

The Transmuted Exponentiated U-quadratic Distribution for Lifetime Modeling

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Abstract: In this work, a new lifetime model called transmuted exponentiated U-quadratic distribution is proposed. Various properties are computed and studied such as moments, moment generating function and Shannon entropy. It is shown that the usual maximum likelihood estimate fail to exist for the transmuted exponentiated U-quadratic distribution due to some irregularities, and suggested an alternative method for their parameter estimation called alternative maximum likelihood method and assessed by simulation studies. A real data application is provided for illustration.

Keywords: Exponentiated-Uquadratic, Moments, Entropy, Maximum likelihood estimation, Alternative Maximum likelihood estimation

1 Introduction

In distribution theory, demands to solve problems in practical applications encountered practitioners and applied researchers to propose so many models so that lifetime data can be assessed and investigated in a better way. In another word, there is a high need to introduce useful models to explore the real-life phenomenon.

One objective for proposing, extending or generalizing probability models is to point out how the lifetime phenomenon arises in various fields of studies such as physics, medical science, computer science, Economics, communication, biology, engineering, life-science among others. For example, the well-known classical distributions such as exponential, BurrXII, and Weibull are not able to show broad flexibility in modeling data with U-shaped density (or bimodal density), in this work, the new model has the ability to accommodate U-shape and bimodal density.

Extending distributions by the transmutation method is one of the popular methods in literature and were considered by many authors in recent years. For example, Transmuted Weibull [1], Transmuted log-logistic [12], Transmuted exponentiated-exponential [3], Transmuted Frechet [10], Transmuted quasi-Lindley [14], Transmuted modified inverse Weibull [7], Transmuted exponentiated-modified Weibull [24], Transmuted generalized linear exponential [6], Transmuted additive Weibull [4], Transmuted generalized Rayleigh (GR) [23],

Transmuted exponentiated-gamma [22], Transmuted exponentiated-Frechet [21], Transmuted Gompertz [20], Transmuted Marshall-Olkin Fréchet [16], Transmuted Weibull Power Function [17], Transmuted New Weibull-Pareto [18] among others.

The transmuted family of distributions has been receiving attention over the last few years, and defined according to [28] as follows

Definition 1 Let $G(x)$ be a baseline cumulative distribution function, $g(x)$ be the density function of $G(x)$, let $\lambda \leq |1|$, then according to the Quadratic Rank Transmutation Map (QRTM), the cumulative distribution function (cdf) and density function of the transmuted - G family satisfy the relationship

$$F(x) = (1 + \lambda)G(x) - \lambda G(x)^2 \quad x \in \mathbb{R} \quad (1)$$

and

$$f(x) = g(x)[(1 + \lambda) - 2\lambda G(x)] \quad x \in \mathbb{R} \quad (2)$$

respectively.

Here, we let $G(x)$ to be the cdf of the exponentiated U-quadratic (EUq) distribution studied by [29], given by

$$G(x) = \left(\frac{\alpha}{3} ((x - \beta)^3 + (\beta - a)^3) \right)^\theta, \quad x \in [a, b] \quad (3)$$

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where $\theta > 0$, $a \in (-\infty, \infty)$, $b \in (a, \infty)$, $\alpha = \frac{12}{(b-a)^3}$ and $\beta = \frac{a+b}{2}$. The corresponding pdf of (3) is

$$g(x) = \theta \alpha^\theta 3^{1-\theta} (x-\beta)^2 ((x-\beta)^3 + (\beta-a)^3)^{\theta-1} \quad (4)$$

The paper is organized as follows. In section 2 we provide the transmuted exponentiated U-quadratic distribution (TEUq) and some essential properties. In section 3 the maximum likelihood estimation and alternative maximum likelihood estimation are discussed, the performance of the AMLE method is assessed by simulation studies. Section 4 provides the application of TEUq to real data. Conclusions in section 5.

2 The TEUq and Properties

In this section, we study the TEUq distribution and some of its important properties such as quantile, moments and Shannon entropy. By using equation (3) in (1) we have the cdf of the TEUq as

$$F(x) = (1+\lambda) \left(\frac{\alpha}{3} ((x-\beta)^3 + (\beta-a)^3) \right)^\theta - \lambda \left(\frac{\alpha}{3} ((x-\beta)^3 + (\beta-a)^3) \right)^{2\theta}, \quad x \in [a, b], \quad (5)$$

where $\theta > 0$, $\lambda \leq |1|$, $a \in \mathbb{R}$, $b > a$, $\alpha = 12/(b-a)^3$ and $\beta = (a+b)/2$. The corresponding density function $f(x)$ and hazard rate function $h(x)$ are given respectively by

$$f(x) = \theta \alpha^\theta 3^{1-\theta} (x-\beta)^2 ((x-\beta)^3 + (\beta-a)^3)^{\theta-1} \cdot \left[(1+\lambda) - 2\lambda \left(\frac{\alpha}{3} ((x-\beta)^3 + (\beta-a)^3) \right)^\theta \right] \quad x \in [a, b] \quad (6)$$

$$h(x) = \frac{\theta \alpha^\theta 3^{1-\theta} (x-\beta)^2 ((x-\beta)^3 + (\beta-a)^3)^{\theta-1} \left[(1+\lambda) - 2\lambda \left(\frac{\alpha}{3} ((x-\beta)^3 + (\beta-a)^3) \right)^\theta \right]}{\left[1 - \left[(1+\lambda) \left(\frac{\alpha}{3} ((x-\beta)^3 + (\beta-a)^3) \right)^\theta - \lambda \left(\frac{\alpha}{3} ((x-\beta)^3 + (\beta-a)^3) \right)^{2\theta} \right] \right]} \quad x \in (a, b) \quad (7)$$

