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Evolution Of Special Ruled Surfaces Via The Evolution Of Their Directrices In Euclidean 3-Space E^3

R. A. Hussien and Taha Youssef**

Math. Dept., Faculty of Science, Assiut Univ., Assiut, 71516, Egypt

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Abstract: In this paper, evolutions of ruled surfaces that are generated by the normal and binormal vector fields of space curve (normal and binormal surfaces) are presented. These evolutions of the ruled surfaces depend on the evolutions of their directrices. Geometric visualization of these ruled surfaces are presented. In addition, the conditions which make these surfaces of types inextensible, developable and minimal are obtained.

Keywords: Curve evolution; Surface evolution; Special Ruled Surfaces; Normal and binormal surface; Inextensible surface.

1 Introduction

Recently, the study of the motion of inelastic plane curves has arisen in a number of diverse engineering applications. Chirikjian and Burdick [1] described the motion of a planar hyperredundant (or snake-like) robot as the flow of a plane curve, while Brockett [2] explicitly proposed the idea of an inelastic string machine as a robotic device. Inelastic plane curves, i.e., plane curves whose lengths are preserved. Inextensible curve and surface flows also arise in the context of many problems in computer vision [3], [4] and computer animation [5], and even structural mechanics [6].

There are many applications in image processing and computer vision, such as scale space by linear and nonlinear diffusions [7–10], image enhancement through anisotropic diffusions [9], [11–14] and image segmentation by active contours [15–18]. The level set formulation [19] has provided good means to implement these flows. Extending these motions to manifolds embedded in spaces of higher dimensions can be beneficial for many applications.

The Subject of how space curves evolve in time is of great interest and has been investigated by many authors. in [20], Hasimoto showed that the nonlinear Schrödinger equation describing the motion of an isolated non−stretching thin vortex filament. Lamb [21] used the Hasimoto transformation to connect other motions of curves to the mKdV and sine−Gordon equations. Nakayama, et al [22] obtained the sine−Gordon equation by considering a nonlocal motion. Also, Nakayama and Wadati [23] presented a general formulation of evolving curves in two dimensions and its their connection to mKdV hierarchy. Nassar, et al [24–27] studied the evolution of plane curves, the motion of hypersurfaces and the evolution of space curves in R^n. R. Mukherjee and R. Balakrishnan [28] applied their method to the sine-Gordon equation and obtained links to five new classes of space curves in addition to the two which were found by Lamb [21]. For each class, they displayed the rich variety of moving curves associated with the one−soliton, the breather, the two−soliton and the soliton−antisoliton solutions. In the case of the motion of surfaces, K. Nakayama and M. Wadati [29] formulated the motion of surfaces in 3-dimensional space using differential geometry. They obtained the time evolutions of the metric and the curvature tensor. D. Y. Kwon and F. C. Park [30], studied the evolution of inelastic plane curves. Also D. Y. Kwon and F.C. Park [31], studied inextensible flows of curves and developable surfaces. T. Körpinar et al [32] studied new inextensible flows of tangent developable surfaces in Euclidian 3−space E^3.

In this paper, we shall derive a pair of coupled nonlinear partial differential equations (CNLPDEs) governing the time evolution of the curvature and torsion of the evolving curve. Applications to some curves are presented. Then we construct normal and binormal surfaces associated to these curves. Geometric

* Corresponding author e-mail: rawya_hussein@yahoo.com tahaussef@yahoo.com

visualization of the normal and binormal surfaces are displayed via solving the Gauss-Weingarten equations for a specified coefficients of the first and second fundamental forms using fundamental theorem of surfaces. The essence of this paper is that, we linked the motion of surfaces with the motion of curves, i.e., if the curve moves, then the normal and binormal surfaces move.

The article is organized as follows. In section 1.1, we introduce the time−evolution equations that are satisfied by the intrinsic quantities of curves. Also, we derive CNPDEs which formulate the problem directly in terms of the curvatures and obtained the exact solution for them. In subsection 1.2, we determine the curve from its curvatures. In section 1.3, we introduce some applications of the curve evolution specified by its local geometry. In section 2, we introduce differential geometry of surfaces. In section 3, we introduce the geometric properties of normal and binormal surfaces. In section 4, we reconstruct the surfaces from the coefficients of the the first and second fundamental forms via numerical integration of the Gauss-Weingarten equations and plotted them.

