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Some Results on Moment of Order Statistics for the Quadratic Hazard Rate Distribution

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Abstract: In this paper, we study the sampling distribution of order statistics of the quadratic hazard rate distribution (QHRD). We consider the single and product moment of order statistics from QHRD and establish some recurrence relations for single and product moments of order statistics. These expressions are used to calculate the mean and variances.

Keywords: Quadratic hazard rate distribution, order statistics, moments, single and product moment of order statistics.

1 Introduction

Order statistics have been used in wide range of problems, including robust statistical estimation and detection of outliers, characterization of probability distributions, goodness of fit-tests, quality control, and analysis of censored sample. The subject of order statistics deals with the properties and applications of these ordered random variable and of functions involving them (see David and Nagaraja [5], Tahir *et al* [15]). Asymptotic theory of extremes and related developments of order statistics are well described in an applausive work of Galambos [7] and the references therein

The use of recurrence relations for the moments of order statistics is quite well known in statistical literature (see for example Arnold *et al.*, [2], Malik *et al.* [12]). For improved form of these results, Samuel and Thomes [13], Arnold et al. [2] have reviewed many recurrence relations and identities for the moments of order statistics arising from several specific continuous distributions such as normal, Cauchy, logistic, gamma and exponential. Recurrence relations for the expected values of certain functions of two order statistics have been considered by Ali and Khan [1]. More recently, Dar and Abdullah [4] study the sampling distribution of order statistics of the two parametric Lomax distribution and derived the exact analytical expressions of entropy, residual entropy and past residual entropy for order statistics of Lomax distribution.

The quadratic hazard rate distribution (QHRD) was introduced by Bain [3]. This distribution generalizes several well-known distributions. Among these distributions are the linear failure (hazard) rate, exponential and Rayleigh distributions. Also, the may have an increasing (decreasing) hazard function or a bathtub shaped hazard function or an upside-down bathtub shaped hazard function. This property enables this distribution to be used in many applications in several areas, such as reliability, life testing, survival analysis and others. Sarhan [14] and Elbatal [06] introduced a generalization of the quadratic hazard rate distribution called the generalized quadratic hazard rate distribution (GQHRD).

A random variable X with range of values $(0,\infty)$ is said to have the Quadric hazard rate distribution (QHRD) with three parameters α,θ,βif its pdf is given by

$$f(x)=(\alpha+\theta x+\beta x^2)e^{-\left(\alpha x+\frac{\theta}{2}x^2+\frac{\beta}{3}x^3\right)},x>0, \tag{1}$$

Where $\alpha\geq 0,\beta\geq 0\ and\ \theta\geq -2\sqrt{\alpha\beta}$ and this restriction on the parameter space is made to be insure that the hazard function with the following form is positive, see Bain [3].

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The cumulative distribution function (cdf) and survival function (sf) associated with (1) is given by

$$F(x) = 1 - e^{-\left(\alpha x + \frac{\theta}{2}x^2 + \frac{\beta}{3}x^3\right)} \quad (2)$$

$$\bar{F}(x) = e^{-\left(\alpha x + \frac{\theta}{2}x^2 + \frac{\beta}{3}x^3\right)}, \quad (3)$$

Respectively, it is easy to see that

$$f(x) = (\alpha + \theta x + \beta x^2)\big(1 - F(x)\big) \quad (4)$$

2 Distribution of Order Statistics

Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from the QHRD and let $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}$ denotes the corresponding order statistics. Then the pdf of $X_{r:n}$, $1 \leq r \leq n$, is given by [see Arnold *et al.* and David and Nagaraja [2], [5]]

$$f_{r:n}(x) = C_{r:n}\{[F(x)]^{r-1}[1 - F(x)]^{n-r}f(x)\}, 0 < x < \infty, \quad (5)$$

Where $C_{r:n} = \frac{n!}{(r-1)!(n-r)!}$.

Using (1), (2) and taking $r = 1$ in (5), yields the pdf of the minimum order statistics for the QHRD

$$f_{1:n}(x) = nA(x)e^{-nB(x)},$$

Where $A(x) = (\alpha + \theta x + \beta x^2)$ and $B(x) = \left(\alpha x + \frac{\theta}{2}x^2 + \frac{\beta}{3}x^3\right)$.

