Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.18576/amis/110503

On the Properties of q-Bernstein-type Polynomials

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Received: 7 Jun. 2016, Revised: 21 Sep. 2016, Accepted: 23 Sep. 2016
Published online: 1 Sep. 2017

Abstract: The aim of this paper is to give a new approach to modified q-Bernstein polynomials for functions depend on the several variables. We derive the recurrence formulas related to the second Stirling numbers and generalized Bernoulli polynomials. Moreover, the interpolation function of these polynomials depend on the several variables and the derivatives of these polynomials and also their generating function are given. Final part of this paper, we get new interesting identities of modified q-Bernoulli numbers and q-Euler numbers applying p-adic q-integral representation on $\mathbb{Z}_p$ and p-adic fermionic q-invariant integral on $\mathbb{Z}_p$, respectively, to the inverse of q-Bernstein polynomials.

Keywords: p-adic q-integral on $\mathbb{Z}_p$; Generating function; Bernstein polynomial of several variables; Shift difference operator; Stirling numbers of the second kind; Bernoulli polynomials of higher order; Mellin transformation.

1 Introduction

The Bernstein polynomials, named after their creater S. N. Bernstein in 1912, have been studied by many researchers for a long time. Recently Acikgoz and Araci have originally defined the generating function of Bernstein polynomials and analysed their interesting properties arising from that generating function, and also the generating function of Bernstein polynomials in two dimensional are defined by the same authors (see [1], [2], [3]). Next, Simsek and Acikgoz have constructed a generating function of (q-) Bernstein type polynomials based on the q-analysis, [40], and gave some new relations related to these polynomials, Hermite polynomials, Bernoulli polynomials of higher order and the second kind Stirling numbers. Interpolation function of (q-) Bernstein type polynomials is defined by applying Mellin transformation to this generating function. In [20], Kim-Choi-Kim have studied on the k-dimensional generalization of q-Bernstein polynomials, in which they have given some interesting properties of the k-dimensional generalization of q-Bernstein polynomials (see[20]). Our generalization of q-Bernstein polynomials are different from the k-dimensional generalization of q-Bernstein polynomials of Kim-Choi-Kim. In the present paper, we also derived some interesting properties of our generalization of q-Bernstein polynomials. Recent works including integral representations and properties of Stirling numbers of the first kind [11], formulae for the q-Bernstein polynomials and q-deformed binomial distributions [16], integral representations for the Gamma function, the Beta Function, and the double Gamma function [27], irregular prime power divisors of the Bernoulli numbers [32], application of a composition of generating functions for obtaining explicit formulas of polynomials [33], hyperharmonic series involving Hurwitz zeta function [34], p-adic q-deformed fermionic integrals in the p-adic integer ring [8] have been investigated extensively.

We are now in a position to give some definitions and some properties of Bernstein polynomials of several variables with their generating function.

Let $C(\mathscr{D}^w)$ denotes the set of continuous functions on $\mathscr{D}^w$, in which $\mathscr{D}^w$ and $\mathscr{D}$ mean $\underbrace{\mathscr{D}\times\mathscr{D}\times\ldots\times\mathscr{D}}_{w-\text{times}}$ and $[0,1]$,

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respectively. For $f \in C(\mathscr{D}^w)$, we have

$$\mathscr{B}_{n_1,n_2,\cdots,n_w}(f;x_1,x_2,\cdots,x_w) := \sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}\cdots\sum_{k_w=0}^{n_w} f\left(\frac{k_1}{n_1},\frac{k_2}{n_2},\cdots,\frac{k_w}{n_w}\right) \times B_{k_1,k_2,\cdots,k_w;n_1,n_2,\cdots,n_w}(x_1,x_2,\cdots,x_w)$$

where $\mathscr{B}_{n_1,n_2,\cdots,n_w}(f;x_1,x_2,\cdots,x_w)$ is called the Bernstein operator of several variables of order $\sum_{i=1}^{w} n_i$ for f. For $k_i, n_i \in \mathbb{N}_0$ with $i = 1,2,\cdots,w$, the Bernstein polynomials of several variables of degree $\sum_{i=1}^{w} n_i$ is defined by

$$B_{k_1,k_2,\cdots,k_w;n_1,n_2,\cdots,n_w}(x_1,x_2,\cdots,x_w) = \prod_{i=1}^{w}\left(\binom{n_i}{k_i} x_i^{k_i}(1-x_i)^{n_i-k_i}\right), \quad (1)$$

where $\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!}$ and $x_i \in \mathscr{D}$ for $i = 1,2,\ldots,w$. These polynomials satisfy the following relation

$$B_{k_1,k_2,\cdots,k_w;n_1,n_2,\cdots,n_w}(x_1,x_2,\cdots,x_w) = \prod_{i=1}^{w} B_{k_i,n_i}(x_i)$$

and they have form a partition of unity; that is:

$$\sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}\cdots\sum_{k_w=0}^{n_w} B_{k_1,k_2,\cdots,k_w;n_1,n_2,\cdots,n_w}(x_1,x_2,\cdots,x_w) = 1.$$

By using the definition of Bernstein polynomials for functions of several variables, it is not difficult to prove the property given above as

$$\sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}\cdots\sum_{k_w=0}^{n_w}\prod_{i=1}^{w} B_{k_i,n_i}(x_i) = 1.$$

Also, $B_{k_1,k_2,\cdots,k_w;n_1,n_2,\cdots,n_w}(x_1,x_2,\cdots,x_w) = 0$ for $k_i > n_i$ with $i = 1,2,\ldots,w$, because $\binom{n_i}{k_i} = 0$. There are $\prod_{i=1}^{w}(n_i+1)$, $\sum_{i=1}^{w} n_i$-th degree Bernstein polynomials.

