Applied Mathematics & Information Sciences
*An International Journal*

http://dx.doi.org/10.12785/amis/090122

# The Second Order Commutative Pairs of a First Order Linear Time-Varying System

*Mehmet Emir Koksal*[*]

Department of Primary Mathematics Education, Mevlana University, Konya, Turkey

Received: 22 Mar. 2014, Revised: 23 Jun. 2014, Accepted: 23 Jun. 2014
Published online: 1 Jan. 2015

**Abstract:** Although several publications have been appeared about the commutativity of linear time-varying systems, no research has appeared so far on the evaluation of higher order commutative pairs of a low-order system. The attempt in this paper is to fill this vacancy partially by giving the explicit results for finding all the second order commutative pairs of a first order linear time-varying system. The derived theoretical results are verified by an example.

**Keywords:** Differential equations, initial conditions, linear time-varying systems, commutativity, cascade connection

## 1 Introduction

Many engineering systems contain series or cascade-connected subsystems of smaller orders, which is a common approach for network synthesis in electrical-electronics engineering [1,2]. These subsystems need to be either the functionally time-varying type as in communication systems [3,4] or time-varying property gets involved as a non-ideal case [5]. It is known that the change of the sequence of the cascade interconnection of two subsystems does not change the functional relation between the input and output of the combined system, which is defined to be commutativity, under certain conditions whilst the system performance concerning sensitivity and disturbance gets better in one of the specific sequence with respect to the other [6]. Hence the role of the commutativity gets importance in system design [7,8].

The concept of commutativity is first introduced in the literature by E. Marshall in 1982 [9]. Later, the commutativity results obtained for first order systems are extended for the second order systems by M. Koksal [10] with the contribution of S.V. Saleh [11]. The explicit commutativity conditions for the first and second order systems are presented in these references [9,10,11] including the important fact that for the commutativity of two linear systems, it is required that either both systems time-invariant or both systems are time-varying.

The research on the commutativity of linear time-varying systems had been continued after 1982 until 1988, when the first exhaustive study on the subject was published [12]. This work has been the basic reference for years since it includes the most general necessary and sufficient conditions for systems of any order but without initial conditions. All the previous results containing the commutativity conditions were shown to be deduced from the main theorem in that paper; namely those for the first order [9], second order [10,11], third order [13], and forth order systems [14].

Further, [12] includes some special results concerning the commutativity of identical time-varying systems with arbitrary time-invariant forward and feedback path gains, which is originally proved for second order systems [7], and the commutativity of Euler systems originally treated in the undistributed report [14].

Later some new results concerning the commutativity conditions for systems having nonzero initial conditions [15] and the role of commutativity on system sensitivity [7] have been appeared. It is shown that two initially relaxed commutative systems may not be commutative when they have nonzero initial conditions and commutativity with arbitrary initial conditions requires additional constraints on the parameters of the time-varying systems. Further, one specific sequence of connection may be more robust than the other when sensitivity and disturbance or noise effects are considered. Although these results are important from both theoretical

[*] Corresponding author e-mail: mekoksal@mevlana.edu.tr

and application points of views, they had not been widely and sufficiently announced at least in an international journal paper until the appearance of the review paper published in 2011 [6]. That tutorial paper originally covers the explicit commutativity conditions of fifth order systems as well.

Due to the new developments in digital technology, especially in the communication area [16,17,18], the research on the commutativity of analogue systems has extended to the area of discrete time systems as well [19].

In the above mentioned literature so far otherwise, no research has appeared on the investigation of higher order commutative pairs of a low order system. This paper aims to fulfil this vacancy by exploring the second order commutative pairs of a first order linear time-varying system. After this introductory section, Section II summarizes the previous results necessary for the reduction of the equations, which is the subject of Section III, for finding the second order commutative pairs of a first order linear time-varying system. Section IV includes an example illustrating and verifying the results obtained in the previous section. Finally the paper ends with its conclusion in Section V.

