer-von Mises (CvM) method is a minimal distance estimation methodology that minimizes the difference between the theoretical cumulative distribution function (cdf) and the empirical distribution function (EDF) so as to determine parameter estimations. This approach focuses more emphasis on the distribution's overall fit than it does on the tails.

Let $x_1, x_2, \ldots, x_n$ be a random sample of size n, and let $x_{(1)} \leq x_{(2)} \leq \cdots \leq x_{(n)}$ denote the corresponding order statistics. The CvM objective function is defined as

$$W^2 = \frac{1}{12n} + \sum_{i=1}^{n}\left[F(x_{(i)}) - \frac{2i-1}{2n}\right]^2, \tag{41}$$

where $F(x)$ denotes the theoretical cumulative distribution function.

For the TNWL distribution, define

$$z_i = a\left[\left(1 + \frac{x_{(i)}}{\beta}\right)^{\alpha} - 1\right]^{b}. \tag{42}$$

The corresponding cumulative distribution function evaluated at $x_{(i)}$ is

$$F(x_{(i)}) = \frac{\exp\{\lambda(1-e^{-z_i})\} - \exp\{\theta(1-e^{-z_i})\}}{e^{\lambda} - e^{\theta}}. \tag{43}$$

Substituting Equation (43) into Equation (41), the CvM objective function for the TNWL distribution is obtained as,

$$\begin{aligned} W^2 = & \frac{1}{12n} + \sum_{i=1}^{n}\left[\frac{\exp\{\lambda(1-e^{-z_i})\} - \exp\{\theta(1-e^{-z_i})\}}{e^{\lambda} - e^{\theta}}\right. \\ & \left. - \frac{2i-1}{2n}\right]^2. \end{aligned} \tag{44}$$

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The CvM estimates of the TNWL parameters are obtained by minimizing the objective function with respect to $a, b, \alpha, \beta, \lambda$, and θ.

6 Application

6.1 First Data Set: The lung cancer survival data

To demonstrate the practical utility of the proposed distribution, we analyzed the lung cancer survival data of 228 patients obtained from [27]. Table 1 presents the summary statistics of survival times (in days) for patients. The observed survival times range from 5 to 1022 days, with a median survival time of approximately 255 days and an average survival time of 305 days, indicating a moderately right-skewed distribution.

Table 1: *Summary statistics of survival times (in days).*

Statistic	**Value**
Minimum	5.0
1st Quartile	166.8
Median	255.5
Mean	305.2
3rd Quartile	396.5
Maximum	1022.0

Table 2: *Estimated hazard ratios with 95% confidence intervals and p-values.*

Variable	**HR**	**95% CI**	***p*-value**
Age (years)	1.020	1.010–1.030	0.001
Sex (female vs. male)	0.590	0.420–0.830	0.003

Table 2 presents the estimated hazard ratios (HRs), 95% confidence intervals (CIs) and corresponding p-values for age and sex. The results indicate that both age and sex are statistically significant predictors of mortality among lung cancer patients. Age was positively associated with mortality, whereas females had a significantly lower hazard compared to males.

The hazard ratio for age indicate that each additional year of age is associated with an approximately 2% increase in the instantaneous hazard of death. Also the female patients experience approximately 41% lower instantaneous mortality hazard than male patients of the same age.

6.1.1 Goodness-of-Fit Test Results of the TNWL Distribution for the first dataset

The goodness-of-fit of the TNWL distribution was evaluated using the lung cancer survival time data and is illustrated in Table 3. And three goodness-of-fit tests used are Kolmogorov–Smirnov (KS) test, Anderson–Darling (AD) test and Cramer–von Mises (CvM) test.

Table 3: *Goodness-of-fit test results for the TNWL distribution.*

Test	**Statistic**	***p*-value**
Kolmogorov–Smirnov	0.0891	0.0535
Anderson–Darling	2.2357	0.0685
Cramér–von Mises	0.4039	0.0705

Since all p-values are greater than 0.05, the null hypothesis is not rejected for any of the goodness-of-fit tests. Therefore, the proposed distribution provides an adequate and statistically acceptable fit to the data.

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6.1.2 Model adequacy check of the TNWL Distribution for the first dataset

Model adequacy was assessed based on the Akaike Information Criterion (AIC) and Bayesian Information Criterion (BIC). The following is how AIC and BIC are calculated,

$$AIC = -2\ell + 2k,$$

$BIC = -2\ell + klog(n)$,
where ℓ represent the log likelihood function, k the number of parameters and n the sample size. We estimate parameters using R's optim package. Better model fits are indicated by smaller values of the AIC and BIC statistics.