Many researchers have studied the Bernstein polynomials of two variables in approximation theory (see [35], [36]). But nothing was known about the generating function of these polynomials. Note that for $k_i, n_i \in \mathbb{N}_0$ and $x_i \in \mathscr{D}$ with $i = 1,2,\ldots,w$, we obtain the generating function for $B_{k_1,k_2,\cdots,k_w;n_1,n_2,\cdots,n_w}(x_1,x_2,\cdots,x_w)$ as follows:

$$F_{k_1,k_2,\cdots,k_w}(t;x_1,x_2,\cdots,x_w) = \prod_{i=1}^{w}\frac{(tx_i)^{k_i}}{k_i!} e^{wt - t\sum_{i=1}^{w} x_i} \quad (2)$$

$$= \sum_{n_1=k_1}^{\infty}\sum_{n_2=k_2}^{\infty}\times\cdots\times \sum_{n_w=k_w}^{\infty} B_{k_1,k_2,\cdots,k_w;n_1,n_2,\cdots,n_w}(x_1,x_2,\cdots,x_w)\prod_{i=1}^{w}\frac{t^{n_i}}{n_i!}$$

where

$$B_{k_1,k_2,\cdots,k_w;n_1,n_2,\cdots,n_w}(x_1,x_2,\cdots,x_w) = \begin{cases} \prod_{i=1}^{w}\binom{n_i}{k_i} x_i^{k_i}(1-x_i)^{n_i-k_i} & \text{if } n_i \geq k_i, \\ 0 & \text{if } n_i < k_i, \end{cases}$$

for $k_i \in \mathbb{N}_0$ and $x_i \in \mathscr{D}$, for $i = 1,2,\ldots,w$.

*Remark.*By substituting $w = 1$ into (2), we get a special case of $F_{k_1,k_2,\cdots,k_w}(t;x_1,x_2,\cdots,x_w)$ which was proved by Acikgoz and Araci (for details, see [1])

$$F_{k_1}(t,x_1) = \frac{(tx_1)^{k_1} e^t}{k_1! e^{tx_1}} = \sum_{n_1=k_1}^{\infty} B_{k_1,n_1}(x_1)\frac{t^{n_1}}{n_1!}.$$

Let $0 < q < 1$. Define the q-number of x by $[x]_q := \frac{1-q^x}{1-q}$ and $[x]_{-q} := \frac{1-(-q)^x}{1+q}$, (see [4],[5],[6],[19],[20],[17],[21],[31],[38],[39],[40] for details and related facts). Note that $\lim_{q\to 1^-}[x]_q = x$. [19] is actually motivated the authors to write this paper and they have extended all results given in [19] to modified q-Bernstein polynomials of several variables.

2 The Modified q-Bernstein Polynomials for Functions of Several Variables

For $0 < q < 1$, we consider

$$F_{k_1,k_2,\cdots,k_w}(t,q;x_1,x_2,\cdots,x_w) = \prod_{i=1}^{w}\frac{\left(t[x]_q\right)^{k_i}}{k_i!} e^{t\sum_{i=1}^{w}[1-x_i]_q}$$

$$= \sum_{n_1=k_1}^{\infty}\sum_{n_2=k_2}^{\infty}\times\cdots\times \sum_{n_w=k_w}^{\infty} B_{k_1,k_2,\cdots,k_w;n_1,n_2,\cdots,n_w}(x_1,x_2,\cdots,x_w;q)\prod_{i=1}^{w}\frac{t^{n_i}}{n_i!}$$

where $k_i, n_i \in \mathbb{N}_0$ and $x_i \in \mathscr{D}$ for $i = 1,2,\ldots,w$. We note that

$$\lim_{q\to 1^-} F_{k_1,k_2,\cdots,k_w}(t,q;x_1,x_2,\ldots,x_w) = F_{k_1,k_2,\cdots,k_w}(t;x_1,x_2,\ldots,x_w).$$

Definition 1.*We define the generating function of modified q-Bernstein polynomials for functions of several variables as follows:*

$$F_{k_1,k_2,\cdots,k_w}(t,q;x_1,x_2,\cdots,x_w) = \prod_{i=1}^{w}\frac{\left(t[x_i]_q\right)^{k_i}}{k_i!} e^{t\sum_{i=1}^{w}[1-x_i]_q} \quad (3)$$

$$= \sum_{n_1=k_1}^{\infty}\sum_{n_2=k_2}^{\infty}\times\cdots\times \sum_{n_w=k_w}^{\infty} B_{k_1,k_2,\cdots,k_w;n_1,n_2,\cdots,n_w}(x_1,x_2,\cdots,x_w;q)\prod_{i=1}^{w}\frac{t^{n_i}}{n_i!}$$

where $k_i, n_i \in \mathbb{N}_0$ and $x_i \in \mathscr{D}$ with $i = 1,2,\ldots,w$.

By using Taylor expansion of $e^{t\sum_{i=1}^{w}[1-x_i]_q}$ and the comparing coefficients on the both sides in (3), we get the following Corollary.

Corollary 1. *For $k_i, n_i \in \mathbb{N}_0$ and $x_i \in \mathscr{D}$ for $i = 1, 2, ..., w$, we have*

$$B_{k_1,k_2,\cdots,k_w;n_1,n_2,\cdots,n_w}(x_1, x_2, \cdots, x_w; q) = \begin{cases} \prod_{i=1}^{w} \binom{n_i}{k_i} [x_i]_q^{k_i} [1-x_i]_q^{n_i-k_i} & \text{if } n_i \geq k_i, \\ 0 & \text{if } n_i < k_i. \end{cases} \quad (4)$$

Theorem 1. ***Recurrence Formula for*** *$B_{k_1,k_2,\cdots,k_w;n_1,n_2,\cdots,n_w}(x_1, x_2, \cdots, x_w; q)$) For $k_i, n_i \in \mathbb{N}_0$, $x_i \in \mathscr{D}$ and $i = 1, 2, ..., w$, we have*

$$B_{k_1,k_2,\cdots,k_w;n_1,n_2,\cdots,n_w}(x_1, x_2, \cdots, x_w; q) = \prod_{i=1}^{w} \left([1-x_i]_q B_{k_i;n_i-1}(x_i;q) + [x_i]_q B_{k_i-1;n_i-1}(x_i;q)\right). \quad (5)$$

Proof. By using the definition of Bernstein polynomials for functions of several variables, we have

$$\begin{aligned} &B_{k_1,k_2,\cdots,k_w;n_1,n_2,\cdots,n_w}(x_1, x_2, \cdots, x_w; q) \\ &= \prod_{i=1}^{w} \left[\binom{n_i-1}{k_i} + \binom{n_i-1}{k_i-1}\right] [x_i]_q^{k_i} [1-x_i]_q^{n_i-k_i} \\ &= \prod_{i=1}^{w} \left([1-x_i]_q B_{k_i;n_i-1}(x_i;q) + [x_i]_q B_{k_i-1;n_i-1}(x_i;q)\right). \end{aligned}$$

This is the desired result.