Observe that when $\lambda = 0$, then TEUq become the exponentiated U-quadratic (EUq) distribution [29]; if $\lambda = 0$ and $\theta = 1$, TEUq become the U-quadratic (Uq) distribution; if $\theta = 1$, TEUq become transmuted U-quadratic (TUq) distribution [47]. Figure 1 show the plots of the $f(x)$ given by (6) and figure 2 show the plots of $h(x)$ given by (7) of the TEUq for some values of λ and θ . We can see from figure 1 and 2 that the density of the TEUq distribution can have U-shape, decreasing then increasing, unimodal then increasing, and bimodal shapes. The hazard function can accommodate data with bathtub failure rate and unimodal then increasing failure rate.

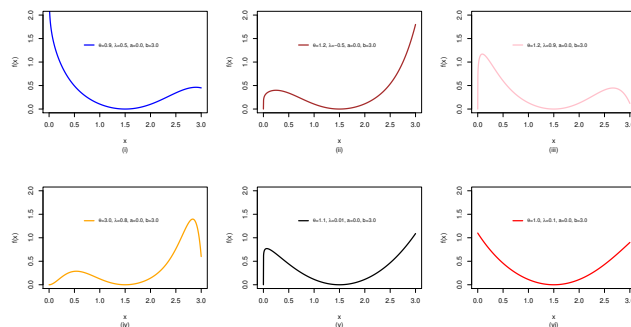


Fig. 1: Plots of the density ($f(x)$) of the TEUq for some values of λ , θ , a and b .

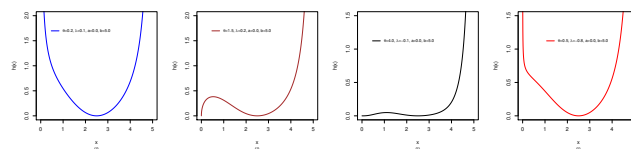


Fig. 2: Plots of the hazard function ($h(x)$) of the TEUq for some values of λ , θ , a and b .

2.1 Quantile function

The quantile function of transmuted exponentiated U-quadratic distribution can be obtained by inverting (5) as

$$Q(u) = (\phi(\lambda, u)^{1/\theta} - (\beta-a)^3)^{1/3} + \beta, \quad u \in (0, 1) \quad (8)$$

where

$$\phi(\lambda, u) = -(1+\lambda) + \frac{\sqrt{(1+\lambda)^2 + 4\lambda u}}{2\lambda}, \quad (9)$$

thus, the median of $X \sim \text{TEUq}$ is obtain as

$$M = (\phi(\lambda, 1/2)^{1/\theta} - (\beta-a)^3)^{1/3} + \beta.$$

Proposition 2.1 Let $P \sim U(0, 1)$, then $X = (\phi(\lambda, p)^{1/\theta} - (\beta-a)^3)^{1/3} + \beta$ is the random variable with TEUq(θ, λ, a, b), where (9) provide $\phi(\lambda, p)$ and $U(0, 1)$ is the uniform distribution.

Moreover, we can use the quantile function in (8) to study the nature of the skewness and kurtosis of the TEUq with respect to the parameters θ and λ by the use of Bowley skewness (B) and Moores kurtosis (M) measures, defined by

$$B = \frac{Q(3/4) + Q(1/4) - 2Q(2/4)}{Q(3/4) - Q(1/4)}$$

and

$$M = \frac{Q(3/8) - Q(1/8) + Q(7/8) - Q(5/8)}{Q(6/8) - Q(2/8)}$$

respectively, where $Q(\cdot)$ is given by (8).

Figure 3 demonstrated the behavior of the skewness and kurtosis of the $TEUq$ as the parameter $\theta > 0$ increases and for $\lambda \leq |1|$. The skewness is decreasing — increasing — decreasing — increasing — decreasing in both θ and λ , while the kurtosis is decreasing — increasing — decreasing — increasing — decreasing in θ and increasing — decreasing — increasing — decreasing in λ .

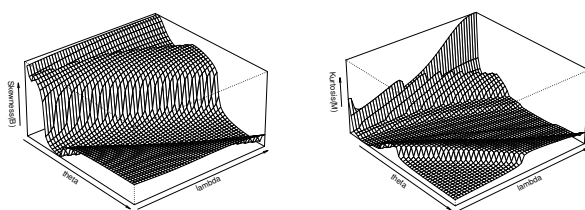


Fig. 3: Plots of the Bowley skewness (B) and Moors kurtosis (M) of the $TEUq$ distribution.

2.2 Moments

Here, we discuss the r^{th} moment, moment generating function of the $TEUq$ distribution. The following lemma is very useful in computations of most of the properties of the $TEUq$ distribution.

Lemma 2.2 Let $c_1, c_2, c_3 \in \mathbb{R}$, let

$$A(c_1, c_2, c_3) = \int_a^b x^{c_1} (x - \beta)^{c_2} ((x - \beta)^3 + (\beta - a)^3)^{c_3} dx, \quad (10)$$

then

$$A(c_1, c_2, c_3) = \sum_{i=0}^{\infty} \sum_{k=0}^{\infty} \psi_{c_2, c_3}(\beta) \frac{b^{c_1+k+1} - a^{c_1+k+1}}{c_1+k+1}, \quad (11)$$

where, $\psi_{c_2, c_3}(\beta) = (\beta - a)^{3c_3-3i} (-\beta)^{c_2+3i-k} \binom{c_3}{i} \binom{c_2+3i}{k}$, and $x \in (-\beta, \beta) \cap (a, 2\beta - a)$.