The suggested TNWL distribution were compared to the Power Generalized DUS Lomax (PGDUSL) distributionn of Thomas and Chacko (2021) [28], Topp-Leone Weibull-Lomax (TLWLx) distribution of Jamal *et al.* (2019) [16], Weibull Inverse Lomax (WIL) distribution of Falgore *et al.* (2019) [15] and Gompertz Lomax (GoLom) distribution of Oguntunde *et al.* (2019) [23]. The Bayesian Information Criterion (BIC) and the Akaike Information Criterion (AIC) are used to compare the performances using goodness of fit statistics.

Table 4: *Maximum likelihood estimates and corresponding standard errors for the fitted lifetime distributions.*

Distribution	MLE	SE
TNWL	$\hat{\lambda} = 3.2548$	1.4854
	$\hat{\theta} = 0.0012$	0.1114
	$\hat{\alpha} = 1.0226$	0.5128
	$\hat{\beta} = 58.8751$	919.2546
	$\hat{a} = 0.3655$	5.5134
	$\hat{b} = 0.9567$	0.4298
PGDUSL	$\hat{\alpha} = 2.8582$	3.09184
	$\hat{\beta} = 0.00345$	0.0000
	$\hat{\theta} = 2.2678$	0.1996
TLWLx	$\hat{\theta} = 25.6836$	4.3401
	$\hat{\alpha} = 0.0976$	0.0374
	$\hat{a} = 9.2700$	3.1273
	$\hat{b} = 0.0035$	0.0015
WIL	$\hat{\alpha} = 0.0542$	0.2927
	$\hat{\beta} = 1.5436$	0.1417
	$\hat{\gamma} = 81.8115$	19.1778
	$\hat{\lambda} = 0.0624$	0.4649
GoLom	$\hat{\alpha} = 1.5576$	0.1437
	$\hat{\beta} = 0.0560$	0.0925
	$\hat{\gamma} = 1.0000$	0.0000
	$\hat{\theta} = 0.0094$	0.0199

Table 5: *Log-likelihood, AIC and BIC values for the fitted lifetime distributions.*

Model	k	ℓ	AIC	BIC
TNWL	6	-1516.579	3045.158	3065.734
PGDUSL	3	-1525.570	3057.130	3067.420
TLWLx	4	-1571.830	3151.670	3165.380
WIL	4	-1571.830	3151.670	3165.380
GoLom	4	-1532.510	3073.020	3086.740

Table 4 summarizes the MLEs and Standard Errors (SE). And Table 5 illustrates log-likelihood, AIC and BIC for various lifetime models. Among them, the TNWL distribution shows the lowest AIC and BIC with the highest

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log-likelihood, indicating the best fit to the lung cancer survival data compared to the other models.

6.2 Second Data Set: The strength of single carbon fibers

In this case we analyze the strength data from Kundu and Raqab (2009) [18] and it represents the strength measured in GPA for single carbon fibers and impregnated 1000-carbon fiber tows. This data set consists of, 63 observations: 1.901, 2.132, 2.203, 2.228, 2.257, 2.350, 2.361, 2.396, 2.397, 2.445, 2.454, 2.474, 2.518, 2.522, 2.525, 2.532, 2.575, 2.614, 2.616, 2.618, 2.624, 2.659, 2.675, 2.738, 2.740, 2.856, 2.917, 2.928, 2.937, 2.937, 2.977, 2.996, 3.030, 3.125, 3.139, 3.145, 3.220, 3.223, 3.235, 3.243, 3.264, 3.272, 3.294, 3.332, 3.346, 3.377, 3.408, 3.435, 3.493, 3.501, 3.537, 3.554, 3.562, 3.628, 3.852, 3.871, 3.886, 3.971, 4.024, 4.027, 4.225, 4.395, 5.020. Table 5 presents the summary statistics of of carbon fiber strength.

Table 6: *Summary statistics of carbon fiber strength.*

Statistic	**Value**
Minimum	1.901
1st Quartile	2.5535
Median	2.996
Mean	3.0593
3rd Quartile	3.4215
Maximum	5.02

It is clear from Table 6 that the majority of values in the carbon fiber strength data are centered between 2.5 and 3.5 GPa, indicating a moderately right-skewed distribution.

6.2.1 Goodness-of-Fit Test Results of the TNWL Distribution for the second dataset

The goodness-of-fit of the TNWL distribution was evaluated using the carbon fibre strength data and is illustrated in Table 7.

Table 7: *Goodness-of-fit test results for the TNWL distribution.*

Test	**Statistic**	***p*-value**
Kolmogorov–Smirnov	0.1153	0.3453
Anderson–Darling	0.6216	0.5370
Cramér–von Mises	0.1243	0.8830

According to the goodness-of-fit results from Table 6, all test p-values are greater than 0.05 which indicate that the TNWL distribution fits the data well. Therefore, the TNWL distribution is appropriate for modeling the carbon fiber strength data since there is no reason to reject the model.