1.1 Time-evolution equations

In this section we briefly review the main results for the evolution of space curves as presented in [28], and extend these results to derive time−evolution equations that is satisfied by the intrinsic quantities of the curve. Let us consider a curve embedded in three-dimensional space described in parametric form by a position vector $\mathbf{r} = \mathbf{r}(s)$, s being the usual arclength variable. The unit tangent vector $\mathbf{t} = \mathbf{r}_s$, the principal normal $\mathbf{n}$ and the binormal $\mathbf{b}$ form an orthonormal triad of unit vectors that satisfy the Frenet–Serret equations [33]:

$$\begin{aligned} \mathbf{t}_s &= \kappa \mathbf{n}, \\ \mathbf{n}_s &= -\kappa \mathbf{t} + \tau \mathbf{b}, \\ \mathbf{b}_s &= -\tau \mathbf{n}. \end{aligned} \tag{1}$$

Here and hereafter, the subscripts denote partial derivatives. κ and τ are the curvature and torsion of the curve.

If this curve moves with time t, then all quantities in Eq. (1) become functions of both s and t. The general temporal evolution in which the triad $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ remains orthonormal adopts the following form [34]

$$\begin{aligned} \mathbf{t}_t &= \alpha \mathbf{n} + \beta \mathbf{b}, \\ \mathbf{n}_t &= -\alpha \mathbf{t} + \gamma \mathbf{b}, \\ \mathbf{b}_t &= -\beta \mathbf{t} - \gamma \mathbf{n}. \end{aligned} \tag{2}$$

As is clear, the parameters α, β and γ (which are the velocities of the moving frame $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$) determine the motion of the curve.

On requiring the compatibility conditions

$$\mathbf{t}_{ts} = \mathbf{t}_{st}, \quad \mathbf{n}_{ts} = \mathbf{n}_{st}, \quad \mathbf{b}_{ts} = \mathbf{b}_{st}. \tag{3}$$

Apply the compatibility conditions Eq. (3) to the systems (1), (2), then

$$\begin{aligned} \kappa_t &= \alpha_s - \tau \beta, \\ \tau_t &= \gamma_s + \kappa \beta, \\ \beta_s &= \kappa \gamma - \tau \alpha. \end{aligned} \tag{4}$$

The temporal evolution of the curvature κ and the torsion τ of the curve may now be expressed in terms of the components of velocity $\{\alpha, \beta\}$ which can be written as coupled nonlinear partial differential equations as follows,

$$\boxed{\begin{aligned} \kappa_t &= \alpha_s - \beta \tau, \\ \tau_t &= \left(\frac{\beta_s + \tau \alpha}{\kappa}\right)_s + \kappa \beta. \end{aligned}} \tag{5}$$

From these equations we note that the component γ in Eq. (2) does not affect the final shape of the evolving curve. For a given $\{\alpha, \beta, \gamma\}$, the motion of the curve is determined from these equations. Mathematica package software (computational software program used in scientific, engineering, mathematical fields and other areas of technical computing) was used for solving the Eqs.(5) which applies the tanh−and sech−methods [35]. The outline for given $\{\alpha, \beta, \gamma\}$ is that we get $\{\kappa, \tau\}$.

1.2 Determining a parametrized curve from its curvature and torsion

One of the basic problems in geometry is to determine exactly the geometric quantities which distinguish one figure from another. For example, line segments are uniquely determined by their lengths, circles by their radii, triangles by side-angle-side, etc. It turns out that this problem can be solved in general for sufficiently smooth regular curves. We will see that a regular curve is uniquely determined by two scalar quantities, called curvature and torsion, as functions of the natural parameter, which follows from the next theorem.

Theorem 1(Fundamental existence and uniqueness theorem for space curves). *Let $\kappa(s)$ and $\tau(s)$ be arbitrary continuous functions on $a \leq s \leq b$. then there exists, except for position in space , one and only one space curve C for which $\kappa(s)$ is the curvature, $\tau(s)$ is the torsion and s is a natural parameter along C [36].*

In the next subsection, we shall show how to recreate curves in the space from their curvature and torsion via numerical integration of Frenet−Seret equations up to its position in space.