Proof:

By the generalized binomial expansion in (10), $((x - \beta)^3 + (\beta - a)^3)^{c_3} = (\beta - a)^{3c_3} \sum_{i=0}^{\infty} \binom{c_3}{i} \left(\frac{x-\beta}{\beta-a}\right)^{3i}$ and $x \in (a, 2\beta - a)$, substituting back to (10) and apply the expansion again we get $(x - \beta)^{c_2+3i} = (-\beta)^{c_2+3i-k} \sum_{k=0}^{\infty} \binom{c_2+3i}{k} x^k$ for $x \in (-\beta, \beta)$. Therefore

$$\begin{aligned} A(c_1, c_2, c_3) &= \sum_{i=0}^{\infty} \sum_{k=0}^{\infty} (\beta - a)^{3c_3-3i} (-\beta)^{c_2+3i-k} \binom{c_3}{i} \binom{c_2+3i}{k} \int_a^b x^{c_1+k} dx \\ &= \sum_{i=0}^{\infty} \sum_{k=0}^{\infty} (\beta - a)^{3c_3-3i} (-\beta)^{c_2+3i-k} \binom{c_3}{i} \binom{c_2+3i}{k} \frac{b^{c_1+k+1} - a^{c_1+k+1}}{c_1+k+1} dx, \end{aligned}$$

and $x \in (-\beta, \beta) \cap (a, 2\beta - a)$.

The r^{th} moment ($E(X^r)$ or μ_r) and moment generating function ($M_X(t)$) of X are provided as follows using (11).

Theorem 2.3 Let $X \sim TEUq(\theta, \lambda, a, b)$, then the r^{th} moment of X (μ_r) can be expressed as

$$\begin{aligned} \mu_r &= \theta \alpha^\theta 3^{1-\theta} (1 + \lambda) A(r, 2, \theta - 1) \\ &\quad - 2\lambda \theta \alpha^\theta 3^{1-\theta} A(r, 2, 2\theta - 1). \end{aligned} \quad (12)$$

where $A(\cdot, \cdot, \cdot)$ can be computed from equation (11).

Theorem 2.4 Let $X \sim TEUq(\theta, \lambda, a, b)$, then the moment generating function of X ($M_X(t)$) can be expressed as

$$\begin{aligned} M_X(t) &= \theta \alpha^\theta 3^{1-\theta} (1 + \lambda) \sum_{r=0}^{\infty} \frac{t^r}{r!} A(r, 2, \theta - 1) \\ &\quad - 2\lambda \theta \alpha^\theta 3^{1-\theta} \sum_{r=0}^{\infty} \frac{t^r}{r!} A(r, 2, 2\theta - 1). \end{aligned}$$

where $A(\cdot, \cdot, \cdot)$ can be computed from equation (11).

Proof: By considering (12) and $M_X(t) = E(e^{tX}) = \sum_{r=0}^{\infty} \frac{t^r}{r!} E(X^r)$.

2.3 Shanon entropy

Entropy is a measure of variation of the uncertainty of a random variable X . The Shannon entropy measure is defined by $E(-\log f(x))$. We provide some lemmas which are essential for the computation of the Shannon entropy of the $TEUq$.

Lemma 2.5 Let X be a random variable with $TEUq$ distribution, then

$$\begin{aligned} E[\log((x - \beta)^3 + (\beta - a)^3)] \\ = \theta \alpha^\theta 3^{1-\theta} \left((1 + \lambda) \frac{\partial}{\partial t} A(0, 2, \theta + t - 1) \Big|_{t=0} - 2\lambda \frac{\partial}{\partial t} A(0, 2, 2\theta + t - 1) \Big|_{t=0} \right), \end{aligned}$$

$$\begin{aligned} E[\log(x - \beta)] &= (1 + \lambda) \theta \alpha^\theta 3^{1-\theta} \frac{\partial}{\partial t} A(0, 2 + t, 2\theta - 1) \Big|_{t=0} \\ &\quad - 2\lambda \theta \alpha^\theta 3^{1-\theta} \frac{\partial}{\partial t} A(0, 2 + t, 2\theta - 1) \Big|_{t=0}. \end{aligned}$$