## 2 General Commutativity Conditions for Second-order Systems

Consider the linear time-varying system $A$ described by the second order differential equation

$$a_2(t)\ddot{y}_A(t)+a_1(t)\dot{y}_A(t)+a_0(t)y_A(t)=x_A(t);\; t\geq 0 \quad (1)$$

with arbitrary initial conditions $y_A(0)$ and $\dot{y}_A(0)$. Where $a_2(t), a_1(t), a_0(t)$ are the time-varying coefficients with $a_2(t)\neq 0$; $x_A(t)$ and $y_A(t)$ being the input and output of the system, respectively; and the dot and doublet dots on the top denote the first and second order derivatives, respectively.

The first set of necessary and sufficient conditions that this system $A$ is commutative with another second or lower order system $B$ are expressed explicitly by

$$\begin{bmatrix} b_2 \\ b_1 \\ b_0 \end{bmatrix} = \begin{bmatrix} a_2 & 0 & 0 \\ a_1 & a_2^{0.5} & 0 \\ a_0 & a_2^{-0.5}\left(2a_1-\dot{a}_2\right)/4 & 1 \end{bmatrix} \begin{bmatrix} c_2 \\ c_1 \\ c_0 \end{bmatrix} \quad (2a)$$

$$-a_2^{0.5}\frac{d}{dt}\left[a_0-\frac{4a_1^2+3\dot{a}_2^2-8a_1\dot{a}_2+8\dot{a}_1a_2-4a_2\ddot{a}_2}{16a_2}c_1\right]=0, \quad (2b)$$

where $b_2(t), b_1(t), b_0(t)$ are the time-varying coefficients of the system $B$ described by

$$b_2(t)\ddot{y}_B(t)+b_1(t)\dot{y}_B(t)+b_0(t)y_B(t)=x_B(t);\; t\geq 0 \quad (3)$$

with the initial conditions $y_B(0)$ and $\dot{y}_B(0)$; $x_B(t)$ and $y_B(t)$ are the input and output of the system $B$, respectively [14].

Note that in Eq. 2, $c_2, c_1, c_0$ are arbitrary constants with $c_1$ satisfying (2b); further with $c_2=0$, (2a) leads to $b_2(t)=0$ and the system $B$ becomes a first order system. The case of $c_2=0$, $c_1=0$ simultaneously is trivial since it leads that $B$ is a time-invariant algebraic system with the constant gain $\frac{1}{c_0}$, $c_0\neq 0$.

For the case of nonzero initial conditions, Eqs. 2a, b are not sufficient for the commutativity and in this case, the second set of necessary and sufficient conditions should hold. These conditions are expressed by

$$\begin{matrix}(n)\\(m)\end{matrix}\begin{bmatrix} y_A \\ A_2^{-1}(y_B-A_1y_A) \end{bmatrix} = \begin{bmatrix} y_B \\ B_2^{-1}(y_A-B_1y_B) \end{bmatrix}\begin{matrix}(m)\\(n)\end{matrix} \quad (4a)$$

for an n-th order system $A$ and an m-th order $(m\leq n)$ system $B$ [6]. Where

$$y_A=\left[y_A(0)\; \dot{y}_A(0)\; \cdots\; y_A^{(n-1)}(0)\right]^T, \quad (4b)$$

$$y_B=\left[y_B(0)\; \dot{y}_B(0)\; \cdots\; y_B^{(m-1)}(0)\right]^T \quad (4c)$$

are the initial conditions of $A$ and $B$, respectively; and the entries of constant matrices $A_1, A_2, B_1, B_2$ of order $m\times n$, $m\times m$, $n\times m$, $n\times n$ are given by

$$\bar{a}_{ij}=\sum_{s=\max(0,i-j)}^{i-1}\frac{(i-1)!}{s!(i-1-s)!}a_{j-i+s}^{(s)},\text{ for } \begin{matrix} i=1,2,\cdots,m, \\ j=1,2,\cdots,n, \end{matrix} \quad (5a)$$