Remark. By setting $w = 1$ and $q \to 1^-$ into (6), we get the familiar identity for $B_{k_1,n_1}(x_1)$ as follows:

$$B_{k_1,n_1}(x_1) = (1-x_1) B_{k_1,n_1-1}(x_1) + x_1 B_{k_1-1,n_1-1}(x_1).$$

(see [1],[3],[40]).

Theorem 2. *For $k_i, n_i \in \mathbb{N}_0$ and $x_i \in \mathscr{D}$ with $i = 1, 2, ..., w$, we have*

$$B_{n_1-k_1,n_2-k_2\cdots,n_w-k_w;n_1,n_2,\cdots,n_w}(1-x_1, 1-x_2, \cdots, 1-x_w; q) = B_{k_1,k_2,\cdots,k_w;n_1,n_2,\cdots,n_w}(x_1, x_2, \cdots, x_w; q)$$

Remark. By substituting $w = 1$ and $q \to 1^-$ into (**??**), we get the well-known identity as follows:

$$B_{n_1-k_1,n_1}(1-x_1) = B_{k_1,n_1}(x_1).$$

(see [1],[3]).

Definition 2. *Let f be a continuous function of several variables on $\mathscr{D}^w$. Then the modified q-Bernstein operator of order $\sum_{i=1}^{w} n_i$ for f is defined by*

$$\mathscr{B}_{n_1,n_2,\cdots,n_w}(f : x_1, x_2, \cdots, x_w; q) = \sum_{k_1=0}^{n_1} \sum_{k_2=0}^{n_2} \cdots \sum_{k_w=0}^{n_w} f\left(\frac{k_1}{n_1}, \frac{k_2}{n_2}, \cdots, \frac{k_w}{n_w}\right) B_{k_1,k_2,\cdots,k_w;n_1,n_2,\cdots,n_w}(x_1, x_2, \cdots, x_w; q)$$

where $x_i \in \mathscr{D}$, $n_i \in \mathbb{N}$.

When we set $f\left(\frac{k_1}{n_1}, \frac{k_2}{n_2}, \cdots, \frac{k_w}{n_w}\right) = 1$ into (**??**), we easily see that,

$$\mathscr{B}_{n_1,n_2,\cdots,n_w}(1 : x_1, x_2, \cdots, x_w; q) = \sum_{k_1=0}^{n_1} \sum_{k_2=0}^{n_2} \cdots \sum_{k_w=0}^{n_w} B_{k_1,k_2,\cdots,k_w;n_1,n_2,\cdots,n_w}(x_1, x_2, \cdots, x_w; q) \quad (6)$$

From the definition of binomial theorem and (6), we get the following Corollary 2 for modified q-Bernstein polynomials for functions of several variables:

Corollary 2. *For any $k_i, n_i \in \mathbb{N}_0$ and $x_i \in \mathscr{D}$ with $i = 1, 2, \cdots, w$, we have*

$$\mathscr{B}_{n_1,n_2,\cdots,n_w}(1 : x_1, x_2, \cdots, x_w; q) = \prod_{i=1}^{w} \left(1 + (1-q)[x_i]_q [1-x_i]_q\right)^{n_i}, \quad (7)$$

we easily see that

$$\lim_{q \to 1} \mathscr{B}_{n_1,n_2,\cdots,n_w}(1 : x_1, x_2, \cdots, x_w; q) = 1.$$

This is a partition of unity for modified Bernstein polynomials for functions of several variables.

Theorem 3. *For $\xi_j \in \mathbb{C}$, $x_j \in \mathscr{D}$ and $n_j \in \mathbb{N}$, with $j = 1, 2, \cdots, w$ and $i = \sqrt{-1}$, we have*

$$B_{k_1,k_2,\cdots,k_w;n_1,n_2,\cdots,n_w}(x_1, x_2, \cdots, x_w; q) = \frac{1}{(2\pi i)^w} \underbrace{\oint_C \oint_C \cdots \oint_C}_{w\text{-times}} \prod_{j=1}^{w} n_j! F_q^{(k_j)}\left(x_j, \xi_j\right) \frac{d\xi_j}{\xi_j^{n_j+1}} \quad (8)$$

where

$$F_q^{(k)}(x,t) = \frac{(t[x]_q)^k}{k!} e^{t[1-x]_q} \text{ (see [40])}$$

and C is a circle around the origin and integration is in the positive direction.

*Proof.*By using the definition of the modified q-Bernstein polynomials of several variables and the basic theory of complex analysis including Laurent series that

$$\underbrace{\oint_C \oint_C \cdots \oint_C}_{w\text{-times}} \prod_{j=1}^{w} F_q^{(k_j)}\left(x_j, \xi_j\right) \frac{d\xi_j}{\xi_j^{n_j+1}}$$

$$= \sum_{l_1=0}^{\infty} \sum_{l_2=0}^{\infty} \cdots \sum_{l_w=0}^{\infty} \oint_C \oint_C \cdots \oint_C \prod_{j=1}^{w} \frac{B_{k_j,l_j}(x_j,q)\,\xi_j^{l_j}}{l_j!} \frac{d\xi_j}{\xi_j^{n_j+1}}$$

$$= (2\pi i)^w \left(\frac{B_{k_1,k_2,\cdots,k_w;n_1,n_2,\cdots,n_w}(x_1,x_2,\cdots,x_w;q)}{n_1!n_2!\cdots n_w!} \right). \quad (9)$$