Lemma 2.6 Let X be a random variable with TEUq distribution, then

$$\begin{aligned}
 & E[\log((1+\lambda) - 2\lambda \left(\frac{\alpha}{3}((x-\beta)^3 + (\beta-a)^3)\right)^\theta)] \\
 &= (1+\lambda)\theta 3^{1-\theta} \sum_{w=0}^{\infty} \frac{(-1)^{2w+1} 2^w \alpha^{\theta(w+1)} \lambda^w}{w 3^{\theta w} (1+\lambda)^w} A(0, 2, \theta(w+1) - 1) \\
 &\quad - 2\theta 3^{1-\theta} \sum_{w=0}^{\infty} \frac{(-1)^{2w+1} \alpha^{\theta(w+1)} 2^w \lambda^{w+1}}{w 3^{\theta w} (1+\lambda)^w} A(0, 2, \theta(w+1)) \\
 &\quad + \log(1+\lambda), \\
 & \text{for } \left| \frac{2\lambda \left(\frac{\alpha}{3}((x-\beta)^3 + (\beta-a)^3)\right)^\theta}{(1+\lambda)} \right| < 1. \\
 & \textbf{Proof:} \text{ we can expand} \\
 & E \left[\log \left[(1+\lambda) - 2\lambda \left(\frac{\alpha}{3}((x-\beta)^3 + (\beta-a)^3)\right)^\theta \right] \right] \\
 &= E \left[\log \left(\left(1 - \frac{2\lambda \left(\frac{\alpha}{3}((x-\beta)^3 + (\beta-a)^3)\right)^\theta}{(1+\lambda)} \right) (1+\lambda) \right) \right], \\
 & \text{for } \left| \frac{2\lambda \left(\frac{\alpha}{3}((x-\beta)^3 + (\beta-a)^3)\right)^\theta}{(1+\lambda)} \right| < 1, \text{ thus,} \\
 & E \left[\log \left(\left(1 - \frac{2\lambda \left(\frac{\alpha}{3}((x-\beta)^3 + (\beta-a)^3)\right)^\theta}{(1+\lambda)} \right) (1+\lambda) \right) \right] \\
 &= \sum_{w=0}^{\infty} \frac{(-1)^{2w+1} \alpha^{\theta w} 2^w \lambda^{w+1}}{w 3^{\theta w} (1+\lambda)^w} E \left[\left((x-\beta)^3 + (\beta-a)^3 \right)^{\theta w} \right] \\
 &\quad + E[\log(1+\lambda)],
 \end{aligned}$$

hence, by considering (10),

$$\begin{aligned}
 & E[\log((1+\lambda) - 2\lambda \left(\frac{\alpha}{3}((x-\beta)^3 + (\beta-a)^3)\right)^\theta)] \\
 &= (1+\lambda)\theta 3^{1-\theta} \sum_{w=0}^{\infty} \frac{(-1)^{2w+1} 2^w \alpha^{\theta(w+1)} \lambda^w}{w 3^{\theta w} (1+\lambda)^w} A(0, 2, \theta(w+1) - 1) \\
 &\quad - 2\theta 3^{1-\theta} \sum_{w=0}^{\infty} \frac{(-1)^{2w+1} \alpha^{\theta(w+1)} 2^w \lambda^{w+1}}{w 3^{\theta w} (1+\lambda)^w} A(0, 2, \theta(w+1)) \\
 &\quad + \log(1+\lambda).
 \end{aligned}$$

Theorem 2.7 Let $X \sim \text{TEUq}(\theta, \lambda, a, b)$, then the Shannon entropy of X can be expressed as

$$\begin{aligned}
 E[-\log f(x)] &= -\log(\theta \alpha^\theta 3^{1-\theta}) - 2(1+\lambda)\theta \alpha^\theta 3^{1-\theta} \frac{\partial}{\partial t} A(0, 2+t, 2\theta-1)|_{t=0} \\
 &\quad + 4\lambda \theta \alpha^\theta 3^{1-\theta} \frac{\partial}{\partial t} A(0, 2+t, 2\theta-1)|_{t=0} \\
 &\quad - (\theta-1)(1+\lambda)\theta \alpha^\theta 3^{1-\theta} \frac{\partial}{\partial t} A(0, 2, \theta+t-1)|_{t=0} \\
 &\quad + 2\lambda \theta \alpha^\theta 3^{1-\theta} (\theta-1) \frac{\partial}{\partial t} A(0, 2, 2\theta+t-1)|_{t=0} \\
 &\quad - (1+\lambda)\theta 3^{1-\theta} \sum_{w=0}^{\infty} \frac{(-1)^{2w+1} 2^w \alpha^{\theta(w+1)} \lambda^w}{w 3^{\theta w} (1+\lambda)^w} A(0, 2, \theta(w+1) - 1) \\
 &\quad + 2\theta 3^{1-\theta} \sum_{w=0}^{\infty} \frac{(-1)^{2w+1} \alpha^{\theta(w+1)} 2^w \lambda^{w+1}}{w 3^{\theta w} (1+\lambda)^w} A(0, 2, \theta(w+1)) - \log(1+\lambda).
 \end{aligned}$$

Proof: From the definition of the Shannon entropy we have the following

$$\begin{aligned}
 E(-\log f(x)) &= -\log(\theta \alpha^\theta 3^{1-\theta}) - 2[\log(x-\beta)] \\
 &\quad - (\theta-1)E[\log((x-\beta)^3 + (\beta-a)^3)] \\
 &\quad - E \left[\log \left((1+\lambda) - 2\lambda \left(\frac{\alpha}{3}((x-\beta)^3 + (\beta-a)^3)\right)^\theta \right) \right],
 \end{aligned}$$

therefore, by using lemma 2.3 and 2.3 we obtain the result.

3 Estimation

In this section, we discussed the failure of the maximum likelihood estimation (MLE) for the TEUq distribution and proposed the use of alternative maximum likelihood estimation (AMLE) for the TEUq distribution.