$$\bar{\bar{a}}_{ij}=\sum_{j=0}^{i-j}\frac{(i-1)!}{s!(i-1-s)!}a_{j-i+n+s}^{(s)},\text{ for } \begin{matrix} i=1,2,\cdots,m, \\ j=1,2,\cdots,i, \end{matrix}$$

$$=0 \text{ for } i=1,2,\cdots,m-1,\; j=i+1,i+2,\cdots,m, \quad (5b)$$

$$\bar{b}_{ij}=\sum_{s=\max(0,i-j)}^{i-1}\frac{(i-1)!}{s!(i-1-s)!}b_{j-i+s}^{(s)},\text{ for } \begin{matrix} i=1,2,\cdots,n, \\ j=1,2,\cdots,m, \end{matrix} \quad (5c)$$

$$\bar{\bar{b}}_{ij}=\sum_{j=0}^{i-1}\frac{(i-1)!}{s!(i-1-s)!}b_{j-i+m+s}^{(s)}=0,\text{ for } \begin{matrix} i=1,2,\cdots,n, \\ j=1,2,\cdots,i, \end{matrix}$$

$$=0,\text{ for } i=1,2,\cdots,n-1,\; j=i+1,i+2,\cdots,n, \quad (5d)$$

respectively [6]. In (5), all the coefficients $a_k(t)$ and $b_k(t)$ are evaluated at the initial time $t=0$.

In particular for the second order system (1) where $n=2$ and for a first order system $B$ where $m=1$ described by

$$b_1(t)\dot{y}_B(t)+b_0(t)y_B(t)=x_B(t);\; t\geq 0 \quad (6)$$

with the initial condition $y_B(0)$, Eq. 5 yields

$$A_1=\left[a_0\; a_1\right],\; A_2=[a_2], \quad (7a, b)$$

$$B_1 = \begin{bmatrix} b_0 \\ \dot{b}_0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} b_1 & 0 \\ \dot{b}_1 + b_0 & b_1 \end{bmatrix}, \tag{7c, d}$$

and Eqs. 4b, c are written as

$$y_A = \begin{bmatrix} y_A(0) \\ \dot{y}_A(0) \end{bmatrix}; \; y_B = [y_B(0)]. \tag{7e, f}$$

Inserting these equations in Eq. 4a, we obtain

$$\begin{bmatrix} y_A(0) \\ \dot{y}_A(0) \\ \frac{1}{a_2}\left[y_B(0) - a_0 y_A(0) - a_1 \dot{y}_A(0)\right] \end{bmatrix} =$$

$$\frac{1}{b_1^2}\begin{bmatrix} b_1^2 y_B(0) \\ b_1\left[y_A(0) - b_0 y_B(0)\right] \\ b_1(\dot{y}_A(0) - \dot{b}_0 y_B(0)) - (\dot{b}_1 + b_0)\left(y_A(0) - b_0 y_B(0)\right) \end{bmatrix} \tag{8}$$

where all time-varying coefficients and their derivatives are evaluated at the initial time $t = 0$.

## 3 Explicit Formulas for the Second-order Systems

With the aid of the commutativity requirements presented in the previous section for a second order system with an order second or lower one, the explicit formulas for finding all the second order commutative pairs of a first order linear time-varying system are derived in this section.

