

Numerical Solutions of the Modified Burgers Equation by Explicit Logarithmic Finite Difference Schemes

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Abstract: In this paper, explicit logarithmic finite difference schemes (E-LFDSs) obtained by using various finite difference approaches instead of the non-linear term of the modified Burgers equation are given for numerical solutions of the modified Burgers equation. A test problem is used to verify the effectiveness and accuracy of the schemes that we have suggested. Comparisons are made with other studies in the literature. All results are consistent with the results obtained in other studies in the literature. L_2 and L_∞ error norms are calculated. The obtained error norms are quite small. The results show that the present numerical approaches are highly successful to solve the modified Burgers equation.

Keywords: Burgers equation, modified Burgers equation, explicit logarithmic finite difference schemes

1 Introduction

Burgers equation was first given by Bateman [2] and later was studied by Burgers [4,5] as a mathematical model for turbulence. The Burgers equation has applications in various fields such as convection and diffusion, number theory, gas dynamics, heat conduction, elasticity since it has an extensive use in engineering and other scientific fields [6]. The one-dimensional generalized Burgers equation is in the form

$$u_t + u^p u_x - \nu u_{xx} = 0 \quad a < x < b, \quad p = 1, 2$$

in which u denotes the velocity for space x and time t and $\nu > 0$ is a constant representing the kinematics viscosity of the fluid. It is known as Burgers equation and modified Burgers equation for $p = 1$ and $p = 2$, respectively.

Many analytical and numerical solutions of such equation were studied by a number of authors using different methods [1]-[29]. An analytical solution of Burgers equation has been given by Benton and Platzman [3]. Miller [21] introduced infinite series solutions of the equation. The equation has been solved numerically by several authors using the explicit and exact-explicit finite difference methods [20], a finite element approach [22], B-spline collocation methods [9], variational iteration method [1] and explicit logarithmic finite difference

schemes [8]. The problem we deal with is in general form:

$$u_t + u^2 u_x - \nu u_{xx} = 0; \quad a < x < b, t > t_0$$

with initial condition

$$u(x, t_0) = f(x); \quad a < x < b$$

and boundary conditions

$$u(a, t) = g_1(t), u(b, t) = g_2(t); \quad t > t_0$$

where x and t are independent variables, $u = u(x, t)$, ν is the viscosity parameter, $f(x)$, $g_1(t)$ and $g_2(t)$ are known functions.

In the literature many numerical method was applied to approximate the solution of the modified Burgers equation by several authors. The collocation method with quintic splines [24], the collocation method with septic splines [25], the sextic B-spline collocation method [15], a non-polynomial spline based method [12], an explicit numerical scheme [6], Petrov-Galerkin method [26] and explicit exponential finite difference schemes [7] have been used to obtain numerical solution of equation by several authors. Time and space splitting techniques have been applied to the Burgers equation and the modified

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Burgers equation, and then the quintic B-spline collocation procedure has been employed to approximate the resulting systems by Saka and Dag [27].

In the present work, main aim is to apply the E-LFDSs for the numerical solutions of modified Burgers equation.

2 Model Problem and Numerical Method

2.1 Model Problem

We consider the modified Burgers equation

$$u_t + u^2 u_x - v u_{xx} = 0; 0 < x < 1, t > 1 \quad (1)$$

with initial condition

$$u(x, 1) = \frac{x}{1 + (1/c_0) \exp(x^2/4v)}; 0 < x < 1 \quad (2)$$

and boundary conditions

$$u(0, t) = 0, u(1, t) = \frac{1/t}{1 + (\sqrt{t}/c_0) \exp(1/4vt)}; t > 1. \quad (3)$$

Following [13] modified Burgers equation (1) has the analytic solution

$$u(x, t) = \frac{x/t}{1 + (\sqrt{t}/c_0) \exp(x^2/4vt)}; 0 < x < 1, t > 1 \quad (4)$$

where c_0 is a constant, $0 < c_0 < 1$.