By using (9), we obtain

$$B_{k_1,k_2,\cdots,k_w;n_1,n_2,\cdots,n_w}(x_1,x_2,\cdots,x_w;q)$$
$$= \frac{1}{(2\pi i)^w} \underbrace{\oint_C \oint_C \cdots \oint_C}_{w\text{-times}} \prod_{j=1}^{w} n_j! F_q^{(k_j)}\left(x_j, \xi_j\right) \frac{d\xi_j}{\xi_j^{n_j+1}}$$

and

$$\underbrace{\oint_C \oint_C \cdots \oint_C}_{w\text{-times}} \prod_{j=1}^{w} F_q^{(k_j)}\left(x_j, \xi_j\right) \frac{d\xi_j}{\xi_j^{n_j+1}}$$
$$= (2\pi i)^w \left(\prod_{j=1}^{w} \frac{[x_j]_q^{k_j} [1-x_j]_q^{n_j-k_j}}{k_j!(n_j-k_j)!} \right). \quad (10)$$

We also obtain from (9) and (10) that

$$\frac{1}{(2\pi i)^w} \underbrace{\oint_C \oint_C \cdots \oint_C}_{w\text{-times}} \prod_{j=1}^{w} n_j! F_q^{(k_j)}$$
$$\left(x_j, \xi_j\right) \frac{d\xi_j}{\xi_j^{n_j+1}} = \prod_{j=1}^{w} \binom{n_j}{k_j} [x_j]_q^{k_j} [1-x_j]_q^{n_j-k_j}. \quad (11)$$

So, from (9) and (11) and Corollary 1, we complete the proof of theorem.

We now give the modified q-Bernstein polynomials for functions of several variables as a linear combination of polynomials of higher order as follows:

Theorem 4.*For $k_i, n_i \in \mathbb{N}_0$, $x_i \in \mathscr{D}$, and $i = 1,2,\ldots,w$, we have*

$$B_{k_1,k_2,\cdots,k_w;n_1,n_2,\cdots,n_w}(x_1,x_2,\cdots,x_w;q)$$
$$= \prod_{i=1}^{w} \left[\left(\frac{n_i - k_i + 1}{k_i} \right) \frac{[x_i]_q}{[1-x_i]_q} \right]$$
$$B_{k_1-1,k_2-1,\cdots,k_w-1;n_1,n_2,\cdots,n_w}(x_1,x_2,\cdots,x_w;q).$$

*Proof.*Using the definition of modified q-Bernstein polynomials for functions of several variables and the property (4), the proof follows.

Theorem 5.*If $n_i, k_i \in \mathbb{N}_0$ and $x_i \in \mathscr{D}$ with $i = 1,2,\ldots,w$, we have*

$$B_{k_1,k_2,\cdots,k_w;n_1,n_2,\cdots,n_w}(x_1,x_2,\cdots,x_w;q)$$
$$= \sum_{l_1=k_1}^{n_1} \sum_{l_2=k_2}^{n_2} \cdots \sum_{l_w=k_w}^{n_w} \prod_{i=1}^{w} \binom{n_i}{l_i} \binom{l_i}{k_i} (-1)^{l_i-k_i} q^{(l_i-k_i)(1-x_i)} [x_i]_q^{l_i}.$$

*Proof.*From the definition of modified q-Bernstein polynomials of several variables and binomial theorem with n_i, $k_i \in \mathbb{N}_0$ and $x_i \in \mathscr{D}$ for $i = 1,2,\ldots,w$, we have

$$B_{k_1,k_2,\cdots,k_w;n_1,n_2,\cdots,n_w}(x_1,x_2,\cdots,x_w;q) = \prod_{i=1}^{w} \binom{n_i}{k_i} [x_i]_q^{k_i} [1-x_i]_q^{n_i-k_i}$$
$$= \sum_{l_1=k_1}^{n_1} \sum_{l_2=k_2}^{n_2} \cdots \sum_{l_w=k_w}^{n_w} \prod_{i=1}^{w} \binom{n_i}{l_i} \binom{l_i}{k_i} (-1)^{l_i-k_i} q^{(l_i-k_i)(1-x_i)} [x_i]_q^{l_i}.$$

This is the desired result.

Theorem 6.*For $n_i, l_i \in \mathbb{N}_0$ and $x_i \in \mathscr{D}$, with $i = 1,2,\ldots,w$, we have*

$$\left(\prod_{i=1}^{w} [x_i]_q \right)^m$$
$$= \prod_{i=1}^{w} \frac{1}{\left([1-x_i]_q + [x_i]_q\right)^{n_i-m}}$$
$$\sum_{k_1=m}^{n_1} \sum_{k_2=m}^{n_2} \cdots \sum_{k_w=m}^{n_w} \prod_{i=1}^{w} \frac{\binom{k_i}{m}}{\binom{n_i}{m}} B_{k_1,k_2,\cdots,k_w;n_1,n_2,\cdots,n_w}(x_1,x_2,\cdots,x_w;q).$$

*Proof.*We easily see from the property of the modified q-Bernstein polynomials of several variables that

$$\sum_{k_1=1}^{n_1} \sum_{k_2=1}^{n_2} \cdots \sum_{k_w=1}^{n_w} \prod_{i=1}^{w} \frac{k_i}{n_i} B_{k_1,k_2,\cdots,k_w;n_1,n_2,\cdots,n_w}(x_1,x_2,\cdots,x_w;q)$$
$$= \prod_{i=1}^{w} [x_i]_q \left([x_i]_q + [1-x_i]_q \right)^{n_i-1}$$

and also

$$\sum_{k_1=2}^{n_1} \sum_{k_2=2}^{n_2} \cdots \sum_{k_w=2}^{n_w} \prod_{i=1}^{w} \frac{\binom{k_i}{2}}{\binom{n_i}{2}} B_{k_1,k_2,\cdots,k_w;n_1,n_2,\cdots,n_w}(x_1,x_2,\cdots,x_w;q)$$
$$= \left(\prod_{i=1}^{w} [x_i]_q \right)^2 \left([x_i]_q + [1-x_i]_q \right)^{n_i-2}.$$

Continuing this method, we have

$$\left(\prod_{i=1}^{w} [x_i]_q \right)^m = \prod_{i=1}^{w} \frac{1}{\left([1-x_i]_q + [x_i]_q\right)^{n_i-m}}$$
$$\times \sum_{k_1=m}^{n_1} \sum_{k_2=m}^{n_2} \cdots \sum_{k_w=m}^{n_w} \prod_{i=1}^{w} \frac{\binom{k_i}{m}}{\binom{n_i}{m}} B_{k_1,k_2,\cdots,k_w;n_1,n_2,\cdots,n_w}(x_1,x_2,\cdots,x_w;q)$$

and after making some algebraic operations, we obtain the desired result.