1.3 Applications

In this subsection, we consider some applications of the curve evolution specified by its local geometry. The set of five geometric parameters $\{\kappa,\tau,\alpha,\beta,\gamma\}$ appearing in the intrinsic Frenet-triad evolution equations (1) and (2) essentially describes a moving curve.

1.3.1 Case (1)

For a curve moving in the space by the velocities

$$\{\alpha,\beta,\gamma\}=\{\kappa,\kappa_s,\frac{\kappa_{ss}+\tau\kappa}{\kappa}\}. \tag{6}$$

The evolution equations for the curvature and the torsion of the curve given from Eq. (5) as follows,

$$\begin{aligned}\kappa_t &= \kappa_s-\kappa_s\tau,\\ \tau_t &= (\frac{\kappa_{ss}+\tau\kappa}{\kappa})_s+\kappa\kappa_s.\end{aligned} \tag{7}$$

The general solutions to this system are given by

$$\begin{aligned}\kappa_1(s,t) &= 2c_2\mathrm{sech}(c_1t+c_2s+c_3),\\ \tau_1(s,t) &= \frac{c_2-c_1}{c_2},\quad c_2\geq 0,\end{aligned} \tag{8}$$

$$\begin{aligned}\kappa_2(s,t) &= c_4,\\ \tau_2(s,t) &= c_5f(c_6t+c_6s+c_7),\end{aligned} \tag{9}$$

where $c_i,(i=1,...,7)$ are arbitrary real constants and f is an arbitrary function. As the first solution, it is easily verified that the set $\{\kappa,\tau,\alpha,\beta,\gamma\}$ satisfies Eq. (4).

The figures 1, 4, 7 and 10 represent snapshots of the evolving space curve obtained by solving the Frenet−Serret Eq. (1) for a specified curvature and torsion using Mathematica [37]. Any moving space curve can be studied from two perspectives, namely the shape of the curve and the evolution of the curve. At every fixed time t, we clearly have a representation of the corresponding static space curve at that instant. The program [37], as it stands, generates static space curves. It was extended slightly to generate the evolution of the space curves with time t.

In all the figures 1, 4, 7 and 10, we have used the total curve length of 20 ($-10\leq s\leq 10$). In practice, the range of variation of t must remain much smaller than that of s so that the length of the curve suffices to display the complete geometric structure corresponding to the solution concerned.

If we put $c_1=0.4, c_2=0.5, c_3=0$ in Eq. (8), then $\kappa=\mathrm{sech}(0.4t+0.5s), \tau=0.2$. we see that $\kappa\to 0$ as $s\to\pm\infty$. Thus, for large values of s, the curve straightens out at both ends as shown in Fig. 1.

1.3.2 Case (2)

We consider the case that the velocities are given by

$$\{\alpha,\beta,\gamma\}=\{0,\kappa,\frac{\kappa_s}{\kappa}\}. \tag{10}$$

From Eq. (5) it follows that the evolution equations for the curvature and the torsion of the curve are given as follows,

$$\begin{aligned}\kappa_t &= -\kappa\tau,\\ \tau_t &= (\frac{\kappa_s}{\kappa})_s+\kappa^2.\end{aligned} \tag{11}$$

The general solution to this system is given by

$$\begin{aligned}\kappa(s,t) &= \sqrt{c_1^2+c_2^2}\mathrm{sech}(c_1t+c_2s+c_3),\\ \tau(s,t) &= c_1\tanh(c_1t+c_2s+c_3).\end{aligned} \tag{12}$$

where c_1,c_2,c_3 are arbitrary real constants. If we take $c_1=1, c_2=1$ and $c_3=0$ in Eq. (12), then $\kappa=\sqrt{2}\mathrm{sech}(t+s)$ and $\tau(s,t)=\tan(t+s)$. We see that $\kappa\to 0$ as $s\to\pm\infty$ and $\tau\to\pm 1$ as $s\to\pm\infty$. Thus, for large values of s, the curve straightens out at both ends as shown in Fig. 10.

In the next section we mention some basic facts in the general theory of surfaces useful for the rest of the paper.