6.2.2 Model adequacy check of the TNWL Distribution for the second dataset

The suggested TNWL distribution were compared to the Gompertz Lomax (GoLom) distribution of Oguntunde *et al.* (2019) [23], Transmuted Weibull-Lomax (TWL) distribution of Afify *et al.* (2015) [1], Weibull Lomax (WL) distribution of Tahir *et al.* (2015) [26] and Mc Lomax (McL) distribution of Lemonte *et al.* (2011) [19]. The parameter estimates values and the performances of the TNWL distribution with the other competing distributions are shown in Table 8 and 9.

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Table 8: *Maximum likelihood estimates and corresponding standard errors for the fitted lifetime distributions.*

Distribution	MLE	SE
TNWL	$\hat{\lambda} = 69.1884$	0.0722
	$\hat{\theta} = 59.3641$	1.5391
	$\hat{\alpha} = 21.9808$	0.9761
	$\hat{\beta} = 1.1555$	0.0041
	$\hat{a} = 6.8204$	0.0637
	$\hat{b} = 86.79047$	3.2637
GoLom	$\hat{\alpha} = 2.1795$	0.5511
	$\hat{\beta} = 0.60905$	0.1476
	$\hat{\gamma} = 3.1874$	0.7851
	$\hat{\theta} = 0.0014$	0.0001
TWL	$\hat{\alpha} = 0.3922$	0.339
	$\hat{\beta} = 0.6603$	1.174
	$\hat{\lambda} = 0.7364$	0.286
	$\hat{a} = 0.5287$	3.32
	$\hat{b} = 8.4451$	4.397
WL	$\hat{\alpha} = 0.2471$	0.06
	$\hat{\beta} = 0.3255$	0.571
	$\hat{a} = 8.003$	49.403
	$\hat{b} = 10.1306$	4.602
McL	$\hat{\alpha} = 45.9249$	59.312
	$\hat{\beta} = 48.3024$	63.047
	$\hat{\lambda} = 353.1435$	375.678
	$\hat{a} = 18.1192$	8.855
	$\hat{b} = 195.4633$	123.217

Table 9: *Log-likelihood, AIC and BIC values for different lifetime distributions.*

Model	k	ℓ	AIC	BIC
TNWL	6	-45.7575	103.5151	115.1060
GoLom	4	-54.2798	116.5597	124.2870
TWL	5	-59.8440	129.6880	140.4035
WL	4	-60.8935	129.7870	138.3594
McL	5	-65.2985	140.5970	151.3125

It is evident from Table 9 that the TNWL distribution clearly achieves the highest log-likelihood value demonstrating better goodness-of-fit than the other models. Furthermore, the TNWL model produces the lowest BIC and AIC values, indicating its effectiveness in attaining an optimal balance between parsimony and model fit.

7 Results and Discussion

This article introduces a new method for extending existing distributions, leading to the development of a novel family of lifetime distributions known as the TN family. We develop new flexible model based on existing Weibull Lomax distribution resulting in the development of TNWL distribution. A detailed mathematical analysis of the model was presented and their parameters were estimated using Maximum Likelihood (ML), Anderson-Darling (AD) and Cramér-von Mises (CvM) methods. The suggested distribution provided a better fit than conventional Weibull and Lomax generalizations when applied to real datasets demonstrating its efficacy and flexibility for modeling lifetime and reliability data.

