Numerical Solution of Linear System of Integro-Differential Equations by using Chebyshev Wavelet Method

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Abstract: In this paper, a numerical method for solving linear system of integro-differential equations has been proposed. The method is based on Chebyshev wavelets approximations. Illustrative examples have been discussed to demonstrate the validity and applicability of the technique and the results have been compared with the exact solution. It is shown that the numerical results are in good agreement with the exact solutions for each problem.

Keywords: system of integro-differential equations, Chebyshev wavelets, operational matrix integration

1 Introduction

The concept of a system of integro-differential equations has motivated a huge amount of research work in recent years. These equations have been found to describe various kind of phenomena such as wind ripple in the desert, nono-hydrodynamics, dropwise consideration and glass-forming process [1,2,3,4]. Several numerical methods have been used, such as the rationalized Haar functions method [5], Galerkin methods with hybrid functions [6], the tau method [7], the differential transform method [8], Runge-Kutta methods [9], the spline approximation method [10], the block pulse functions method [11], the spectral method [12], the finite difference approximation method [13], new homotopy perturbation method [14] and etc. Chebyshev wavelets have been used by many authors for solving various functional. The main idea of using Chebyshev basis is that the problem under study reduces to a system of linear or nonlinear algebraic equations. This can be done by truncated series of orthogonal basis functions for the solution of problem and using the operational matrices. An extension of Chebyshev wavelets method for solving linear systems of integro-differential equations is the novelty of this paper.

2 Chebyshev wavelets preliminaries

Wavelets constitute a family of functions constructed from dilation and translation of a single function called the mother wavelet [15,16,17]. When the dilation parameter, a and the translation parameter,b, vary continuously we have the following family of continuous wavelets as

$$\psi_{a,b}(x) = |a|^{-\frac{1}{2}} \psi(\frac{x-b}{a}), \quad a, b \in \mathbb{R}, a \neq 0.$$ (1)

we take dilation and translation parameters $a^{-k}$ and $nba^{-k}$ respectively where $a > 1, b > 0, n$ and $k$ positive integers, then we have the following family of discrete wavelets

$$\psi_{k,n}(x) = |a|^k \psi(a^k x - nb).$$ (2)

These functions are a wavelet basis for $L^2(\mathbb{R})$ and in special case $a = 2$, and $b = 1$, the functions $\psi_{k,n}(x)$ are an orthonormal basis.

Chebyshev wavelets $\psi_{n,m}(x) = \psi(k,n,m,x)$ have four arguments, $n = 1,2,\ldots,2^k-1$, $k$ is an arbitrary positive integer and $m$ is the order of Chebyshev polynomials of the first kind. They are defined on the interval $[0,1]$, as follows:

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The integration of the product of two Chebyshev wavelets vector functions with respect to the weight function \( W_n(x) \), is derived as
\[
\int_0^1 W_n(x) \psi(x) \psi^T(x) dx = I
\]
where \( I \) is an identity matrix.

A function \( f(x,y) \) defined on \([0,1] \times [0,1]\) can be approximated as the following
\[
f(x,y) \approx \psi^T(x) K \psi(y)
\]
Here the entries of matrix \( K = [k_{ij}]_{2^{k-1}M \times 2^{k-1}M} \) will be obtained by
\[
k_{ij} = \langle \psi_i(x), (f(x,y), \psi_j(y)) W_n(x) \rangle W_n(x), \quad i, j = 1, 2, ..., 2^{k-1}M.
\]
The integration of the vector \( \psi(x) \), defined in (10) can be achieved as
\[
\int_0^1 \psi(t) dt = P \psi(x)
\]
where \( P \) is the \( 2^{k-1}M \) operational matrix of integration [18, 19]. This matrix is determined as follows
\[
P = \frac{1}{2^k} \begin{bmatrix} L & F & F & \cdots & F \\ O & L & F & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \cdots & F \\ O & \cdots & O & L & \end{bmatrix}
\]
Where \( L, F \) and \( O \) are \( M \times M \) matrices given by
\[
L = \begin{bmatrix} \frac{1}{2} & \frac{1}{2^2} & 0 & 0 & \cdots & 0 \\ \frac{1}{2^2} & \frac{1}{2^2} & \frac{1}{2} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \cdots & \frac{1}{2} & 0 \\ \frac{1}{2^2} & \frac{1}{2^2} & \frac{1}{2} & \cdots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & \frac{1}{2} & \frac{1}{2^2} \end{bmatrix}
\]
\[
F = \begin{bmatrix} 2 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{2^2} & \frac{1}{2^2} & \cdots & \frac{1}{2^2} \\ \frac{1}{2^2} & \frac{1}{2^2} & \cdots & \frac{1}{2^2} \end{bmatrix}
\]
\[
O = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}
\]
The property of the product of two Chebyshev wavelets vector functions will be as follows