2 Differential geometry of surfaces

Let $\mathbf{x}=\mathbf{x}(s,t)$ denote the position vector of a generic point P on a surface S in R^3. Then, the vectors $\mathbf{x}_s$ and $\mathbf{x}_t$ are tangential to S at P, at such points at which they are linearly independent,

$$\mathbf{N}=\frac{\mathbf{x}_s\wedge\mathbf{x}_t}{|\mathbf{x}_s\wedge\mathbf{x}_t|}, \tag{13}$$

determines the unit normal vector to S. The first and second fundamental forms of S are defined by

$$\begin{aligned}I &= \langle d\mathbf{x}.d\mathbf{x}\rangle = g_{11}ds^2+2g_{12}dsdt+g_{22}dt^2,\\ II &= \langle -d\mathbf{x}.d\mathbf{N}\rangle = L_{11}ds^2+2L_{12}dsdt+L_{22}dt^2,\end{aligned} \tag{14}$$

where,

$$\begin{aligned}g_{11}&=\langle\mathbf{x}_s,\mathbf{x}_s\rangle, & g_{12}&=\langle\mathbf{x}_s,\mathbf{x}_t\rangle, & g_{22}&=\langle\mathbf{x}_t,\mathbf{x}_t\rangle\\ L_{11}&=\langle\mathbf{x}_{ss},\mathbf{N}\rangle, & L_{12}&=\langle\mathbf{x}_{st},\mathbf{N}\rangle, & L_{22}&=\langle\mathbf{x}_{tt},\mathbf{N}\rangle,\end{aligned} \tag{15}$$

where $\langle,\rangle$ is the Euclidean scaler product.

The Gauss equations associated with s are [33]:

$$\begin{aligned}\mathbf{x}_{ss} &= \Gamma_{11}^1\mathbf{x}_s+\Gamma_{11}^2\mathbf{x}_t+L_{11}\mathbf{N},\\ \mathbf{x}_{st} &= \Gamma_{12}^1\mathbf{x}_s+\Gamma_{12}^2\mathbf{x}_t+L_{12}\mathbf{N},\\ \mathbf{x}_{tt} &= \Gamma_{22}^1\mathbf{x}_s+\Gamma_{22}^2\mathbf{x}_t+L_{22}\mathbf{N}.\end{aligned} \tag{16}$$

The Weingarten equations are

$$\begin{aligned}\mathbf{N}_s &= \frac{g_{12}L_{12}-g_{22}L_{11}}{g}\mathbf{x}_s+\frac{g_{12}L_{11}-g_{11}L_{12}}{g}\mathbf{x}_t,\\ \mathbf{N}_t &= \frac{g_{12}L_{22}-g_{22}L_{12}}{g}\mathbf{x}_s+\frac{g_{12}L_{12}-g_{11}L_{22}}{g}\mathbf{x}_t,\end{aligned} \tag{17}$$

where
$$g = det(g_{ij}) = g_{11}g_{22} - g_{12}^2. \tag{18}$$

The quantities Γ_{ij}^k are called the Christoffel symbols of the second kind, and they are given by

$$\Gamma_{ij}^k = \frac{1}{2}g^{kl}(\frac{\partial}{\partial u^i}g_{lj} + \frac{\partial}{\partial u^j}g_{il} - \frac{\partial}{\partial u^l}g_{ij}), \tag{19}$$

where $\mathbf{x}^1 = s$, $\mathbf{x}^2 = t$,

$$I = g_{ik}d\mathbf{x}^j d\mathbf{x}^k \tag{20}$$

and
$$g^{jk}g_{kl} = \delta_j^i. \tag{21}$$

In the above, the Einstein convention of summation over repeated indices has been adopted.

The Gaussian curvature κ_g and the mean curvature κ_m are

$$\kappa_g = \frac{L}{g} = \frac{L_{11}L_{22} - L_{12}^2}{g_{11}g_{22} - g_{12}^2}, \tag{22}$$

$$\kappa_m = \frac{L_{11}g_{22} - 2L_{12}g_{12} + L_{22}g_{11}}{2g}, \tag{23}$$

where

$$L = det(L_{ij}) = L_{11}L_{22} - L_{12}^2. \tag{24}$$

Applying the compatibility conditions $(\mathbf{x}_{ss})_t = (\mathbf{x}_{st})_s$ and $(\mathbf{x}_{st})_t = (\mathbf{x}_{tt})_s$ to the linear Gauss system (16). Then we have the Gauss and Mainardi-Codazzi system