3.1 Maximum likelihood estimation

In this subsection, we established the fact that the log-likelihood function for the TEUq is unbounded for any sample size $n \geq 1$, and thus the maximum likelihood estimates always fail to exist. The log-likelihood function of the TEUq distribution is

$$\begin{aligned}
 \log \ell(\Theta) &= n \log \theta + n \theta \log \alpha + n(1-\theta) \log 3 \\
 &\quad + 2 \log(x_i - \beta) + (\theta-1) \log((x_i - \beta)^3 + (\beta-a)^3) \\
 &\quad + \log \left[1 + \lambda - 2\lambda \left(\frac{\alpha}{3}((x_i - \beta)^3 + (\beta-a)^3)\right)^\theta \right]
 \end{aligned} \tag{13}$$

where $\Theta = (\theta, \lambda, a, b)^T$. The first partial derivative of the log likelihood function with respect to the parameter θ and λ are given respectively by

$$\begin{aligned}
 \frac{\partial \log \ell(\Theta)}{\partial \theta} &= \frac{n}{\theta} + n \log(\alpha/3) + \sum_{i=1}^n \log((x_i - \beta)^3 + (\beta-a)^3) \\
 &\quad - 2\lambda \sum_{i=1}^n \frac{\left(\frac{\alpha}{3}((x_i - \beta)^3 + (\beta-a)^3)\right)^\theta \log\left(\frac{\alpha}{3}((x_i - \beta)^3 + (\beta-a)^3)\right)}{1 + \lambda - 2\lambda \left(\frac{\alpha}{3}((x_i - \beta)^3 + (\beta-a)^3)\right)^\theta}
 \end{aligned} \tag{14}$$

$$\frac{\partial \log \ell(\Theta)}{\partial \lambda} = \sum_{i=1}^n \frac{1 - 2 \left(\frac{\alpha}{3}((x_i - \beta)^3 + (\beta-a)^3)\right)^\theta}{1 + \lambda - 2\lambda \left(\frac{\alpha}{3}((x_i - \beta)^3 + (\beta-a)^3)\right)^\theta}. \tag{15}$$

Proposition 3.1 Let $\ell(\Theta|\mathbf{x})$ denote the likelihood function for an independent and identically distributed (i.i.d) random sample of size $n \geq 1$ say $x_1, x_2, \dots, x_n$, drawn from TEUq (θ, λ, a, b) distribution, then (i) $\log \ell(\Theta|\mathbf{x}) \rightarrow \infty$ for $\theta > 1$ and (ii) $\log \ell(\Theta|\mathbf{x}) \rightarrow -\infty$ for $\theta < 1$.

Proof: Let $\ell(\Theta|\mathbf{x})$ denote the likelihood function given by (13), let $X_1 \leq X_2 \leq \dots \leq X_n$ be the order statistics for an independent and identically distributed random sample from TEUq (θ, λ, a, b) , then, for $n = 1$ (or $x_i = x_j$) or for $x_i \neq x_j$, there always exist at least one x_i corresponding to the minimum order statistics X_1 , in this case $a = x_1$, and $(x_1 - \beta)^3 + (\beta - a)^3 = 0$, therefore, $\log((x_i - \beta)^3 + (\beta - a)^3)|_{x_1} \rightarrow -\infty$, hence the proof.

The following propositions provide another fact that maximum likelihood estimates of TEUq fail to exist.

Proposition 3.2 Let $\ell(\Theta|\mathbf{x})$ denote the likelihood function for an independent and identically distributed (i.i.d) random sample of size $n \geq 1$ say $x_1, x_2, \dots, x_n$, drawn from TEUq (θ, λ, a, b) distribution, then, $\frac{\partial}{\partial \theta} \log \ell(\Theta|\mathbf{x})$ is unbounded $\forall \theta > 0$.

Proof: Consider (14) and Proposition 3.1

Proposition 3.3 Let $\ell(\Theta|x)$ denote the likelihood function for an independent and identically distributed (i.i.d) random sample of size $n \geq 1$ say $x_1, x_2, \dots, x_n$, drawn from TEUq (θ, λ, a, b) distribution, then, $\frac{\partial}{\partial \lambda} \log \ell(\Theta|x)_{|x_n \& \lambda=1} \rightarrow \infty$.

Proof: Consider (15).

The following corollary shows that the maximum likelihood estimates fail to exist for generalizations of the TEUq distributions.

Corollary 3.4 If $\omega = \{f(x|\xi) : \xi \in \Lambda\}$ is a family of distributions that contains the TEUqs as a subfamily, then the maximum likelihood estimate of the parameter vector ξ based on an i.i.d. sample of size $n \geq 1$ drawn from $f(x|\xi)$ does not exist.

Proof: The fact that ω contains the TEUqs as a subfamily guarantees the existence of $\Lambda^* \subset \Lambda$ such that $\omega_0 = \{f(x|\xi) : \xi \in \Lambda^*\}$ is the family of TEUqs. Let $\ell(\xi|x)$ denote the likelihood function for $f(x|\xi)$, $\xi \in \Lambda$. The fact that the log likelihood function for the TEUq is unbounded guarantees that $\log \ell(\xi|x)$ is unbounded on Λ^* .

Several authors have considered the problem of non existence of maximum likelihood estimation for probability distributions. [38] discussed the non-existence of maximum likelihood estimates for the extended exponential power distribution and showed that the result holds for its generalization and re-parametrization. [32] discussed that the maximum likelihood of the Weibull-Pareto distribution does not exist when the shape parameter of the Weibull is less than one. [36] prove that the finite maximum likelihood estimates for the three-parameter Burr XII distribution do not always exist because the distribution tends towards non-degenerated limiting forms as parameters tend to their boundaries.