\[ \psi(x) \psi^T(x) Y \approx Y \psi(x) \]  (19)

where \( Y \) is a given vector and \( Y \) is a \( 2^{k-1}M \times 2^{k-1}M \) matrix. This matrix is called the operational matrix of product.

3 Chebyshev wavelets applied to systems of integro-differential equations

A system of integro-differential can be considered in general as follows:

\[ \frac{d\mathbf{x}(t)}{dt} = \mathbf{F}(t, \mathbf{x}(t)) + \int_0^t \mathbf{K}(t, s, \mathbf{x}(s)) ds, \]  (20)

where

\[ \mathbf{x}(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T, \mathbf{K}(t, s, \mathbf{x}(t)) = (k_1(t, s, \mathbf{x}(s)), k_2(t, s, \mathbf{x}(s)), \ldots, k_n(t, s, \mathbf{x}(s)))^T. \]

If \( \mathbf{K}(t, s, \mathbf{x}(s)) \) and \( \mathbf{F}(t, \mathbf{x}(t)) \) be linear, the system (20) can be represented as the following simple form

\[ \frac{d\mathbf{x}(t)}{dt} = f_i(t) + \sum_{j=1}^m w_{ij}(t)x_j(t) + \int_0^t k_{ij}(t, s)x_j(s) ds, \]

\[ x_i(0) = \alpha, \quad i = 1, 2, \ldots, n, \quad m = 1, 2, \ldots \]  (21)

For solving system (21), by Chebyshev wavelets method, consider the following approximations:

\[ \frac{d\mathbf{x}_i(t)}{dt} = C_i^T \psi(t), \quad i = 1, 2, \ldots, n \]  (22)

Where \( C_i, i = 1, 2, \ldots, n \) are \( 2^{k-1}M \times 1 \) matrices given by

\[ C = [c_{1,0}, c_{1,1}, \ldots, c_{1,2^{k-1}M-1}, c_{2,0}, c_{2,1}, \ldots, c_{2,2^{k-1}M-1}, \ldots, c_{2^{k-1}M-1,0}, c_{2^{k-1}M-1,1}, \ldots, c_{2^{k-1}M-1,2^{k-1}M-1}]^T \]  (23)

and \( \psi(t) \) is defined in (10). Also consider the following approximations

\[ x_i(t) \simeq C_i^T P \psi(t) + D_i^T \psi(t), \]

\[ f_i(t) \simeq C_i^T \psi(t), \]

\[ w_{ij}(t)x_j(t) \simeq Y_{ij}^T \psi(t), \]

\[ k_{ij}(t, s) \simeq \psi^T(t)K_{ij}\psi(s). \]  (24)

where \( K_{ij} \) are the \( 2^{k-1}M \times 2^{k-1}M \) matrices, \( F_i \) are the \( 2^{k-1}M \times 1 \) matrices, and \( Y_{ij} \) are column vectors with the entries of the vectors \( C_i \) for \( i = 1, 2, \ldots, n, j = 1, 2, \ldots, m \). Substitution of approximations (22) and (24) into the system (21), will be resulted to:

\[ C_i^T \psi(t) \simeq F_i^T \psi(t) + \sum_{j=1}^m Y_{ij}^T \psi(t) + \sum_{j=1}^m \int_0^t \psi^T(s)K_{ij}\psi(s)(C_i^T P \psi(s) + D_i^T \psi(s)) ds = \]

\[ F_i^T \psi(t) + \sum_{j=1}^m Y_{ij}^T \psi(t) + \sum_{j=1}^m \int_0^t \psi^T(s)(C_i^T P + D_i^T) \psi(s) ds = \]

\[ F_i^T \psi(t) + \sum_{j=1}^m Y_{ij}^T \psi(t) + \sum_{j=1}^m \psi^T(t)K_{ij}\tilde{Y}_j \psi(t), \]

\[ i = 1, 2, \ldots, n, \quad m = 1, 2, \ldots \]  (25)

where \( \tilde{Y}_j \) are \( 2^{k-1}M \times 1 \) matrices and \( P \) is the \( 2^{k-1}M \times 2^{k-1}M \) operational matrix of integration.