We have seen from the theorem given above, it is possible to write $\left(\prod_{i=1}^{w}[x_i]_q\right)^m$ as a linear combination of modified q-Bernstein polynomials of several variables by using the degree evaluation formulae and mathematical induction method.

For $k \in \mathbb{N}_0$, the Bernoulli polynomials of degree k are defined by

$$\underbrace{\left(\frac{t}{e^t-1}\right)\left(\frac{t}{e^t-1}\right)\times\cdots\times\left(\frac{t}{e^t-1}\right)}_{k-times}e^{xt} = \left(\frac{t}{e^t-1}\right)^k e^{xt}$$

$$= \sum_{n=0}^{\infty} B_n^{(k)}(x)\frac{t^n}{n!},$$

and $B_n^{(k)} = B_n^{(k)}(0)$ are called the n-th Bernoulli numbers of order k. It is well known that the second kind Stirling numbers are defined by $\frac{(e^t-1)^k}{k!} := \sum_{n=0}^{\infty} S(n,k)\frac{t^n}{n!}$ for $k \in \mathbb{N}$ (see [19],[40]). By using the above relations, we can give the following theorem:

Theorem 7. *For $k_i, n_i, \in \mathbb{N}_0$ and $x_i \in \mathscr{D}$ with $i = 1,2,\cdots,w$, we have*

$$B_{k_1,k_2,\cdots,k_w;n_1,n_2,\cdots,n_w}(x_1,x_2,\cdots,x_w;q)$$
$$= \sum_{l_1=0}^{n_1}\sum_{l_2=0}^{n_2}\cdots\sum_{l_w=0}^{n_w}\prod_{i=1}^{w}[x_i]_q^{l_i}\binom{n_i}{l_i}B_{l_i}^{(k_i)}\left([1-x_i]_q\right)S(n_i-l_i,k_i).$$

Proof. By using the generating function of modified q-Bernstein polynomials of several variables, we have

$$\prod_{i=1}^{w}\frac{(t[x_i]_q)^{k_i}}{k_i!}e^{t\left(\sum_{i=1}^{w}[1-x_i]_q\right)} =$$
$$\prod_{i=1}^{w}[x_i]_q^{k_i}\left(\sum_{n_1=0}^{\infty}S(n_1,k_1)\frac{t^{n_1}}{n_1!}\right)\cdots\left(\sum_{n_w=0}^{\infty}S(n_w,k_w)\frac{t^{n_w}}{n_w!}\right)\times$$
$$\left(\sum_{l_1=0}^{\infty}B_{l_1}^{(k_1)}\left([1-x_1]_q\right)\frac{t^{l_1}}{l_1!}\right)\cdots\left(\sum_{l_w=0}^{\infty}B_{l_w}^{(k_w)}\left([1-x_w]_q\right)\frac{t^{l_w}}{l_w!}\right).$$

By using the Cauchy product for sums given above

$$B_{k_1,k_2,\cdots,k_w;n_1,n_2,\cdots,n_w}(x_1,x_2,\cdots,x_w;q)$$
$$= \sum_{l_1=0}^{n_1}\sum_{l_2=0}^{n_2}\cdots\sum_{l_w=0}^{n_w}\prod_{i=1}^{w}[x_i]_q^{l_i}\binom{n_i}{l_i}B_{l_i}^{(k_i)}\left([1-x_i]_q\right)S(n_i-l_i,k_i).$$

By comparing the last two relations, we have the desired result.

Let Δ be the shift difference operator defined by $\Delta f(x) = f(x+1) - f(x)$. By using the mathematical induction method we have

$$\Delta^n f(0) = \sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}f(k), \tag{12}$$

for $n \in \mathbb{N}$ and using (12) in the generating function of second kind Stirling numbers,

$$\sum_{n=0}^{\infty}S(n,k)\frac{t^n}{n!} = \frac{1}{k!}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}e^{lt}$$
$$= \sum_{n=0}^{\infty}\left(\frac{1}{k!}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}l^n\right)\frac{t^n}{n!}. \tag{13}$$

By comparing the coefficients on both sides, we have

$$S(n,k) = \frac{1}{k!}\sum_{l=0}^{k}\binom{k}{l}(-1)^{k-l}l^n. \tag{14}$$

When we compared Eq. (12) and Eq. (14), becomes

$$S(n,k) = \frac{\Delta^k 0^n}{k!}. \tag{15}$$

For $n_i, k_i \in \mathbb{N}$, by using the equation (15), we obtain the relation

$$B_{k_1,k_2,\cdots,k_w;n_1,n_2,\cdots,n_w}(x_1,x_2,\cdots,x_w;q)$$
$$= \sum_{l_1=0}^{n_1}\sum_{l_2=0}^{n_2}\cdots\sum_{l_w=0}^{n_w}\prod_{i=1}^{w}[x_i]_q^{l_i}\binom{n_i}{l_i}B_{l_i}^{(k_i)}\left([1-x_i]_q\right)\frac{\Delta^{k_i}0^{n_i-l_i}}{k_i!}$$

which is the relation of the q-Bernstein polynomials of several variables in terms of Bernoulli polynomials of order k and second Stirling numbers with shift difference operator.

Let $(Eh)(x) = h(x+1)$ be the shift operator. Then the q-difference operator is defined by

$$\Delta_q^n = \prod_{i=0}^{n-1}\left(E - q^i I\right) \tag{16}$$

where I is the identity operator (see [19]).