2.2 Numerical Method

We indicate the finite difference approximation of $u(x, t)$ at the grid point (x_i, t_n) by u_i^n in which $x_i = ih$ ($i = 0, 1, \dots, N$), $t_n = nk$ ($n = 0, 1, 2, \dots$), h is the grid size in x -direction and k represents the increment in time. Let $F(u)$ is any continuous differential function and multiplying equation (1) by $\frac{\partial F}{\partial u}$ the following equation is obtained:

$$\frac{\partial F}{\partial u} \frac{\partial u}{\partial t} = F'(u) \left(-u^2 \frac{\partial u}{\partial x} + v \frac{\partial^2 u}{\partial x^2} \right) \quad (5)$$

and

$$\frac{\partial F}{\partial t} = F'(u) \left(-u^2 \frac{\partial u}{\partial x} + v \frac{\partial^2 u}{\partial x^2} \right). \quad (6)$$

If we use the forward difference approach instead of $\frac{\partial F}{\partial t}$ the finite difference representation of equation (6) is obtained as:

$$F(u_i^{n+1}) - F(u_i^n) = k F'(u_i^n) \left[-\left(u^2 \frac{\partial u}{\partial x}\right)_i^n + v \left(\frac{\partial^2 u}{\partial x^2}\right)_i^n \right] \quad (7)$$

in which k is the time step.

Let $F(u) = e^u$ then the E-LFDS for equation (1) is obtained as:

$$u_i^{n+1} = u_i^n + \ln \left\{ 1 - k \left(u^2 \frac{\partial u}{\partial x}\right)_i^n + vk \left(\frac{\partial^2 u}{\partial x^2}\right)_i^n \right\} \quad (8)$$

In this E-LFDS, instead of the dependent variable and its derivatives their approximate values are written using the various finite difference approaches. These approximations will result various explicit equations.

If central difference approach is used instead of $\frac{\partial^2 u}{\partial x^2}$ in equation (8)

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_i^n \cong \frac{u_{i-1}^n - 2u_i^n + u_{i+1}^n}{h^2}, 1 \leq i \leq N-1 \quad (9)$$

and then the following finite difference approaches are used instead of the non-linear term $u^2 \frac{\partial u}{\partial x}$

Approach-I.

$$\left(u^2 \frac{\partial u}{\partial x}\right)_i^n \cong (u_i^n)^2 \left(\frac{u_{i+1}^n - u_{i-1}^n}{2h}\right),$$

Approach-II.

$$\left(u^2 \frac{\partial u}{\partial x}\right)_i^n \cong \left(\frac{u_i^n + u_{i+1}^n}{2}\right)^2 \left(\frac{u_{i+1}^n - u_{i-1}^n}{2h}\right),$$

Approach-III.

$$\left(u^2 \frac{\partial u}{\partial x}\right)_i^n \cong \left(\frac{u_{i-1}^n + u_i^n}{2}\right)^2 \left(\frac{u_{i+1}^n - u_{i-1}^n}{2h}\right),$$

Approach-IV.

$$\left(u^2 \frac{\partial u}{\partial x}\right)_i^n \cong \left(\frac{u_{i+1}^n + u_i^n + u_{i-1}^n}{3}\right)^2 \left(\frac{u_{i+1}^n - u_{i-1}^n}{2h}\right),$$

respectively, where $1 \leq i \leq N-1$. We obtained the following E-LFDSs

$$\text{E-LFDS-I. } u_i^{n+1} = u_i^n + \ln \left\{ 1 - \frac{k}{2h} (u_i^n)^2 (u_{i+1}^n - u_{i-1}^n) + rv (u_{i+1}^n - 2u_i^n + u_{i-1}^n) \right\} \quad (10)$$

$$\text{E-LFDS-II. } u_i^{n+1} = u_i^n + \ln \left\{ 1 - \frac{k}{2h} \left(\frac{u_i^n + u_{i+1}^n}{2}\right)^2 (u_{i+1}^n - u_{i-1}^n) + rv (u_{i+1}^n - 2u_i^n + u_{i-1}^n) \right\}, \quad (11)$$