$$\begin{aligned} L = {} & g_{11}((\Gamma_{22}^1)_s - (\Gamma_{12}^1)_t + \Gamma_{22}^1\Gamma_{11}^1 + \Gamma_{22}^2\Gamma_{12}^1 - \Gamma_{12}^1\Gamma_{12}^1 - \Gamma_{12}^2\Gamma_{22}^1) \\ & + g_{12}((\Gamma_{22}^2)_s - (\Gamma_{12}^2)_t + \Gamma_{22}^1\Gamma_{11}^2 - \Gamma_{12}^1\Gamma_{12}^2), \end{aligned} \tag{25}$$

$$\begin{aligned} \frac{\partial L_{11}}{\partial t} - \frac{\partial L_{12}}{\partial s} &= L_{11}\Gamma_{12}^1 + L_{12}(\Gamma_{12}^2 - \Gamma_{11}^1) - L_{22}\Gamma_{11}^2, \\ \frac{\partial L_{12}}{\partial t} - \frac{\partial L_{22}}{\partial s} &= L_{11}\Gamma_{22}^1 + L_{12}(\Gamma_{22}^2 - \Gamma_{12}^1) - L_{22}\Gamma_{12}^2. \end{aligned} \tag{26}$$

Theorem 2.*(Fundamental existence and uniqueness theorem Of Surfaces) Let g_{11},g_{12} and g_{22} be functions of s and t of class C^2 and let L_{11},L_{12} and L_{22} be functions of s and t of class C^1 all defined on an open set containing (s_0,t_0) such that for all (s, t),*
(i) $g_{11}g_{22} - g_{12}^2 > 0, \quad g_{11} > 0, \quad$ and $g_{22} > 0$
(ii) $g_{11}, g_{12}, g_{22}, L_{11}, L_{12}$ and L_{22} satisfy the compatibility equations (25),(26). Then there exists a patch $\mathbf{x} = \mathbf{x}(s,t)$ of class C^3 defined in a neighborhood of (s_0,t_0) for which $g_{11}, g_{12}, g_{22}, L_{11}, L_{12}$ and L_{22} are the first and second fundamental coefficients. The surface represented by $\mathbf{x} = \mathbf{x}(s,t)$ is unique except for position in space [36].

3 Geometric properties of the normal and binormal surfaces

A ruled surface is a surface that can be swept out by moving a line in space. It therefore has a parameterization of the form
$$\mathbf{x}(s,u) = \mathbf{r}(s) + u\ell(s), \tag{27}$$

where $\mathbf{r}$ is called the ruled surface directrix (also called the base curve) and ℓ is the director curve. The straight lines themselves are called rulings. The rulings of a ruled surface are asymptotic curves. Furthermore, the Gaussian curvature on a ruled regular surface is everywhere nonpositive. If the curves $\mathbf{r} = \mathbf{r}(s)$ and $\ell = \ell(s)$ move with time t, then Eq. (27) becomes

$$\mathbf{x}(s,u,t) = \mathbf{r}(s,t) + u\ell(s,t) \tag{28}$$

3.1 Normal surface

We may take each of the frame vectors of the curve $\mathbf{r}$ and reapply it as the vector ℓ to describe a ruled surface uniquely determined by the shape of the curve and its variations in space. If the generatrix vector ℓ is the normal $\mathbf{n}$ of the curve $\mathbf{r}$, then Eq. (28) define a normal surface with the parametric representation

$$\mathbf{x}(s,u,t) = \mathbf{r}(s,t) + u\mathbf{n}(s,t). \tag{29}$$

Definition 1.*A surface evolution $\mathbf{x}(u,s,t)$ and its flow $\frac{\partial \mathbf{x}}{\partial t}$ are said to be inextensible if its first fundamental quantities $\{g_{11}, g_{12}, g_{22}\}$ satisfies [31]*

$$\frac{\partial g_{11}}{\partial t} = \frac{\partial g_{12}}{\partial t} = \frac{\partial g_{22}}{\partial t} = 0. \tag{30}$$

The tangent vectors for the surface S are

$$\begin{aligned} \mathbf{x}_s(s,u,t) &= (1-u\kappa)\mathbf{t}(s,t) + \tau u\mathbf{b}(s,t), \\ \mathbf{x}_u(s,u,t) &= \mathbf{n}(s,t). \end{aligned} \tag{31}$$