For $f \in C([0,1])$ and $n \in \mathbb{N}$, we have

$$\Delta_q^n f(0) = \sum_{k=0}^{n}\binom{n}{k}_q(-1)^k q^{\binom{n}{2}}f(n-k), \tag{17}$$

where $\binom{n}{k}_q$ is the Gaussian binomial coefficient defined by

$$\binom{n}{k}_q = \frac{[n]_q[n-1]_q\cdots[n-k+1]_q}{[k]_q!}. \tag{18}$$

Theorem 8. *For $n_i, l_i \in \mathbb{N}_0$ and $x_i \in \mathscr{D}$ for $i = 1,2,\ldots,w$, we have*

$$\prod_{i=1}^{w}\frac{1}{\left([1-x_i]_q+[x_i]_q\right)^{n_i-l_i}}\sum_{k_1=m}^{n_1}\sum_{k_2=m}^{n_2}\cdots\sum_{k_w=m}^{n_w}\left(\prod_{i=1}^{w}\frac{\binom{k_i}{m}}{\binom{n_i}{m}}\right)$$
$$\times B_{k_1,k_2,\cdots,k_w;n_1,n_2,\cdots,n_w}(x_1,x_2,\cdots,x_w;q)$$
$$= \sum_{l_1=0}^{m}\sum_{l_2=0}^{m}\cdots\sum_{l_w=0}^{m}q^{\sum_{i=1}^{w}\binom{l_i}{2}}\prod_{i=1}^{w}\binom{x_i}{l_i}[l_i]_q!S(m,l_i;q).$$

*Proof.*To prove this theorem, we let $F_q(t)$ be the generating function of the q-extension of the second kind Stirling numbers as follows:

$$F_q(t) := \frac{q^{-\binom{k}{2}}}{[k]_q!}\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}_q q^{\binom{k-i}{2}}e^{[i]_q t} = \sum_{n=0}^{\infty} S(n,k;q)\frac{t^n}{n!}$$

From the above, we have

$$\begin{aligned} S(n,k;q) &= \frac{q^{-\binom{k}{2}}}{[k]_q!}\sum_{i=0}^{k}(-1)^i q^{\binom{i}{2}}\binom{k}{i}_q [k-i]_q^n \\ &= \frac{q^{-\binom{k}{2}}}{[k]_q!}\Delta_q^k 0^n \end{aligned} \tag{19}$$

where $[k]_q! = [k]_q[k-1]_q\cdots[2]_q[1]_q$. It is easy to see that

$$[x]_q^n = \sum_{k=0}^{n} q^{\binom{k}{2}}\binom{x}{k}_q [k]_q! S(n,k;q) \tag{20}$$

and in similar way that

$$\left(\prod_{i=1}^{w}[x_i]_q\right)^m = \sum_{l_1=0}^{m}\sum_{l_2=0}^{m}\cdots\sum_{l_w=0}^{m} q^{\sum_{i=1}^{w}\binom{l_i}{2}}\prod_{i=1}^{w}\binom{x_i}{l_i}[l_i]_q! S(m,l_i;q). \tag{21}$$

Then, we obtain the desired result from (20) and (21).

3 Interpolation Function of Modified q-Bernstein Polynomials for Functions of Several Variables

The classical Bernoulli numbers interpolate by Riemann zeta function, which has profound effect on Analytic numbers theory and complex analysis. The values of the negative integer points, also found by Euler, are rational numbers and play a vital and important role in the theory of modular forms. Many generalization of the Riemann zeta function, such as Dirichlet series, Dirichlet L-functions and L-functions, are known in [18], [24], [25], [26], [9], [10]. So, we construct interpolation function of modified q-Bernstein polynomials of several variables.

For $s \in \mathbb{C}$ and $x_i \neq 1$ with $i = 1,2,...,w$, by applying Mellin transformation to Eq. (3), we procure

$$\begin{aligned} &D_q(s,k_1,k_2\cdots k_w;x_1,x_2\cdots x_w) \\ &= \frac{1}{\Gamma(s)}\int_0^\infty t^{s-(k_1+k_2+\cdots+k_w)-1}F_{k_1,k_2,\cdots,k_w}(-t,q;x_1,x_2,\cdots,x_w)\,dt \\ &= \frac{1}{\Gamma(s)}\int_0^\infty t^{s-(k_1+k_2+\cdots+k_w)-1}\prod_{i=1}^{w}\frac{\left(-t[x_i]_q\right)^{k_i}}{k_i!}e^{-t\sum_{i=1}^{w}[1-x_i]_q}dt \\ &= (-1)^{\sum_{i=1}^{w}k_i}\prod_{i=1}^{w}\frac{[x_i]_q^{k_i}}{k_i!}\left(\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}e^{-t[1-x_i]_q}dt\right) \\ &= (-1)^{\sum_{i=1}^{w}k_i}\prod_{i=1}^{w}\frac{[x_i]_q^{k_i}}{k_i!}[1-x_i]_q^{-s}. \end{aligned}$$

From the above, we give the definition of interpolation function for Corollary 1 as follows:

Definition 3.*Let $s \in \mathbb{C}$ and $x_i \neq 1$ with $i = 1,2,...,w$. We define interpolation function of the polynomials $B_{k_1,k_2,\cdots,k_w;n_1,n_2,\cdots,n_w}(x_1,x_2,\cdots,x_w;q)$ as*

$$D_q(s,k_1,k_2\cdots k_w;x_1,x_2\cdots x_w) = (-1)^{\sum_{i=1}^{w}k_i}\prod_{i=1}^{w}\frac{[x_i]_q^{k_i}}{k_i!}\left([1-x_i]_q\right)^{-s}. \tag{22}$$

*Remark.*By substituting $w=1$ into (22), we get

$$D_q(s,k_1) = (-1)^{k_1}\frac{[x_1]_q^{k_1}}{k_1!}[1-x_1]_q^{-s}$$

where $D_q(s,k_1)$ is introduced by Simsek and Acikgoz cf. [40].