$$\text{E-LFDS-III. } u_i^{n+1} = u_i^n + \ln \left\{ 1 - \frac{k}{2h} \left(\frac{u_{i-1}^n + u_i^n}{2}\right)^2 (u_{i+1}^n - u_{i-1}^n) + rv (u_{i+1}^n - 2u_i^n + u_{i-1}^n) \right\}, \quad (12)$$

$$\text{E-LFDS-IV. } u_i^{n+1} = u_i^n + \ln \left\{ 1 - \frac{k}{2h} \left(\frac{u_{i+1}^n + u_i^n + u_{i-1}^n}{3}\right)^2 (u_{i+1}^n - u_{i-1}^n) + rv (u_{i+1}^n - 2u_i^n + u_{i-1}^n) \right\}, \quad (13)$$

respectively, where $r = \frac{k}{h^2}$, $1 \leq i \leq N-1$, $n = 0, 1, 2, \dots$

The values of u_i^0 ($i = 0, 1, 2, \dots, N$) are known from initial and boundary conditions. Hence the right side of the equality is known in every time step.

3 Local Truncation Error and Consistency

In order to analyze the local truncation errors of the numerical schemes (10)-(13), the nonlinear terms of the schemes have been linearized by replacing the quantities u_i^n , $\frac{u_i^n + u_{i+1}^n}{2}$, $\frac{u_{i-1}^n + u_i^n}{2}$ and $\frac{u_{i+1}^n + u_i^n + u_{i-1}^n}{3}$ by local constants $\tilde{U}_1$, $\tilde{U}_2$, $\tilde{U}_3$ and $\tilde{U}_4$, respectively. For simplicity, we let $\tilde{U}_1 = \tilde{U}_2 = \tilde{U}_3 = \tilde{U}_4 = \tilde{U}$ hence the numerical schemes (10)-(13), convert into

$$u_i^{n+1} = u_i^n + \ln \left\{ 1 - \frac{k}{2h} (\tilde{U})^2 (u_{i+1}^n - u_{i-1}^n) + rv (u_{i+1}^n - 2u_i^n + u_{i-1}^n) \right\}. \quad (14)$$

Since the scheme (14) is logarithmic, the examination will be improved by expanding the logarithmic term of the scheme into a Taylor's series. Hilal et al. [14] applied the same procedure to calculate the local truncation error of exponential finite difference schemes and examine their stability. If the logarithmic term of the scheme expand to Taylor series and use the first term of the expansion the schemes can be written as:

$$u_i^{n+1} = u_i^n - \frac{k}{2h} (\tilde{U})^2 (u_{i+1}^n - u_{i-1}^n) + rv (u_{i+1}^n - 2u_i^n + u_{i-1}^n). \quad (15)$$

Expansion of the terms U_i^{n+1} , U_{i+1}^n and U_{i-1}^n about the point (x_i, t_n) by Taylor's series and substitution into

$$T_i^n = U_i^{n+1} - U_i^n + \frac{k}{2h} (\tilde{U})^2 (U_{i+1}^n - U_{i-1}^n) - rv (U_{i+1}^n - 2U_i^n + U_{i-1}^n)$$

leads to

$$T_i^n = \left[\frac{\partial U}{\partial t} + (\tilde{U})^2 \frac{\partial U}{\partial x} - v \frac{\partial^2 U}{\partial x^2} \right]_i^n + \frac{k}{2} \left(\frac{\partial^2 U}{\partial t^2} \right)_i^n + \frac{h^2}{6} (\tilde{U})^2 \left(\frac{\partial^3 U}{\partial x^3} \right)_i^n + \dots$$

Therefore the principal part of the local truncation error is as follows:

$$\frac{k}{2} \left(\frac{\partial^2 u}{\partial t^2} \right)_i^n + \frac{h^2}{6} (\tilde{U})^2 \left(\frac{\partial^3 u}{\partial x^3} \right)_i^n$$

Hence the local truncation error is $T_i^n = O(k) + O(h^2)$.