Let the first order system be defined as in Eq. 6. Since $b_2(t) = 0$ the constant $c_2$ in Eq. (2a) is identically zero. The remaining 2 equations in (2a) can be solved for the constants $c_1$ and $c_0$ as

$$\begin{bmatrix} c_1 \\ c_0 \end{bmatrix} = \begin{bmatrix} a_2^{0.5} & 0 \\ a_2^{-0.5}\left(2a_1 - \dot{a}_2\right) & 1 \end{bmatrix}^{-1} \begin{bmatrix} b_1 \\ b_0 \end{bmatrix}$$

$$= \begin{bmatrix} a_2^{-0.5} b_1 \\ -a_2^{-1}\left(0.5a_1 - 0.25\dot{a}_2\right) b_1 + b_0 \end{bmatrix}. \tag{9}$$

The first row of this equation implies

$$a_2 = \frac{b_1^2}{c_1^2}. \tag{10}$$

Using this equation and its first derivative in the second equality of (9), $a_1$ is evaluated as

$$a_1 = \frac{b_1}{c_1^2}\left[2b_0 + \dot{b}_1 - 2c_0\right] \tag{11}$$

which is the second explicit equation for the coefficients of the second order system A.

Finally, to find the remaining coefficient $a_0(t)$ Eq. 2b is used. Since the second order commutative pair is looked for, $a_2(t) \neq 0$ and hence Eq. 2b can be written as

$$a_0 = K + \frac{4a_1^2 + 3\dot{a}_2^2 - 8a_1\dot{a}_2 + 8\dot{a}_1 a_2 - 4a_2\ddot{a}_2}{16a_2} \tag{12a}$$

where $K$ is the arbitrary constant of integration. At last, substituting the values of $a_2$ and its first and second derivatives obtained from Eq. 10, and the values of $a_1$ and its derivative obtained from (11) into the equation (12a), we obtain the last coefficient

$$a_0 = K + \frac{1}{c_1}\left[c_0^2 + b_0^2 - 2c_0 b_0 + b_1\dot{b}_0\right] \tag{12b}$$

which is also expressed explicitly in terms of the coefficients of the first order differential equation (6) defining the subsystem $B$. Note that all the constants $c_1, c_0$ and $K$ in Eqs. 10-12 can be arbitrarily chosen to obtain second order commutative pairs $A$ described by Eq. 1 of the first order system $B$ described by Eq. 6. However, the commutativity property involved here is valid when the subsystem $B$ and its commutative pairs $A$ are relaxed or at their zero-states, that is; when all the initial conditions are zero.

To obtain the commutativity conditions with non-zero initial conditions, Eq. 8 is used. The first line obviously requires the same initial states $y_A(0) = y_B(0)$, with which the second line gives the expression for $\dot{y}_A(0)$. Hence the initial conditions of the subsystem $A$ must be expressed in terms of the initial condition of $B$ as follows;

$$y_A(0) = y_B(0), \tag{13a}$$

$$\dot{y}_A(0) = \frac{1 - b_0(0)}{b_1(0)} y_B(0). \tag{13b}$$

For the commutativity of $A$ and $B$ a final equation is required which is implied by the third line in Eq. 8. In fact, inserting the values of $y_A(0)$ and $\dot{y}_A(0)$ as expressed in Eqs. 13a and 13b in the third line and organizing the terms, we obtain the constraint equation

$$\frac{1 - a_0}{a_2} - \frac{a_1}{a_2}\frac{1 - b_0}{b_1} = \left(\frac{1 - b_0}{b_1}\right)^2 + \frac{d}{dt}\left(\frac{1 - b_0}{b_1}\right) \tag{14}$$

that must be satisfied at $t = 0$ for nonzero initial conditions.

So far, the coefficients of the second order system A and its initial conditions are all expressed in terms of those of $B$ as seen in Eqs. 10-12 and 13, respectively. Using (10) and (11), the constraint equations (14) can also be expressed for the constant $K$ by using the parameters of $B$ only; the result is

$$K = 1 - \left(\frac{1 - c_0}{c_1}\right)^2 + \frac{1 - c_1}{c_1^2}\left[(c_0 - b_0)^2 + b_1\dot{b}_0\right] \tag{15}$$

where $b_1, b_0, \dot{b}_0$ are evaluated at $t = 0$.

We now summarize the results obtained in this section in the form of a theorem.