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References

[1] A. Afify, Z. Nofal, H. M. Yousof, Y. Gebaly and N. Butt, The transmuted Weibull Lomax distribution: Properties and application, Pak. J. Stat. Oper. Res., 11, 135–152 (2015).
[2] C. Alexander, G. M. Cordeiro, E. M. M. Ortega and J. M. Sarabia, Generalized beta generated distributions, Comput. Stat. Data Anal., 56, 1880–1897 (2012).
[3] M. Alizadeh, M. Emadi, M. Doostparast, G. M. Cordeiro, E. M. M. Ortega and R. R. Pescim, A new family of distributions: The Kumaraswamy odd log-logistic, properties and applications, Hacet. J. Math. Stat., 44(6), 1491–1512 (2015).
[4] M. Alizadeh, M. H. Tahir, G. M. Cordeiro, M. Mansoor, M. Zubair and G. G. Hamedani, The Kumaraswamy Marshal–Olkin family of distributions, J. Egypt. Math. Soc., 23(3), 546–557 (2015).
[5] M. Alizadeh, G. M. Cordeiro, E. de Brito and C. G. B. Demétrio, The beta Marshall–Olkin family of distributions, J. Stat. Distrib. Appl., 2(1), 1–18 (2015).
[6] A. Alzaatreh, C. Lee and F. Famoye, A new method for generating families of continuous distributions, Metron, 71(1), 63–79 (2013).
[7] M. Amini, S. M. T. K. Mir Mostafaee and J. Ahmadi, Log-gamma-generated families of distributions, Statistics, 48, 913–932 (2014).
[8] M. Bourguignon, R. B. Silva and G. M. Cordeiro, The Weibull-G family of probability distributions, J. Data Sci., 12, 53–68 (2014).
[9] M. H. Chen, J. G. Ibrahim and D. Sinha, A new Bayesian model for survival data with a surviving fraction, J. Am. Stat. Assoc., 94(447), 909–919 (1999).
[10] G. M. Cordeiro, M. Alizadeh and E. M. M. Ortega, The exponentiated half-logistic family of distributions: Properties and applications, J. Probab. Stat., 2014, Article ID 864396, 21 pages (2014).
[11] G. M. Cordeiro and M. de Castro, A new family of generalized distributions, J. Stat. Comput. Simul., 81(7), 883–898 (2011).
[12] G. M. Cordeiro, E. M. Ortega and D. C. da Cunha, The exponentiated generalized class of distributions, J. Data Sci., 11(1), 1–27 (2013).
[13] G. M. Cordeiro, E. M. Ortega, B. V. Popović and R. R. Pescim, The Lomax generator of distributions: Properties, minification process and regression model, Appl. Math. Comput., 247, 465–486 (2014).
[14] N. Eugene, C. Lee and F. Famoye, Beta-normal distribution and its applications, Commun. Stat. Theory Methods, 31(4), 497–512 (2002).
[15] J. Y. Falgore, S. I. Doguwa and A. Isah, The Weibull-Inverse Lomax (WIL) distribution with application on bladder cancer, Biostat. Biom. Open Access J., 8(6), 211–218 (2019).
[16] F. Jamal, H. M. Reyad, M. A. Nasir, C. Chesneau, M. A. A. Shah and S. O. Ahmed, Topp-Leone Weibull-Lomax distribution: Properties, regression model and applications, NED Univ. J. Res., 20(1), 45–65 (2019).
[17] D. Kumar, U. Singh and S. K. Singh, A method of proposing new distribution and its application to bladder cancer patients data, J. Stat. Appl. Probab. Lett., 2(3), 235–245 (2015).
[18] D. Kundu and M. Z. Raqab, Estimation of $R = P(Y < X)$ for three-parameter Weibull distribution, Stat. Probab. Lett., 79, 1839–1846 (2009).
[19] A. J. Lemonte and G. M. Cordeiro, An extended Lomax distribution, Statistics, 47(4), 800–816 (2013).
[20] A. W. Marshall and I. Olkin, A new method for adding a parameter to a family of distributions with application to the exponential and Weibull families, Biometrika, 84(3), 641–652 (1997).
[21] S. K. Maurya and S. Nadarajah, Poisson generated family of distributions: A review, Sankhyā B, 83(2), 359–388 (2021).
[22] G. S. Mudholkar and D. K. Srivastava, Exponentiated Weibull family for analyzing bathtub failure-rate data, IEEE Trans. Reliab., 42(2), 299–302 (1993).
[23] P. E. Oguntunde, M. A. Khaleel, M. T. Ahmed, A. O. Adejumo and O. A. Odetunmibi, A new generalization of the Lomax distribution with increasing, decreasing, and constant failure rate, Math. Theory Model., 9(7), 32–45 (2019).
[24] M. H. Tahir, G. M. Cordeiro, M. Alizadeh, M. Mansoor, M. Zubair and G. G. Hamedani, The odd generalized exponential family of distributions with applications, J. Stat. Distrib. Appl., 2(1), 1–28 (2015).
[25] M. H. Tahir, G. M. Cordeiro, A. Alzaatreh, M. Mansoor and M. Zubair, The Logistic-X family of distributions and its applications, Commun. Stat. Theory Methods, 45(24), 7326–7349 (2016).
[26] M. H. Tahir, G. M. Cordeiro, M. Mansoor and M. Zubair, The Weibull-Lomax distribution: Properties and applications, Hacet. J. Math. Stat., 44(2), 455–474 (2015).
[27] T. M. Therneau and P. M. Grambsch, *Modeling Survival Data: Extending the Cox Model*, Springer-Verlag, New York, USA, 1–350 (2000).
[28] B. Thomas and V. M. Chacko, Power generalized DUS transformation in Weibull and Lomax distributions, J. Appl. Probab. Stat., 16(2), 85–101 (2021).

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[29] K. Zografos and N. Balakrishnan, On families of beta- and generalized gamma-generated distributions and associated inference, Stat. Methodol., 6(4), 344–362 (2009).



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