Since $\lim_{k, h \rightarrow 0} [O(k) + O(h^2)] = 0$ present schemes are consistent. And the schemes are first order in time and second order in space.

4 Stability Analysis

To investigate the stability of schemes, we will use the von Neumann stability analysis in which the growth factor of a typical Fourier mode is defined as follows:

$$u_i^n = u_m^n = \gamma^n e^{i\beta m h} \quad (16)$$

where $i = \sqrt{-1}$. von Neumann stability analysis is used to analyze the stability of finite difference schemes applied to linear partial differential equations. So we will investigate the stability of linear form of the schemes. By substituting the (16) equality into the (15) linear form of the schemes, we get the growth factor as follows:

$$\gamma = 1 - 4rv \sin^2 \frac{\beta h}{2} - i \left(\frac{k}{h} (\tilde{U})^2 \sin \beta h \right).$$

Stability condition in von-Neumann method is $|\gamma| \leq 1$. Therefore stability condition of the schemes is

$$\left(4rv \sin^2 \frac{\beta h}{2} \right)^2 - 8rv \sin^2 \frac{\beta h}{2} + \left(\frac{k}{h} (\tilde{U})^2 \sin \beta h \right)^2 \leq 0.$$

5 Rate and Order of Convergence

5.1 Rate of Convergence

The accuracy of the method is measured in terms of the error norm defined by

$$E = \left[\frac{\sum_{i=0}^N |U_i - u_i|^2}{\sum_{i=0}^N |U_i|^2} \right]^{\frac{1}{2}} \quad (17)$$

in which U and u indicate exact and computed numerical solutions, respectively. The rates of convergence of the method, computed by using

$$rate = \frac{\log(E^h/E^{\frac{h}{2}})}{\log(2)}$$

where E^h and $E^{\frac{h}{2}}$ are the errors defined in eq. (17) with the grid size h and $\frac{h}{2}$, respectively.

5.2 Order of Convergence

Definition 1.[23] A sequence $\{x^{(k)}\}$ generated by a numerical method is said to converge to α with order $p \geq 1$ if,

$$\exists C > 0 : \frac{|x^{(k+1)} - \alpha|}{|x^{(k)} - \alpha|^p} \leq C, \quad \forall k_0 \leq k,$$

where $k_0 \geq 0$ is a suitable integer. In such a case, the method is said to be of order p .

To estimate p , we suppose $\varepsilon_{k+1} = |x^{(k+1)} - \alpha|$, $\varepsilon_k = |x^{(k)} - \alpha|$, and so, the previous inequality will be $\varepsilon_{k+1} \leq C\varepsilon_k^p$ [14]. Therefore:

$$p \approx \frac{\log \left(\frac{\varepsilon_{k+1}}{\varepsilon_k} \right)}{\log \left(\frac{\varepsilon_k}{\varepsilon_{k-1}} \right)}.$$

6 Numerical Results and Discussion

Numerical solutions of test problem have been obtained by E-LFDSs. To show the accuracy of the results, L_2 and L_∞ error norms:

$$L_2 = \|u - U_N\|_2 = \sqrt{h \sum_{j=0}^N |u_j - (U_N)_j|^2},$$

$$L_\infty = \|u - U_N\|_\infty = \max_j |u_j - (U_N)_j|$$

have been used. Numerical solutions have been obtained at different times for different values of ν , h , k and $c_0 = 0.5$. The obtained results have been shown in Tables 1-5 and Figure 1-2. Table 1 presents L_2 and L_∞ error norms with $\nu = 0.001$, $h = 0.0125$ and $k = 0.01$ at different times. As it is seen from the table, L_2 and L_∞ error norms are small enough. L_2 and L_∞ error norms of E-LFDSs at $t_f = 2$ for $\nu = 0.01$ and $k = 0.001$ for different values of h are given in Table 2. It can be seen from Table 2 that the values of L_2 and L_∞ decrease with decrease of h . The obtained L_2 and L_∞ error norms of present study are compared with other methods [15, 24] for $\nu = 0.005$, $h = 0.005$ and $k = 0.001$ at times $t = 2, 6, 10$ in Table 3. While Table 4 compares the L_2 and L_∞ error norms of the present study with other methods [15, 25, 27] for $\nu = 0.01$, $h = 0.02$ and $k = 0.01$ at times $t = 2, 6, 10$. Table 5 compares the L_2 and L_∞ error norms of the present study with other methods [15, 24, 25] for $\nu = 0.001$, $h = 0.005$ and $k = 0.01$ at times $t = 2, 6, 10$. The Tables 3-5 clearly show that both of the L_2 and L_∞ error norms are better or as good as the others found in the literature. Rate of convergence for $\nu = 0.01$, $k = 0.01$ and h changes as N changes at $t = 6$ is shown in Table 6. From the table, we observe that the present schemes are second order accurate in space. From this table, it can be seen that errors approach to zero as h decreases, which shows that the schemes are consistent. Rate of convergence for $\nu = 0.001$, $k = 0.01$ and k changes as K changes at $t = 6$ is shown in Table 7. From the table, we observe that the present schemes are first order accurate in time. From this table, it can be seen that errors approach to zero as h decreases, which shows that the schemes are consistent. Figures 1 and 2 show behavior of the numerical solutions for $\nu = 0.01$ and $\nu = 0.005$ with $h = 0.05$, $k = 0.01$ at times $t = 1, 2, 4, 6, 8, 10$ for E-LFDS-I. The top and bottom curves are at $t = 1$ and $t = 10$, respectively. It can be seen from the figures that the curve of the numerical solution decays as the time increases. Note that as the viscosity parameter ν gets small the decay gets fast.

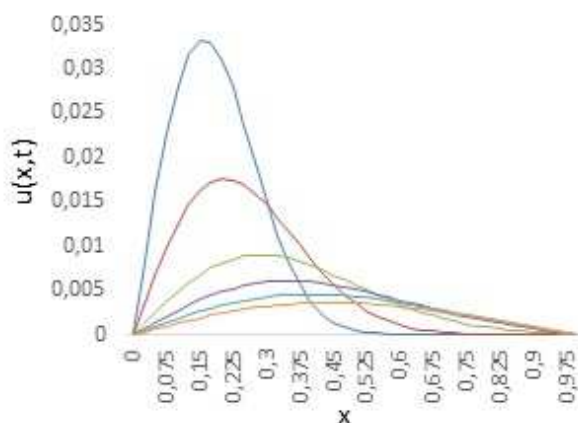


Fig. 1: Numerical solutions for $\nu = 0.01$ with $h = 0.05$, $k = 0.01$ at times $t = 1, 2, 4, 6, 8, 10$ for E-LFDS-I.

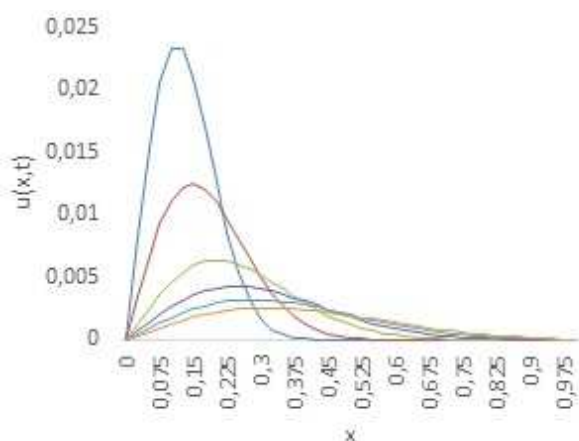


Fig. 2: Numerical solutions for $\nu = 0.005$ with $h = 0.05$, $k = 0.01$ at times $t = 1, 2, 4, 6, 8, 10$ for E-LFDS-I.