**Theorem** (Koksal 3) All the second order linear time-varying pairs $(A's)$ as described by the differential equations in (1) of a first order linear time-varying system $(B)$ described by the differential equation in (6) can be obtained by using Eqs. 10, 11, 12 for the coefficients of $A$, Eqs. 13a and 13b for the initial conditions of $A$, where $c_1 \neq 0$ and $c_0$ are arbitrary constants and $K$ is satisfying the constraint equations in (15) at the initial time $t = 0$. If $B$ is initially relaxed so must be $A's$, that is, the constraint relation (15) between the arbitrary constants is not needed, that is $K$ is also arbitrary.

## 4 Example

To illustrate the concepts introduced in this paper and to verify the obtained results consider the first order system described by

$$(1+t)\dot{y}_B(t)+2(1-\sin t)y_B(t)=x_B(t);\ y_B(0)=-2,\ t\geq 0 \quad (16)$$

Choosing the arbitrary constants $c_1 = 1, c_0 = 1$; and $K = 1$ as to satisfy Eq. 15 for the commutativity with nonzero initial conditions, the following commutative pair $A$ of $B$ is obtained by using Eqs. 10, 11, 12;

$$(1+t)^2\ddot{y}_A(t)+(1+t)(3-4\sin t)\dot{y}_A(t)$$

$$+\left[2+4\sin^2 t-4\sin t-2(1+t)\cos t\right]y_A(t)=x_A(t), \quad (17a)$$

$$y_A(0)=-2,\ \dot{y}_A(0)=2. \quad (17b)$$

Note that the initial conditions of $A$ are chosen as to satisfy Eq. 13.

When the interconnections $AB$ and $BA$ are excited by a unit step function, it is observed that both of the outputs $AB$ and $BA$ are identical as shown in Fig. 1 (Output $AB$, $BA$). When $y_A(0) = -2$, $\dot{y}_A(0) = 2$, the initial conditions in Eq. 13 are not satisfied. Therefore, the commutativity is not valid anymore, as shown in Fig. 1 (Output $AB$, Output $BA$).

To illustrate the case where the initial conditions of $A$ are chosen as in Eq. 13, but the choice of $K$ does not satisfy Eq. 15, the example is simulated by choosing $K = 2$. This changes $a_0(t)$ by unity. Hence with $y_A(0) = y_B(0) = -2$, $\dot{y}_A(0) = 2$ and $a_0$ increased by 1, it is observed that $A$ and $B$ do not commute at all as shown in Fig. 2 (Output $AB$, Output $BA$). When the initial conditions are zero, Eq. 15 need not be satisfied for commutativity. Hence with zero initial conditions and $K = 2$, $AB$ and $BA$ gives the same output response as shown in Fig. 2 (Output $AB$, $BA$).