Similarly using (1), (2) and taking $r = n$ in (5), yields the pdf of the largest order statistics for the QHRD

$$f_{n:n}(x) = n\,A(x)\sum_{i=0}^{n-r}\binom{n-r}{i}(-1)^i e^{-(i+1)B(x)}.$$

The joint pdf of $X_{r:n}$ and $X_{s:n}$ for $1 \leq r < s \leq n$ is given by [see Arnold *et al.* [2]]

$$f_{r,s:n}(x,y) = C_{r,s:n}\{[F(x)]^{r-1}[F(y) - F(x)]^{s-r-1}[1 - F(y)]^{n-s}f(x)f(y)\} \quad (6)$$

For $-\infty < x < y < \infty$ and $C_{r,s:n} = \frac{n!}{(r-1)!(s-r-1)!(n-s)!}$.

Theorem 2.1: Let $F(x)$ and $f(x)$ be the cdf and pdf of the QHRD distribution. Then the density function of the r^{th} order statistics say $f_{r:n}(x)$ is given by:

$$f_{r:n}(x) = C_{r:n}A(x)\sum_{i=0}^{n-r+1}\binom{n-r+1}{i}(-1)^i\left(1 - e^{-B(x)}\right)^{r+1-i} \quad (7)$$

Proof: Using (4) in (5), we have

$$f_{r:n}(x) = C_{r:n}A(x)(1 - F(x))^{n-r+1}(F(x))^{r+1} \quad (8)$$

The proof follows by expanding the terms $(1 - F(x))^{n-r+1}$ using the binomial expansion.

Theorem 2.2: Let $X_{r:n}$ and $X_{s:n}$ for $1 \leq r < s \leq n$ be the r^{th} and s^{th} order statistics from QHRD. Then the joint pdf of $X_{r:n}$ and $X_{s:n}$ is given by

$$f_{r;s:n}(x) = A(y)A(x)e^{-B(x)}C_{r,s:n}\sum_{i=0}^{s-r-1}\sum_{j=0}^{n-s+1}\binom{s-r-1}{i}\binom{n-s+1}{j}(-1)^{i+j} \times \left(1 - e^{-B(x)}\right)^{r+1-i} \times \left(1 - e^{-B(y)}\right)^{s-r-1-i+j}$$

Proof: Equation (6) can be written as

$$f_{r;s:n}(x) = C_{r,s:n} \sum_{i=0}^{s-r-1} \binom{s-r-1}{i} (-1)^i \big(F(x)\big)^{r-1+i} \big(F(y)\big)^{s-r-1-i} \times \big(1-F(y)\big)^{n-s} f(x)f(y) \qquad (9)$$

The proof can be easily obtained by using (4) into (9)

3 Single and Product Moments

In this section, we derive explicit expressions for both of the single and product moments of order statistics from the QHRD.

Theorem 3.1: Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from the QHRD.In addition, let $X_{1:n} \le X_{2:n} \le \cdots \le X_{n:n}$ denote the corresponding order statistics. Then the k^{th} moments of the rth order statistics for $k = 1, 2, \ldots$ denoted by $\mu_{r:n}^{(k)}$ is given by

$$\mu_{r:n}^{(k)} = C_{r:n} U_{i,j,l,m} \left\{ \alpha \frac{\Gamma(k+2l+3m+1)}{(\alpha j)^{k+2l+3m+1}} + \theta \frac{\Gamma(k+2l+3m+2)}{(\alpha j)^{k+2l+3m+2}} + \beta \frac{\Gamma(k+2l+3m+3)}{(\alpha j)^{k+2l+3m+3}} \right\},$$

Where,

$$U_{i,j,l,m} = \sum_{i=0}^{n-r+1} \sum_{j=0}^{r-1+i} \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \binom{n-r+1}{i} \binom{r-1+i}{j} (-1)^{i+j+l+m} \frac{j^{l+m} \theta^l \beta^m}{2^l 3^m l!\, m!}$$

and Γ is a gamma function.

Proof: We know that

$$\mu_{r:n}^{(k)} = E(X_{r:n}^k) = \int_0^\infty x^k f_{r:n}(x) dx \qquad (10)$$

$$= C_{r:n} \int_0^\infty x^k [F(x)]^{r-1} [1-F(x)]^{n-r} f(x) dx .$$

Using (4), one gets

$$\mu_{r:n}^{(k)} = C_{r:n} \sum_{i=0}^{n-r+1} \sum_{j=0}^{r-1+i} \binom{n-r+1}{i} \binom{r-1+i}{j} (-1)^{i+j} \int_0^\infty A(x) x^k e^{-jB(x)} dx \qquad (11)$$

Now,

$$\int_0^\infty A(x) x^k e^{-jB(x)} dx = \sum_{l=0}^\infty \sum_{m=0}^\infty (-1)^{l+m} \frac{j^{l+m} \theta^l \beta^m}{2^l 3^m l!\, m!} \left\{ \alpha \frac{\Gamma(k+2l+3m+1)}{(\alpha j)^{k+2l+3m+1}} + \theta \frac{\Gamma(k+2l+3m+2)}{(\alpha j)^{k+2l+3m+2}} + \beta \frac{\Gamma(k+2l+3m+3)}{(\alpha j)^{k+2l+3m+3}} \right\}.$$

Using this value in (11), we get the desired result.