According to the Galerkin method by multiplying \( w_n(t) \psi^T(t), \) in both sides of the system (25) and then applying \( \int_0^t (\cdot) dt, \) linear system in terms of the entries of \( C_i, i = 1, 2, \ldots, n, \) will be obtained. The elements of vector functions \( C_i, i = 1, 2, \ldots, n, \) can be computed by solving this system.

4 Examples

In this section we present two examples. These examples are considered to illustrate the Chebyshev wavelets approach for systems of integro-differential equations.

Example 1. [14]: Consider the following system of integro-differential equations with the exact solutions \( x_1(t) = t^e \) and \( x_2(t) = e^{-t}, \)

\[ \frac{dx_1(t)}{dt} = r(t) - 2t^3 - 2t^2 - 6 + (3t^2 - 6t + 7)x_1(t) + \]

\[ 2r^2(t + 1)x_2(t) + \int_0^t ((s^3 - t^3)x_1(s) + r^2(s^2 - t^2)x_2(s)) ds, \]

\[ x_1(0) = 1, \]

\[ \frac{dx_2(t)}{dt} = -r^4 - 3t^2 + 2(2t - 1)x_1(t) + \]

\[ (2r^4 + 2r^3 + 2t^2 - 1)x_2(t) + \]

\[ \int_0^t ((t^2 - s^2)x_1(s) + r^2(s^2 + t^2)x_2(s)) ds, \]

\[ x_2(0) = 1. \]  (26)
Lets
\[
\frac{dx_1(t)}{dt} \simeq C_1^T \psi(t),
\]
\[
x_1(t) \simeq C_1^T P \psi(t) + x_1(0) \simeq C_1^T P \psi(t) + D_1^T \psi(t),
\]
\[
\frac{dx_2(t)}{dt} \simeq C_2^T \psi(t),
\]
\[
x_2(t) \simeq C_2^T P \psi(t) + x_2(0) \simeq C_2^T P \psi(t) + D_2^T \psi(t),
\]
\[t^4 - t^3 - 2t^2 - 6 \simeq E_1^T \psi(t),
\]
\[(3t^2 - 6t + 7)x_1(t) \simeq Y_1^T \psi(t),
\]
\[2t^2(t + 1)x_2(t) \simeq Y_2^T \psi(t),
\]
\[(2t^4 + 2t^3 + 2t^2 - 1)x_2(t) \simeq Y_4^T \psi(t),
\]
\[(s^3 - t^3) \simeq \psi^T(t)K_1 \psi(s),
\]
\[(s^2 - t^2) \simeq \psi^T(t)K_2 \psi(s),
\]
\[(s^3 - t^2) \simeq \psi^T(t)K_3 \psi(s),
\]
\[\psi^T(t)K_4 \psi(s)(27)\]

Substitution into the system (26), leads to the following system
\[
C_1^T \psi(t) = F_1^T \psi(t) + Y_2^T \psi(t) +
\]
\[
\psi^T(t)K_1 \int_0^t \psi(s)(C_1^T P + D_1^T) \psi(s) ds +
\]
\[
\psi^T(t)K_2 \int_0^t \psi(s)(C_1^T P + D_1^T) \psi(s) ds =
\]
\[
F_1^T \psi(t) + Y_1^T \psi(t) + Y_2^T \psi(t) + \psi^T(t)K_1 Y_1 P \psi(t) +
\]
\[
\psi^T(t)K_2 Y_2 P \psi(t),
\]
\[
C_2^T \psi(t) = F_2^T \psi(t) + Y_4^T \psi(t) +
\]
\[
\psi^T(t)K_3 \int_0^t \psi(s)(C_2^T P + D_2^T) \psi(s) ds +
\]
\[
\psi^T(t)K_4 \int_0^t \psi(s)(C_2^T P + D_2^T) \psi(s) ds =
\]
\[
F_2^T \psi(t) + Y_1^T \psi(t) + Y_4^T \psi(t) + \psi^T(t)K_3 Y_1 P \psi(t) +
\]
\[
\psi^T(t)K_4 Y_4 P \psi(t),
\]
\[\text{(28)}\]

Multiply \(W_n(t) \psi^T(t)\), on both sides of the system(28), applying \(\int_0^t(\cdot) dt\), and then solve the system.