3.2 Alternative maximum likelihood estimation

Here, we proposed the alternative maximum likelihood estimation method discussed by [37] as the suitable method of estimating the parameters of TEUq distribution. in this method we set $a = x_1$, then we excluded all the data points correspond to x_1 and use the usual maximum likelihood method to estimate the other parameters by the numerical solutions of the nonlinear equations given by (16) and (17). These equations cannot be solved analytically, and statistical software (*nlnmb*, *nlm*, *maxBFGS* or *optimx* in R-software) can be used to solve them numerically via iterative methods.

$$\begin{aligned} \frac{\partial \log \ell(\Theta)}{\partial \theta} &= \frac{n}{\theta} + n \log(\alpha/3) \\ &+ \sum_{x_i \neq x_1} \log((x_i - \beta)^3 + (\beta - a)^3) \\ &- 2\lambda \sum_{x_i \neq x_1} \frac{\left(\frac{\alpha}{3}((x_i - \beta)^3 + (\beta - a)^3)\right)^\theta w_i(x)}{1 + \lambda - 2\lambda \left(\frac{\alpha}{3}((x_i - \beta)^3 + (\beta - a)^3)\right)^\theta}, \quad (16) \end{aligned}$$

$$\frac{\partial \log \ell(\Theta)}{\partial \lambda} = \sum_{x_i \neq x_1} \frac{1 - 2\left(\frac{\alpha}{3}((x_i - \beta)^3 + (\beta - a)^3)\right)^\theta}{1 + \lambda - 2\lambda \left(\frac{\alpha}{3}((x_i - \beta)^3 + (\beta - a)^3)\right)^\theta},$$

where $w_i(x) = \log\left(\left(\frac{\alpha}{3}((x_i - \beta)^3 + (\beta - a)^3)\right)\right)$.

For the asymptotic interval estimation and hypothesis tests of the parameters θ and λ , we need 2×2 Fisher information matrix denoted by $(J(\Theta))$, under the usual condition that are fulfilled for the parameters θ and λ in the interior of the parameter space but not on the boundary. The asymptotic distribution of $\sqrt{n}(\hat{\Theta} - \Theta)$ is $N_2(0, I^{-1}(\Theta))$, which is a Normal 2-variate with zero mean and variance covariance $I(\Theta)$. This condition is also applicable if $I(\Theta)$ is substitute by the information matrix evaluated at $\hat{\Theta}$, that is $J(\hat{\Theta})$. The Normal 2-variate distribution $N_2(0, J^{-1}(\Theta))$ can be used to establish an approximate confidence interval and region for the model parameters θ and λ . The 2×2 information matrix is defined by $J(\Theta) = -[\partial^2 \ell / \partial \theta \partial \theta^T]$.

In the following theorem, we provide the existence of the roots of AMLE of θ for the TEUq distribution. It has been considered for the MLEs of several distributions by many authors, for example, exponential Poisson (EP) distribution [33], exponential geometric (EG) [31], generalized BurrXII Poisson (GBXIIIP) [34], Mustapha-type II (MuII) distribution [45] and the complementary exponentiated BurrXII-Poisson (CEBXIIP) [35], recently the generalized Half-logistic Poisson (GHLP) [44] among others.

Theorem 3.5 Let $g_\theta(\theta; \lambda, a, b, x_i)$ denote the function on the right-hand side of the equation (16), where λ is the true values of the parameter and $\lambda \neq 1$, then, $g_\theta(\theta; \lambda, a, b, x_i) = 0$ has at least one root and the root lie in the interval (t_1, t_2) , where

$$t_1 = \frac{-n}{n \log \frac{\alpha}{3} + \sum_{x_i \neq x_1} \log((x_i - \beta)^3 + (\beta - a)^3) + \frac{\sum_{x_i \neq x_1} \log\left(\frac{\alpha}{3}((x_i - \beta)^3 + (\beta - a)^3)\right)}{1 - \lambda}}$$

and

$$t_2 = \frac{-n}{n \log \frac{\alpha}{3} + \sum_{x_i \neq x_1} \log((x_i - \beta)^3 + (\beta - a)^3)}$$

Proof:

$$\begin{aligned} \text{Let } \zeta_\theta &= n \log(\alpha/3) + \sum_{x_i \neq x_1} \log((x_i - \beta)^3 + (\beta - a)^3) + \\ &\sum_{x_i \neq x_1} \frac{\left(\frac{\alpha}{3}((x_i - \beta)^3 + (\beta - a)^3)\right)^\theta \log\left(\frac{\alpha}{3}((x_i - \beta)^3 + (\beta - a)^3)\right)}{1 + \lambda - 2\lambda \left(\frac{\alpha}{3}((x_i - \beta)^3 + (\beta - a)^3)\right)^\theta}, \end{aligned}$$

Table 1: AMLEs, standard deviations, Bias and MSE for some various values of parameters for the TEUq.