The applications of above theorem can be explained as:

For $k = 1$, we obtain the mean of the rth order statistic as:

$$\mu_{r:n} = C_{r:n} U_{i,j,l,m} \left\{ \alpha \frac{\Gamma(2l+3m+2)}{(\alpha j)^{2l+3m+2}} + \theta \frac{\Gamma(2l+3m+3)}{(\alpha j)^{2l+3m+3}} + \beta \frac{\Gamma(2l+3m+4)}{(\alpha j)^{2l+3m+4}} \right\}.$$

Now for $k = 2$, one can get the second order moment of the rth order statistic as

$$\mu_{r:n}^{(2)} = C_{r:n} U_{i,j,l,m} \left\{ \alpha \frac{\Gamma(2l+3m+3)}{(\alpha j)^{2l+3m+3}} + \theta \frac{\Gamma(2l+3m+4)}{(\alpha j)^{2l+3m+4}} + \beta \frac{\Gamma(2l+3m+5)}{(\alpha j)^{2l+3m+5}} \right\}.$$

Therefore, the variance of the rth order statistic can be obtained as:

$$V(X_{r:n}) = C_{r:n}U_{i,j,l,m}\left[\begin{array}{c}\left\{\alpha\frac{\Gamma(2l+3m+3)}{(\alpha j)^{2l+3m+3}}+\theta\frac{\Gamma(2l+3m+4)}{(\alpha j)^{2l+3m+4}}+\beta\frac{\Gamma(2l+3m+5)}{(\alpha j)^{2l+3m+5}}\right\}\\ -C_{r:n}U_{i,j,l,m}\left\{\left(\alpha\frac{\Gamma(2l+3m+2)}{(\alpha j)^{2l+3m+2}}+\theta\frac{\Gamma(2l+3m+3)}{(\alpha j)^{2l+3m+3}}+\beta\frac{\Gamma(2l+3m+4)}{(\alpha j)^{2l+3m+4}}\right)^2\right\}\end{array}\right]$$

Similarly, the third and fourth order moments of the rth order statistic can be obtained as:

$$\mu_{r:n}^{(3)} = C_{r:n}U_{i,j,l,m}\left\{\alpha\frac{\Gamma(2l+3m+4)}{(\alpha j)^{2l+3m+4}}+\theta\frac{\Gamma(2l+3m+5)}{(\alpha j)^{2l+3m+5}}+\beta\frac{\Gamma(2l+3m+6)}{(\alpha j)^{2l+3m+6}}\right\}.$$

$$\mu_{r:n}^{(4)} = C_{r:n}U_{i,j,l,m}\left\{\alpha\frac{\Gamma(2l+3m+5)}{(\alpha j)^{2l+3m+5}}+\theta\frac{\Gamma(2l+3m+6)}{(\alpha j)^{2l+3m+6}}+\beta\frac{\Gamma(2l+3m+7)}{(\alpha j)^{2l+3m+7}}\right\}.$$

The mean, variance and other statistical measure of the extreme order statistics are always of great interest. Taking $r = 1$, one can obtain the mean of smallest order statistics:

$$\mu_{1:n} = nW_{l,m}\left\{\begin{array}{r}\alpha\frac{\Gamma(2l+3m+2)}{(n\alpha)^{2l+3m+2}}+\theta\frac{\Gamma(2l+3m+3)}{(n\alpha)^{2l+3m+3}}\\ +\beta\frac{\Gamma(2l+3m+4)}{(n\alpha)^{2l+3m+4}}\end{array}\right\},$$

Where $W_{l,m} = \sum_{l=0}^{\infty}\sum_{m=0}^{\infty}(-1)^{l+m}\frac{n^{l+m}\theta^l\beta^m}{2^l3^m l!m!}$