Substituting $s = -(n_1+n_2+\cdots+n_w)$ into Eq. (22), we have

$$\begin{aligned} &D_q(-n_1-n_2-\cdots-n_w,k_1,k_2\cdots k_w;x_1,x_2\cdots x_w) \\ &= (-1)^{\sum_{i=1}^{w}k_i}\prod_{i=1}^{w}\frac{[x_i]_q^{k_i}}{k_i!}[1-x_i]_q^{n_i} \\ &= \prod_{i=1}^{w}\frac{(-1)^{k_i}n_i!}{(n_i+k_i)!}\prod_{i=1}^{w}\binom{n_i+k_i}{k_i}[x_i]_q^{k_i}[1-x_i]_q^{(n_i+k_i)-k_i} \\ &= \prod_{i=1}^{w}\frac{(-1)^{k_i}n_i!}{(n_i+k_i)!}B_{k_1,k_2,\cdots,k_w;n_1+k_1,n_2+k_2,\cdots,n_w+k_w}(x_1,x_2,\cdots,x_w;q). \end{aligned}$$

So, we arrive at the following theorem.

Theorem 9.*The following equality holds true:*

$$\begin{aligned} &D_q(-n_1-n_2-\cdots-n_w,k_1,k_2\cdots k_w;x_1,x_2\cdots x_w) \\ &= \prod_{i=1}^{w}\frac{(-1)^{k_i}n_i!}{(n_i+k_i)!}B_{k_1,k_2,\cdots,k_w;n_1+k_1,n_2+k_2,\cdots,n_w+k_w}(x_1,x_2,\cdots,x_w;q). \end{aligned}$$

By using (22), we have

$$D_q(s,k_1,k_2\cdots k_w;x_1,x_2\cdots x_w) \to D(s,k_1,k_2\cdots k_w;x_1,x_2\cdots x_w) \text{ as } q \to 1.$$

Thus one has

$$D(s,k_1,k_2\cdots k_w;x_1,x_2\cdots x_w) = (-1)^{\sum_{i=1}^{w}k_i}\prod_{i=1}^{w}\frac{x_i^{k_i}}{k_i!}(1-x_i)^{-s}. \tag{23}$$

By substituting $x_i = 1$ with $i = 1,2,...,w$ within the above, we have

$$D(s,k_1,k_2\cdots k_w;x_1,x_2\cdots x_w) = \infty.$$

We now evaluate the i-th s-derivative of $D(s,k_1,k_2\cdots k_w;x_1,x_2\cdots x_w)$ as follows: For $x_j\neq 1$ with $i=1,2,...,w$

$$\frac{\partial^i}{\partial s^i}D(s,k_1,k_2\cdots k_w;x_1,x_2\cdots x_w) = \log^i\left(\frac{1}{1-x_i}\right)D(s,k_1,k_2\cdots k_w;x_1,x_2\cdots x_w) \quad (24)$$

which seems to be interesting.

*Remark.*By taking $w=1$, $q\to 1^-$ into (23), we arrive at the following relation which was proved by Simsek and Acikgoz [40],

$$\frac{\partial^i}{\partial s^i}D(s,k_1;x_1)=\log^i\left(\frac{1}{1-x_1}\right)D(s,k_1;x_1).$$

4 p-adic Integral Representation of q-Bernstein-type polynomials

Throughout this section, we will use the following notations: $\mathbb{Z}_p$ denotes the ring of p-adic rational integers, $\mathbb{Q}_p$ denotes the field of p-adic rational numbers, $\mathbb{C}_p$ denotes the completion of algebraic closure of $\mathbb{Q}_p$. Let v_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p=p^{-v_p(p)}=p^{-1}$. When we mention about q-extension, we say that q is considered in many ways such as an indeterminate, a complex number $q\in\mathbb{C}$, or p-adic number $q\in\mathbb{C}_p$. If $q\in\mathbb{C}$ we assume that $|q|<1$. If $q\in\mathbb{C}_p$we normally assume that $|q-1|_p<p^{-\frac{1}{p-1}}$ so that $q^x=\exp(x\log q)$ for $|x|_p\leq 1$ *cf.* [4], [5], [7], [8], [9], [10], [31], [13], [14]. Let $UD(\mathbb{Z}_p)$ be the set of uniformly differentiable function. For $f\in UD(\mathbb{Z}_p)$, the p-adic q-integral on $\mathbb{Z}_p$ was originally defined by Kim [31] as follows:

$$\begin{aligned}I_q(f)&=\int_{\mathbb{Z}_p}f(x)\,d\mu_q(x)\\&=\lim_{n\to\infty}\sum_{x=0}^{p^n-1}f(x)\,\mu_q(x+p^n\mathbb{Z}_p)\\&=\lim_{n\to\infty}\frac{1}{[p^n]_q}\sum_{x=0}^{p^n-1}f(x)\,q^x.\end{aligned}$$

As q tends to 1^- in (**??**), we get known identity (p-adic Volkenborn Integral) as

$$\int_{\mathbb{Z}_p}f(x)\,d\mu_1(x)=\lim_{n\to\infty}\frac{1}{p^n}\sum_{x=0}^{p^n-1}f(x) \text{ (see [13], [14]).}$$

As $I_{-q}(f)=\lim_{q\to -q}I_q(f)$ symbolically, which yields, for p an odd prime, to

$$I_{-q}(f)=\int_{\mathbb{Z}_p}f(x)\,d\mu_{-q}(x)=\lim_{n\to\infty}\frac{1}{[p^n]_{-q}}\sum_{x=0}^{p^n-1}(-1)^x f(x)\,q^x \quad (25)$$

is known as fermionic p-adic q-invariant integral in the p-adic integer ring. And also, letting q to 1^- in (25), it reduces to

$$\int_{\mathbb{Z}_p}f(x)\,d\mu_{-1}(x)=\lim_{n\to\infty}\sum_{x=0}^{p^n-1}(-1)^x f(x) \quad (26)$$

(see [12], [13], [22]).