7 Conclusions

Explicit logarithmic finite difference schemes obtained by using various finite difference approaches instead of the non-linear term $u^2 \frac{\partial u}{\partial x}$ of the modified Burgers equation have been given for numerical solutions of the modified Burgers equation. A test problem was used to verify the effectiveness and accuracy of the schemes that we have suggested. Comparisons have been made with other studies in the literature. L_2 and L_∞ error norms have been calculated. The obtained error norms are quite small. The results clearly show that the present schemes are accurate, reliable and convenient alternative schemes. Therefore, the present schemes can be recommended to obtain numerical solutions of many other nonlinear equations commonly used in the literature.

Table 1: $L_2 \times 10^3$ and $L_\infty \times 10^3$ error norms with $\nu = 0.001$, $h = 0.0125$ and $k = 0.01$.

t	$L_2 \times 10^3$				$L_\infty \times 10^3$			
	E-LFDS-I	E-LFDS-II	E-LFDS-III	E-LFDS-IV	E-LFDS-I	E-LFDS-II	E-LFDS-III	E-LFDS-IV
2	0.07162	0.07164	0.07159	0.07162	0.26391	0.26429	0.26350	0.26391
3	0.06347	0.06353	0.06340	0.06347	0.23107	0.23136	0.23074	0.23106
4	0.05725	0.05732	0.05718	0.05725	0.19484	0.19508	0.19456	0.19482
5	0.05289	0.05296	0.05281	0.05289	0.16843	0.16864	0.16819	0.16842
6	0.04954	0.04960	0.04946	0.04953	0.14886	0.14904	0.14866	0.14885
7	0.04679	0.04685	0.04672	0.04678	0.13365	0.13381	0.13348	0.13364
8	0.04445	0.04451	0.04439	0.04445	0.12132	0.12145	0.12117	0.12131
9	0.04241	0.04247	0.04235	0.04241	0.11100	0.11112	0.11087	0.11099
10	0.04061	0.04066	0.04055	0.04060	0.10281	0.10293	0.10268	0.10280

Table 2: $L_2 \times 10^3$ and $L_\infty \times 10^3$ error norms with $\nu = 0.01$, $k = 0.001$ and $t_f = 2$.

h	$L_2 \times 10^3$				$L_\infty \times 10^3$			
	E-LFDS-I	E-LFDS-II	E-LFDS-III	E-LFDS-IV	E-LFDS-I	E-LFDS-II	E-LFDS-III	E-LFDS-IV
$h = 0.05$	0.43131	0.43161	0.43102	0.43158	0.87507	0.87895	0.87057	0.87502
$h = 0.025$	0.39056	0.39088	0.39023	0.39062	0.82809	0.83051	0.82555	0.82809
$h = 0.0125$	0.38123	0.38142	0.38104	0.38124	0.81787	0.81911	0.81659	0.81787
$h = 0.01$	0.38014	0.38029	0.37998	0.38014	0.81788	0.81886	0.81688	0.81788

Table 3: Comparison of the error norms L_2 and L_∞ with those in other studies in the literature at $t = 2, 6, 10$ for $h = 0.005$, $k = 0.001$ and $\nu = 0.005$.

	$t = 2$		$t = 6$		$t = 10$	
	$L_2 \times 10^3$	$L_\infty \times 10^3$	$L_2 \times 10^3$	$L_\infty \times 10^3$	$L_2 \times 10^3$	$L_\infty \times 10^3$
E-LFDS-I	0.22642	0.57989	0.16461	0.32987	0.13960	0.22886
E-LFDS-II	0.22647	0.58023	0.16470	0.33004	0.13968	0.22897
E-LFDS-III	0.22638	0.57954	0.16451	0.32970	0.13953	0.22875
E-LFDS-IV	0.22643	0.57989	0.16461	0.32987	0.13960	0.22886
[24]	0.25786	0.72264	0.22569	0.43082	0.18735	0.30006
[15] (SBCM1)	0.22890	0.58623	-	-	0.14042	0.23019
[15] (SBCM2)	0.23397	0.58424	-	-	0.13747	0.22626

Table 4: Comparison of the error norms L_2 and L_∞ with those in other studies in the literature at $t = 2, 6, 10$ for $h = 0.02$, $k = 0.01$ and $\nu = 0.01$.