Sample size	Actual values				Estimated values		Standard deviations		Bias		Mean square error	
n	a	b	λ	θ	$\hat{\lambda}$	$\hat{\theta}$	$sd(\hat{\lambda})$	$sd(\hat{\theta})$	$Bias_{\hat{\lambda}}$	$Bias_{\hat{\theta}}$	$MSE_{\hat{\lambda}}$	$MSE_{\hat{\theta}}$
20	-3.0	4.0	-0.5	0.1	-0.1233	0.1685	0.1825	0.0535	0.3767	0.0685	0.1752	0.0076
	-3.0	4.0	0.1	2.5	0.1367	2.5563	0.2175	0.8427	0.0367	0.0563	0.0486	0.7133
	-3.0	4.0	0.9	1.2	0.4039	1.0623	0.2537	0.2349	-0.4961	-0.1378	0.3105	0.0741
	0.0	5.0	0.5	1.5	0.2144	1.3715	0.2395	0.3672	-0.2856	-0.1285	0.1389	0.1513
	0.0	5.0	0.2	0.5	0.2806	0.6161	0.2507	0.1479	0.0801	0.1161	0.0693	0.0353
30	-3.0	4.0	-0.5	0.1	-0.1173	0.1555	0.1781	0.0488	0.3827	0.0555	0.1752	0.0055
	-3.0	4.0	0.1	2.5	0.1121	2.4340	0.1975	0.6722	0.0121	-0.0655	0.0391	0.4561
	-3.0	4.0	0.9	1.2	0.4960	1.0758	0.2403	0.1934	-0.4040	-0.1242	0.2210	0.0528
	0.0	5.0	0.5	1.5	0.2425	1.3542	0.2479	0.2982	-0.2575	-0.1457	0.1278	0.1101
	0.0	5.0	0.2	0.5	0.2841	0.5846	0.2454	0.1158	0.0841	0.0846	0.0673	0.0206
50	-3.0	4.0	-0.5	0.1	-0.1171	0.1442	0.1687	0.0244	0.3829	0.0442	0.1751	0.0025
	-3.0	4.0	0.1	2.5	0.0879	2.3173	0.1723	0.4937	-0.0121	-0.1827	0.0298	0.2772
	-3.0	4.0	0.9	1.2	0.5967	1.0971	0.2054	0.1533	-0.3033	-0.1030	0.1342	0.0341
	0.0	5.0	0.5	1.5	0.2824	1.3503	0.2460	0.2434	-0.2177	-0.1496	0.1079	0.0817
	0.0	5.0	0.2	0.5	0.2808	0.5606	0.2297	0.0919	0.0807	0.0606	0.0593	0.0120
100	-3.0	4.0	-0.5	0.1	-0.1176	0.1380	0.1519	0.0174	0.3824	0.0379	0.1693	0.0017
	-3.0	4.0	0.1	2.5	0.0615	2.2669	0.1307	0.3356	-0.0384	-0.2316	0.0186	0.1689
	-3.0	4.0	0.9	1.2	0.7089	1.1283	0.1437	0.1079	-0.1911	-0.0717	0.0572	0.0168
	0.0	5.0	0.5	1.5	0.3409	1.3744	0.2221	0.1881	-0.1591	-0.1256	0.0746	0.0512
	0.0	5.0	0.2	0.5	0.2620	0.5374	0.1940	0.0668	0.0622	0.0374	0.0415	0.0059
200	-3.0	4.0	-0.5	0.1	-0.1039	0.1362	0.1224	0.0138	0.3960	0.0362	0.1628	0.0015
	-3.0	4.0	0.1	2.5	0.0450	2.2669	0.0970	0.2429	-0.0550	-0.2331	0.0125	0.1133
	-3.0	4.0	0.9	1.2	0.7856	1.1563	0.0956	0.0758	-0.1144	-0.0437	0.0222	0.0077
	0.0	5.0	0.5	1.5	0.3951	1.4104	0.1768	0.1431	-0.1048	-0.0896	0.0422	0.0285
	0.0	5.0	0.2	0.5	0.2380	0.5212	0.1586	0.0497	0.0387	0.0212	0.0267	0.0029

then, $\lim_{\theta \rightarrow 0^+} \zeta_{\theta} = n \log(\alpha/3) + \sum_{x_i \neq x_1} \log((x_i - \beta)^3 + (\beta - a)^3) + \sum_{x_i \neq x_1} \frac{\log(\frac{\alpha}{3}((x_i - \beta)^3 + (\beta - a)^3))}{1 - \lambda}$

therefore,

$$g_{\theta}(\theta; \lambda, a, b, x_i) > \frac{n}{\theta} + \lim_{\theta \rightarrow 0^+} \zeta_{\theta} = \frac{n}{\theta} + n \log(\alpha/3) + \sum_{x_i \neq x_1} \log((x_i - \beta)^3 + (\beta - a)^3) + \sum_{x_i \neq x_1} \frac{\log(\frac{\alpha}{3}((x_i - \beta)^3 + (\beta - a)^3))}{1 - \lambda} > 0$$

if

$$\theta > \frac{-n}{n \log \frac{\alpha}{3} + \sum_{x_i \neq x_1} \log((x_i - \beta)^3 + (\beta - a)^3) + \frac{\sum_{x_i \neq x_1} \log(\frac{\alpha}{3}((x_i - \beta)^3 + (\beta - a)^3))}{1 - \lambda}}.$$

on the other hand, $\lim_{\theta \rightarrow \infty} \zeta_{\theta} = n \log(\alpha/3) + \sum_{x_i \neq x_1} \log((x_i - \beta)^3 + (\beta - a)^3)$, therefore,

$$g_{\theta}(\theta; \lambda, a, b, x_i) < \frac{n}{\theta} + \lim_{\theta \rightarrow \infty} \zeta_{\theta} = \frac{n}{\theta} + n \log(\alpha/3) + \sum_{x_i \neq x_1} \log((x_i - \beta)^3 + (\beta - a)^3) < 0 \text{ if } \theta < \frac{-n}{n \log \frac{\alpha}{3} + \sum_{x_i \neq x_1} \log((x_i - \beta)^3 + (\beta - a)^3)}, \text{ hence,}$$