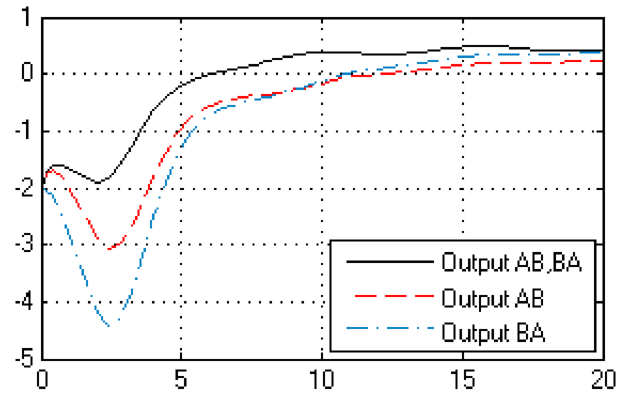


**Fig. 1:** The outputs of $AB$ and $BA$ fort the systems in Example 1, $K = 1$.

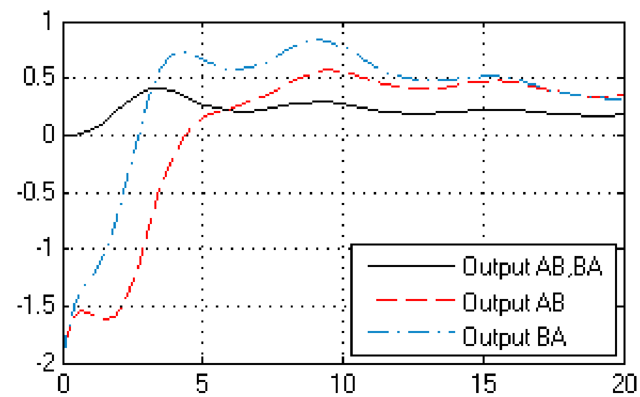


**Fig. 2:** The outputs of $AB$ and $BA$ fort the systems in Example 1, $K = 2$
.

The example verifies that with zero initial conditions, the second order commutativity pairs of $B$ can be found by choosing all the constants $c_1, c_0, K$ arbitrarily. However, when the initial conditions are nonzero, both Eq. 13 imposing constraints on the choice of initial conditions of $A$ and Eq. 15 specifying the value of the constant $K$ must be satisfied for the commutativity of $A$ and $B$.

Before concluding the section it might deserve to note that only very few classes of linear time varying systems have compact analytic solutions. These are known to be solvable classes [20]; among them $A_1$ class has constant eigenvalues, $A_h$ class has eigenvalues of constant multiples of a common function $h(t)$. In a recent publication, another solvable class having eigenvalues in certain types of polynomials has been reported [21]. Since both Eq. 17 describing System $A$ and the third order system resulting by cascading it with system $B$ defined by Eq. 16 do not belong to any one of the known solvable classes, the results obtained by MATLAB Simulink toolbox cannot be verified by obtaining compact analytical solutions. To meet this aim partially, the series solution of the cascaded system $AB$ (or $BA$) for the case $y_A(0) = y_B(0) = -2$, $\dot{y}_A(0) = 2$, and unit step excitation is obtained by using routine series solution techniques. The explicate result for the output containing the first six terms is

$$y(t) \cong -2+2t-4t^2+\frac{35}{6}t^3-\frac{17}{2}t^4+\frac{481}{40}t^5-\frac{3961}{240}t^6. \tag{18}$$

With this solution, the simulation results are shown to be well verified for small values of $t$; namely for $t = 0, 0.1, 0.2$ the series solution yields $y = -2, -1.8349, -1.7241$ while the simulation results are $y = -2, -1.8349, -1.7239$. For higher values of $t$, the results of the series solution deviate from those of the simulations, which is natural due to the truncation at the sixth term in the series solution of Eq. 18.

## 5 Conclusions

The explicit formulas are derived to find all the second order commutative pairs of a first order linear time-varying system. The time-varying coefficients of the second order system are obtained from those of the first order system by using three constants which can all be arbitrarily chosen in case of zero initial conditions. For the case of the non-zero initial condition of the first order system, however, both of the initial conditions of the second order system and one of the above mentioned constants are not arbitrary anymore and they should be chosen, so as to satisfy the constraints developed in this paper so that both systems are commutative. The results are verified by an example.