The elements of vector functions \(C_1\) and \(C_2\) can be obtained as follows
\[
C_1 = [2.197072263, 0.7532217714, 0.0931434913, 0.007745016935, 0.00050077305657, 0.00003345920292, 0.000000500023009867, 0.0000017896575785]^T,
\]
\[
C_2 = [-0.0808321301, 0.2774592787, -0.03418569156, 0.002900083585, -0.0001498235218, 0.0000000299796862, 0.0000004561044304, 0.000001798662095]^T.
\]

Therefore, the following solutions will result.
\[
x_1(t) \simeq C_1^T P \psi(t) + x_1(0) \simeq C_1^T P \psi(t) + D_1^T \psi(t) = 0.001651417113 x^3 - 0.002730551472 x^6 + 0.01312030509 x^3 + 0.03868281601 x^4 + 0.1674220136 x^3 + 0.4998030468 x^2 + 0.9999898624 x + 0.9999999367,
\]
\[
x_2(t) \simeq C_2^T P \psi(t) + x_2(0) \simeq C_2^T P \psi(t) + D_2^T \psi(t) = 0.001505744420 x^2 - 0.003421435813 x^6 - 0.002759244740 x^6 + 0.03843721505 x^4 - 0.1656548697 x^5 + 0.4998700759 x^6 - 0.999841587 x + 0.9999999364.
\]

Table 1 shows some values of the solutions and absolute errors at some and plots of the exact and approximate solutions are shown in Figure 1.
Example 2. [14]: Consider the following system of integro-differential equations with the exact solutions $x_1(t) = \cosh(t)$ and $x_2(t) = \sinh(t)$.

\[
\frac{dx_1(t)}{dt} = -t^3 - 6t + 1 + x_1(t) + (7 - 2t)x^2(t) + \int_0^t (s+t)x_1(s) + (s-t)x_2(s) ds, \quad x_1(0) = 1,
\]

\[
\frac{dx_2(t)}{dt} = -3t^2 + t - 6 + (7 - 2t)x_1(t) + x_2(t) + \int_0^t (s-t)^3 x_1(s) + (s-t)x_2(s) ds, \quad x_2(0) = 0. \quad (29)
\]

By applying the Chebyshev wavelets method and solving the resulted linear system, the following results would be achieved.

\[
C_1 = \begin{bmatrix}
0.6913440397 & 0.5116838793 & 0.0272796418 \\
0.004340530682 & -0.0001604934677 & -0.00006543048876 \\
-0.000016705880723 & -0.000003004951770
\end{bmatrix}^T,
\]

\[
C_2 = \begin{bmatrix}
1.499774940 & 0.2344433640 & 0.06157643030 \\
0.001498845797 & 0.0000161905546 & -0.00007406505452 \\
-0.000016397378437 & -0.000003021159578
\end{bmatrix}^T.
\]

Therefore, we have the following approximate solutions

\[
x_1(t) = -0.005515113281x^2 + 0.01329204418x^6 - 0.01266244896x^3 + 0.04796640532x^4 - 0.001878448025x^3 + 0.5001752830x^2 - 0.00001296863644x + 0.00000016705880723
\]

\[
x_2(t) = -0.005413291216x^2 + 0.01210581944x^6 - 0.00459296698x^5 + 0.006532724767x^4 + 0.1646412918x^3 + 0.0002430555256x^2 + 0.9999694387x + 1.065316727x \times 10^{-7}.
\]

Some values of exact, approximate solutions and absolute errors are presented in Table 2 and the plots of exact and approximate solutions are shown in Figure 2.

5 Conclusion

In this research, we have presented the Chebyshev wavelet method for solving system of integro-differential equations. The Chebyshev operational matrix of integration is used to solve these systems. The present method reduces system of integro-differential equations into a set of algebraic equations. Illustrative examples have been discussed to demonstrate the validity and applicability of the technique and the results have been compared with the exact solution.

References


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