In addition, second order moment of the smallest order statistic can be obtained as:

$$\mu_{1:n}^{(2)} = nW_{l,m}\left\{\begin{array}{r}\alpha\frac{\Gamma(2l+3m+3)}{(n\alpha)^{2l+3m+3}}+\theta\frac{\Gamma(2l+3m+4)}{(n\alpha)^{2l+3m+4}}\\ +\beta\frac{\Gamma(2l+3m+5)}{(n\alpha)^{2l+3m+5}}\end{array}\right\}.$$

Therefore

$$V(X_{1:n}) = nW_{l,m}\left[\begin{array}{c}\left\{\alpha\frac{\Gamma(2l+3m+3)}{(n\alpha)^{2l+3m+3}}+\theta\frac{\Gamma(2l+3m+4)}{(n\alpha)^{2l+3m+4}}+\beta\frac{\Gamma(2l+3m+5)}{(n\alpha)^{2l+3m+5}}\right\}\\ -nW_{l,m}\left\{\left(\alpha\frac{\Gamma(2l+3m+2)}{(n\alpha)^{2l+3m+2}}+\theta\frac{\Gamma(2l+3m+3)}{(n\alpha)^{2l+3m+3}}+\beta\frac{\Gamma(2l+3m+4)}{(n\alpha)^{2l+3m+4}}\right)^2\right\}\end{array}\right]$$

Similarly one can obtain the mean, second order moment and hence variance of the largest order statistics $(r = n)$.

Theorem 3.2: Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from the QHRD and let $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}$ denote the corresponding order statistics. Then for $1 \leq r \leq n$, we have the following moment relation:

$$\mu_{r:n}^k = (n-r+1)\left\{\frac{\alpha}{k+1}\left(\mu_{r:n}^{k+1}-\mu_{r-1:n}^{k+1}\right)+\frac{\theta}{k+2}\left(\mu_{r:n}^{k+2}-\mu_{r-1:n}^{k+2}\right)+\frac{\beta}{k+3}\left(\mu_{r:n}^{k+3}-\mu_{r-1:n}^{k+3}\right)\right\}$$

Proof:

$$\mu_{r:n}^{(k)} = \int_0^\infty x^k f_{r:n}(x)dx$$

$$= C_{r:n}\int_0^\infty x^k[F(x)]^{r-1}[1-F(x)]^{n-r}f(x)dx$$

Using (4), one gets

$$\mu_{r:n}^{(k)} = C_{r:n}\int_0^{\infty} x^k A(x)[F(x)]^{r-1}[1-F(x)]^{n-r+1}dx.$$

By using integration by parts, we obtain the desired result.

Theorem 3.3: For $1 \le r \le s \le n, n \in N,$ we have

$$\mu_{r;s:n}^{(k_1,k_2)} = (n-s+1)\left\{\frac{\alpha}{k_2+1}\left(\mu_{r,s:n}^{k_1,k_2+1} - \mu_{r,s-1:n}^{k_1,k_2+1}\right) + \frac{\theta}{k_2+1}\left(\mu_{r,s:n}^{k_1,k_2+2} - \mu_{r,s-1:n}^{k_1,k_2+2}\right) + \frac{\beta}{k_2+3}\left(\left(\mu_{r,s:n}^{k_1,k_2+3} - \mu_{r,s-1:n}^{k_1,k_2+3}\right)\right)\right\} \quad (12)$$

Proof: Using (6), we have

$$\mu_{r;s:n}^{(k_1,k_2)} = C_{r:s;n}\int_0^{\infty}\int_x^{\infty} x^{k_1}y^{k_2}\,[F(x)]^{r-1}[F(y)-F(x)]^{s-r-1}[1-F(y)]^{n-s}f(x)f(y)\,dydx$$

Or

$$\mu_{r;s:n}^{(k_1,k_2)} = C_{r:s;n}\int_0^{\infty} x^{k_1}\,[F(x)]^{r-1}f(x)I_X dx \quad (13)$$

Where,

$$I_X = \int_x^{\infty} y^{k_2}\,[F(y)-F(x)]^{s-r-1}[1-F(y)]^{n-s}f(y)\,dy.$$

Using (4), we get

$$I_X = \int_x^{\infty} A(y)y^{k_2}\,[F(y)-F(x)]^{s-r-1}[1-F(y)]^{n-s+1}dy.$$

Now integrating by parts and then substituting I_X in (13), we get the desired result.

4 Conclusion

In this paper, we study the sampling distribution from the order statistics of quadratic hazard rate distribution. In addition, we consider the single and product moment of order statistics from QHRD. We establish recurrence relation for single and product moments of order statistics.

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