$g_{\theta}(\theta; \lambda, a, b, x_i) = 0$ for $\theta \in (t_1, t_2)$,

where

$$t_1 = \frac{-n}{n \log \frac{\alpha}{3} + \sum_{x_i \neq x_1} \log((x_i - \beta)^3 + (\beta - a)^3) + \frac{\sum_{x_i \neq x_1} \log(\frac{\alpha}{3}((x_i - \beta)^3 + (\beta - a)^3))}{1 - \lambda}}$$

and

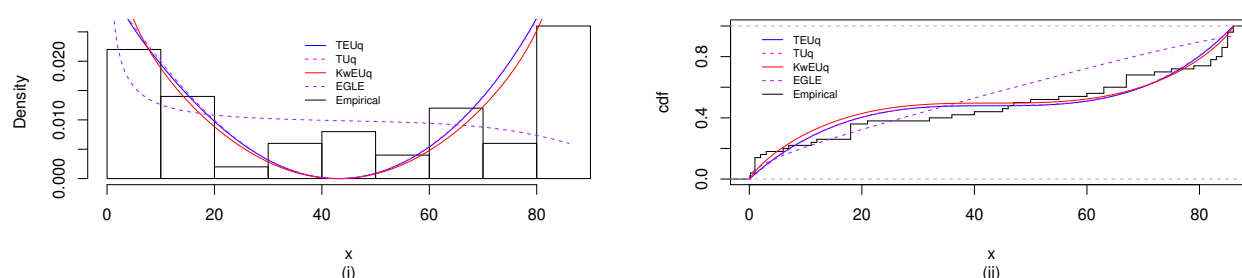
$$t_2 = \frac{-n}{n \log \frac{\alpha}{3} + \sum_{x_i \neq x_1} \log((x_i - \beta)^3 + (\beta - a)^3)}$$

3.3 Simulation study

In this subsection, we assessed the proposed AMLEs by simulation studies. We generate 10,000 samples of size $n = (20, 30, 50, 100, 200)$, the estimated values, standard deviations (sd), bias and mean square error (MSE) of the estimates are computed using R-software. The results presented in Table 1 is the estimated values, standard deviation (sd), bias, and mean square error (MSE). From table 1 it is clear that the estimated values of the parameters converge to their actual values in most cases, also the standard deviations and the mean square error decrease as the sample size increases.

Table 2: MLEs, $\ell(\Theta)$, KS and P-value for the given data set.

Model	α	β	λ	a	b	γ	θ	δ	$\ell(\Theta)$	KS	P-value
TEUq	—	—	-0.1900	0.100	86.000	—	0.9262	—	-212.86	0.1219	0.414
TUq	—	—	-0.1136	0.100	86.000	—	—	—	-223.18	0.1263	0.371
EUq	—	—	—	0.100	86.000	—	0.9692	—	-219.93	0.4072	7.3e-08
KwEUq	—	—	0.8977	0.100	86.000	—	0.8977	0.8081	-212.43	0.1473	0.207
GU	0.267	51.942	—	0.090	86.713	—	—	—	-207.33	0.1520	0.198
EGLE	—	—	—	3.3e-3	1.7e-4	4.564	0.112	—	-224.34	0.1475	0.206
GLE	—	—	—	9.6e-3	4.5e-4	0.730	—	—	-235.93	0.1598	0.139
GLFR	—	—	—	3.8e-3	3.1e-4	—	0.533	—	-233.15	0.1620	0.129

**Fig. 4:** Plots of the histogram with estimated density and empirical cdf with estimated cdf of the competing distribution for the given data set

4 Illustration

Here, we fitted the TEUq distribution using AMLEs. The AMLEs are computed by the solution of the (16) and (17) using R-software. The data set used is the lifetimes of fifty devices provided in [30] and recently studied by [43]. The data set are: .1, .2, 1, 1, 1, 1, 1, 2, 3, 6, 7, 11, 12, 18, 18, 18, 18, 21, 32, 36, 40, 45, 46, 47, 50, 55, 60, 63, 63, 67, 67, 67, 67, 72, 75, 79, 82, 82, 83, 84, 84, 84, 85, 85, 85, 85, 85, 86, 86.

We also compare the fit of the TEUq using the Kolmogorov Smirnov (KS) test statistics with its sub models and some other popular models such as gamma-uniform (GU) [42], generalized linear exponential (GLE) [39], exponentiated generalized linear exponential (EGLE) [40], generalized linear failure rate (GLFR) [41], exponentiated generalized linear exponential (EGLE) [40], transmuted U-quadratic (TUq) distribution [47], and Kumaraswamy exponentiated U-quadratic (KwEUq) [46].

The numerical results are provided in table 2 showing that TEUq fit the data better than the other models. Figure 4 provides the plots of the histogram and cumulative distribution function of the empirical and estimated TEUq distribution.

5 Conclusions

We have proposed a new lifetime model called transmuted exponentiated U-quadratic (TEUq) distribution. Various properties of the TEUq are derived and studied such as the explicit form of the moments, moment generating function and Shannon entropy. Moreover, it is shown that the usual maximum likelihood estimates does not hold for the TEUq and suggested the use of the alternative maximum likelihood estimation method, the alternative maximum likelihood estimators was assessed by simulation studies. The effectiveness of the TEUq was demonstrated by an application to a real data set showing that TEUq fit the data better than some other existing distributions as measured by the Kolmogorov Smirnov test.

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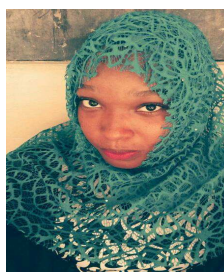
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