The Bernoulli numbers was generated by the following generating function: For $t\in\mathbb{C}$ (with $|t|<2\pi$)

$$\sum_{n=0}^{\infty}B_n\frac{t^n}{n!}=\frac{t}{e^t-1} \text{ (see [13], [14], [15], [23]).}$$

Next, it was shown that the Bernoulli numbers can be generated by p-adic Volkenborn integral as follows

$$B_n=\int_{\mathbb{Z}_p}x^n\,d\mu_1(x) \text{ for } n\in\mathbb{Z}_+:=\mathbb{N}\cup\{0\}$$

, where $\mathbb{N}$ is the set of natural numbers.

The following may be defined as a new q-extension of Bernoulli numbers

$$\beta_n(q)=\int_{\mathbb{Z}_p}q^x[x]_q^n\,d\mu_q(x).$$

Observe that

$$\lim_{q\to 1^-}\beta_n(q)=B_n.$$

Recall that

$$\sum_{n=k}^{\infty}B_{k,n}(x;q)\frac{t^n}{n!}=\frac{\left(t[x]_q\right)^k}{k!}e^{t[1-x]_q}$$

is called q-Bernstein-type polynomials. From this, we have

$$B_{k,n}(x;q)=\binom{n}{k}[x]_q^k[1-x]_q^{n-k}.$$

Throughout this section, we will assume that $x\in(0,1)$. So we can write

$$[x]_q^k=\frac{q^xB_{k,n}(x;q)}{\binom{n}{k}\left(1-[x]_q\right)^{n-k}}=\frac{q^x}{\binom{n}{k}}B_{k,n}(x;q)\sum_{l=0}^{\infty}\binom{n-k+l-1}{l}[x]_q^l.$$

Further

$$\frac{\binom{n}{k}}{B_{k,n}(x;q)}=\sum_{l=0}^{\infty}\binom{n-k+l-1}{l}q^x[x]_q^{l-k}. \quad (27)$$

Applying p-adic q-integral on $\mathbb{Z}_p$ in the both sides of (27), it yields to

$$\int_{\mathbb{Z}_p}\frac{\binom{n}{k}}{B_{k,n}(x;q)}d\mu_q(x)=\sum_{l=k}^{\infty}\binom{n-k+l-1}{l}\beta_{l-k}(q).$$

Therefore we get the following theorem.

Theorem 10. *For $k=0,1,2,\cdots,n$ and $n\in\mathbb{Z}_+$, we have*

$$\int_{\mathbb{Z}_p}\binom{n}{k}B^{-1}_{k,n}(x;q)\,d\mu_q(x)=\sum_{l=k}^{\infty}\binom{n-k+l-1}{l}\beta_{l-k}(q)$$

where $B^{-1}_{k,n}(x;q)$ is the inverse of $B_{k,n}(x;q)$.

As q tends to 1^- in Theorem 10, we have the following Corollary.

Corollary 3. *For $k=0,1,2,\cdots,n$ and $n\in\mathbb{Z}_+$, we have*

$$\int_{\mathbb{Z}_p}\binom{n}{k}B^{-1}_{k,n}(x)\,d\mu(x)=\sum_{l=k}^{\infty}\binom{n-k+l-1}{l}B_{l-k}$$

where $B^{-1}_{k,n}(x)$ is the inverse of $B_{k,n}(x)$.

The generating function of Euler polynomials has the following series expansion at $t=0$:

$$\sum_{n=0}^{\infty}E_n(x)\frac{t^n}{n!}=\frac{2}{e^t+1}e^{xt}\quad(|t|<\pi).$$

The Euler numbers are defined by $E_n(1/2)=2^nE_n$. The Euler polynomials can be generated through Equation (26)

$$E_n(x)=\int_{\mathbb{Z}_p}(x+y)^n\,d\mu_{-1}(y)\tag{28}$$

(for details, see [6], [8], [9], [13], [15], [22], [28], [29], [30]).

In [22], Kim defined the following q-Euler numbers

$$E_{n,q}=\int_{\mathbb{Z}_p}q^{-x}[x]_q^n\,d\mu_{-q}(x).$$

It is clear that

$$\lim_{q\to1^-}E_{n,q}=E_n(0).$$

By (25) and (27), we arrive at the following theorem.

Theorem 11. *For $k=0,1,2,\cdots,n$ and $n\in\mathbb{Z}_+$, we have*

$$\int_{\mathbb{Z}_p}\binom{n}{k}B^{-1}_{k,n}(x;q)\,d\mu_{-q}(x)=\sum_{l=k}^{\infty}\binom{n-k+l-1}{l}E_{l-k,q}$$

where $B^{-1}_{k,n}(x;q)$ is the inverse of $B_{k,n}(x;q)$.

As q tends to 1^- in Theorem 11, we have the following Corollary.

Corollary 4. *For $k=0,1,2,\cdots,n$ and $n\in\mathbb{Z}_+$, we have*

$$\int_{\mathbb{Z}_p}\binom{n}{k}B^{-1}_{k,n}(x)\,d\mu_{-1}(x)=\sum_{l=k}^{\infty}\binom{n-k+l-1}{l}E_{l-k}(0)$$

where $B^{-1}_{k,n}(x)$ is the inverse of $B_{k,n}(x)$.

5 Conclusion

In the paper, we have investigated a new approach to modified q-Bernstein polynomials for functions depend on the several variables, and then derived the recurrence formulas related to the second Stirling numbers and generalized Bernoulli polynomials. Moreover, the interpolation function of these polynomials depend on the several variables and the derivatives of these polynomials and also their generating function are given. Final part of this paper, we have got new interesting identities of modified q-Bernoulli numbers and q-Euler numbers by applying p-adic q-integral representation on $\mathbb{Z}_p$ and p-adic fermionic q-invariant integral on $\mathbb{Z}_p$, respectively, to the inverse of q-Bernstein polynomials.

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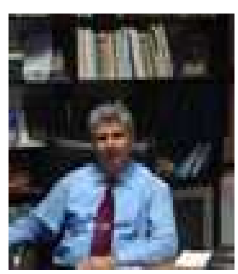

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