	$t = 2$		$t = 6$		$t = 10$	
	$L_2 \times 10^3$	$L_\infty \times 10^3$	$L_2 \times 10^3$	$L_\infty \times 10^3$	$L_2 \times 10^3$	$L_\infty \times 10^3$
E-LFDS-I	0.37553	0.80874	0.32924	0.52579	0.55848	1.28125
E-LFDS-II	0.37585	0.81075	0.32980	0.52579	0.55877	1.28125
E-LFDS-III	0.37519	0.80665	0.32865	0.52579	0.55818	1.28125
E-LFDS-IV	0.37556	0.80874	0.32922	0.52579	0.55847	1.28125
[25]	0.79043	1.70309	0.51672	0.76105	0.80026	1.80239
[27] (QBCA1)	0.37911	0.81254	0.32941	0.52579	0.55848	1.28125
[27] (QBCA2)	0.39473	0.88383	0.31588	0.53910	0.52425	1.28125
[15] (SBCM1)	0.38474	0.82611	-	-	0.55985	1.28127
[15] (SBCM2)	0.41321	0.81502	-	-	0.55095	1.28127

Table 5: Comparison of the error norms L_2 and L_∞ with those in other studies in the literature at $t = 2, 6, 10$ for $h = 0.005$, $k = 0.01$ and $\nu = 0.001$.

	$t = 2$		$t = 6$		$t = 10$	
	$L_2 \times 10^3$	$L_\infty \times 10^3$	$L_2 \times 10^3$	$L_\infty \times 10^3$	$L_2 \times 10^3$	$L_\infty \times 10^3$
E-LFDS-I	0.06695	0.25831	0.04939	0.14774	0.04070	0.10258
E-LFDS-II	0.06697	0.25846	0.04942	0.14782	0.04072	0.10263
E-LFDS-III	0.06693	0.25815	0.04936	0.14766	0.04068	0.10253
E-LFDS-IV	0.06695	0.25831	0.04939	0.14774	0.04070	0.10258
[24]	0.06703	0.27967	0.06046	0.17176	0.05010	0.12129
[25]	0.18355	0.81862	0.08142	0.21348	0.05512	0.13943
[15] (SBCM1)	0.06843	0.26233	-	-	0.04080	0.10295
[15] (SBCM2)	0.07220	0.25975	-	-	0.03871	0.09882

Table 6: Rate of convergence for $\nu = 0.01$, $k = 0.01$ and $t = 6$.

N	E-LFDS-I		E-LFDS-II		E-LFDS-III		E-LFDS-IV	
	Rate	Order	Rate	Order	Rate	Order	Rate	Order
4	0.32010	-	0.31999	-	0.32020	-	0.32006	-
8	0.21166	1.51233	0.21145	1.51331	0.21207	1.50988	0.21174	1.51157
16	0.11234	1.88410	0.11217	1.88509	0.11255	1.88423	0.11236	1.88448
32	0.05699	1.97122	0.05693	1.97031	0.05705	1.97283	0.05699	1.97157
64	0.02864	1.98987	0.02863	1.98847	0.02866	1.99058	0.02865	1.98918

Table 7: Rate of convergence for $\nu = 0.001$, $h = 0.02$ and $t = 6$.

K	E-LFDS-I		E-LFDS-II		E-LFDS-III		E-LFDS-IV	
	Rate	Order	Rate	Order	Rate	Order	Rate	Order
128	0.03899	-	0.03896	-	0.03902	-	0.03899	-
256	0.03892	1.00180	0.03890	1.00154	0.03896	1.00154	0.03893	1.00154
512	0.03889	1.00077	0.03886	1.00103	0.03892	1.00103	0.03889	1.00103
1024	0.03886	1.00077	0.03884	1.00051	0.03889	1.00077	0.03887	1.00052
2048	0.03885	1.00026	0.03883	1.00026	0.03888	1.00026	0.03886	1.00026

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