## References


[1] D. Vasileska and S. M. Goodnick, Computational electronics, Morgan & Claypool Publishers, ISBN 1-59829-056-8 (2006).
[2] R. C. Jaeger and T. N. Blalock Microelectronic Circuit Design 3rd Ed. McGraw–Hill, Boston, ISBN 978-0-07-319163-8 (2006).
[3] G. M. Miller and J. S. Bensley, Modern Electronic Communication, 7th Ed. Prentice Hall, New Jersey, USA (2002).
[4] E. Witrant, C. Canudas-De-Wit, D. Georges, and M. Alamir, Remote stabilization via time-varying communication network delays: application to TCP networks, Proceedings of the 2004 IEEE International Conference on Control Applications **1** 474-479 (2004).
[5] J. I. Montojo and L. B. Milstein, Error rate for PSK and QAM modulations for non-ideal OFDM systems with noisy channel estimates and receive diversity, IEEE Transactions on Communication **59** 2701-2715 (2011).
[6] M. Koksal and M. E. Koksal, Commutativity of linear time-varying differential systems with non-zero initial conditions: a review and some new extensions, Mathematical Problems in Engineering **2011** 1-25 (2011).
[7] M. Koksal, Effects of commutativity on system sensitivity, Proc. of the 6th Int. Symposium on Networks, Systems and Signal Processing, ISYNT2 89, Zagreb, Yugoslavia 61-62 (1989).
[8] M. Koksal, Effects of Initial Conditions on the commutativity and effects of commutativity on the system sensitivity (in Turkish), Proc. of. the 2nd National Cong. of Electrical Engineers, Ankara, Turkey 570-573 (1987).
[9] E. Marshall, Commutativity of time-varying systems, Electro. Letters **18** 539-540 (1977).
[10] M. Koksal,. Corrections on ‘Commutativity of second-order time-varying systems’, Int. J. of Cont. **1** 273-274 (1983).
[11] S. V. Saleh, Comments on ‘Commutativity of second-order time-varying systems’, Int. J. of Cont. **37** 1195 (1983).
[12] M. Koksal, Exhaustive study on the commutativity of time-varying systems, Int. J. of Cont. **5** 1521-1537 (1988).
[13] M. Koksal, General conditions for commutativity of time-varying systems, Proc. of IASTED Int. Conf. on Telecommunication and Control (TELECON’84), 229-225 (1984).
[14] M. Koksal, A Survey on the Commutativity of Time-Varying Systems, Technical Report, no. GEEE/CAS-85/1, Gaziantep, Turkey (1985)
[15] M. Koksal, Effects of nonzero initial conditions on the commutativity of linear time-varying systems, Proc. of Int. AMSE Conf. on Modelling and Simulation, Istanbul, Turkey, AMSE Press **1A** 49-55 (1988).
[16] M. Rice, Digital Communications: A Discrete-Time Approach, Prentice Hall, ISBN 13: 9780130304971-2009 (2008).
[17] S. Haykin, Digital Communication Systems, 1st Edition, Wiley (2013).
[18] M. Timmers, D. Vanderhaegen, D. Van Bruyssel, et al., Transmitter-Controlled Adaptive Modulation: Concepts and Results, Bell Labs Technical Journal, **18** 153-169 (2013).
[19] M. E. Koksal and M. Koksal, Commutativity of cascade connected discrete time linear time-varying systems, Proc. of the 2013 Automatic Control National Meeting TOK’2013, Malatya, Turkey, 1128-1131 (2013).
[20] M. Y. Wu, Solution of certain classes of linear time-varying systems, Int. J. of Cont. **31** 11-20 (1980).
[21] H. Guneyli, N. Watsuji, and A. Nacaroglu, A new solution method for time varying linear systems, Proc. of Electrical, Electronics and Computers Symposium FEEB-2011, Elazığ, Turkey, **2** 68-171 (2011).

**Mehmet Emir Koksal** is Assistant Professor of Mathematics at Mevlana (Rumi) University. He received his PhD degree in Applied Mathematics from Gebze Institute of Technology in 2009. After taking his PhD, he studied as a post-doc in Florida Institute of Technology from 2009 to 2010. His studies were supported by the scientific and technological research council of Turkey during this period. He has worked for various private universities in Turkey. He has been working for Mevlana University since 4 years. His research interests are in new operator-difference schemes in Hilbert space for partial differential equations (PDEs) and applications of PDEs in control theory, non-uniform transmission lines, fluid dynamics and material sciences. He has many published articles in reputed international journals of mathematical and engineering sciences.