Using Eqs. (13) and (31), then the unit normal vector field on S has the form

$$\mathbf{N} = \frac{\mathbf{x}_u \wedge \mathbf{x}_s}{|\mathbf{x}_u \wedge \mathbf{r}_s|} = \frac{(u\kappa - 1)\mathbf{b} + \tau u\mathbf{t}}{\sqrt{(u\kappa - 1)^2 + \tau^2u^2}}. \tag{32}$$

Using Eq. (15), then the coefficients of the first fundamental form are

$$\begin{aligned} g_{11} &= (1-u\kappa)^2 + (\tau u)^2, \\ g_{12} &= 0, \\ g_{22} &= 1. \end{aligned} \tag{33}$$

Using the above system and Eq. (30), one can see that the normal surface is inextensible if

$$-2u\kappa_t(1-u\kappa) + 2u^2\tau\tau_t = 0. \tag{34}$$

Using Eq. (15), then the coefficients of the second fundamental form are

$$L_{11} = \frac{-u^2\kappa_s\tau + u\tau_s(u\kappa - 1)}{\sqrt{(u\kappa-1)^2+u^2\tau^2}},$$
$$L_{12} = \frac{-\tau}{\sqrt{(u\kappa-1)^2+u^2\tau^2}}, \qquad (35)$$
$$L_{22} = 0.$$

So, the Gaussian curvature κ_g and the mean curvature κ_m are

$$\kappa_g = \frac{-\tau^2}{((1-u\kappa)^2+u^2\tau^2))^2}, \quad \kappa_m = \frac{-u^2\kappa_s\tau + u\tau_s(u\kappa-1)}{2((1-u\kappa)^2+\tau^2u^2)^{3/2}}. \qquad (36)$$

From the above equations, the normal surface $\mathbf{x}(s,u,t)$ is developable ($\kappa_g = 0$) if

$$\tau = 0, \qquad (37)$$

and minimal $(\kappa_m = 0)$ if

$$-u^2\kappa_s\tau + u\tau_s(u\kappa-1) = 0, \qquad (38)$$

a short calculation shows that the above equation can be written as

$$u^2\frac{\partial}{\partial s}\left(\frac{\tau}{\kappa}\right) - \frac{\tau_s}{\kappa^2} = 0 \qquad (39)$$

3.2 *Binormal surface*

If the generatrix vector ℓ is the binormal $\mathbf{b}$ of the curve $\mathbf{r}$, then from Eq.28, we get the binormal surface. The parametric representation for this surface is

$$\mathbf{x}(s,v,t) = \mathbf{r}(s,t) + v\mathbf{b}(s,t). \qquad (40)$$

The tangent vectors for S are

$$\mathbf{x}_s(s,v,t) = \mathbf{t}(s,t) - \tau v\mathbf{n}(s,t), \quad \mathbf{x}_v(s,v,t) = \mathbf{b}(s,t). \qquad (41)$$

The unit normal vector field on the surface S is given by

$$\mathbf{N} = \frac{\mathbf{n}+\tau v\mathbf{t}}{\sqrt{1+\tau^2v^2}}. \qquad (42)$$

A short calculation shows that the coefficients of the first fundamental form are

$$g_{11} = 1+v^2\tau^2, \quad g_{12} = 0, \quad g_{22} = 1.$$

Using the above system and Eq. (30), one can see that the binormal surface is inextensible if

$$\frac{\partial\tau}{\partial t} = 0, \text{ so} \quad \tau = \tau_0 = \text{constant} \quad (\text{we choose } \tau_0 = 0). \qquad (43)$$

Using Eq. (15), then the coefficients of the second fundamental form are

$$L_{11} = \frac{\tau^2v^2\kappa + \kappa - \tau_s v}{\sqrt{1+v^2\tau^2}}, \; L_{12} = \frac{-\tau}{\sqrt{1+v^2\tau^2}}, \; L_{22} = 0.$$

So, the Gaussian curvature κ_g and the mean curvature κ_m are

$$\kappa_g = \frac{-\tau^2}{(1+\tau^2v^2)^2}, \quad \kappa_m = \frac{\tau^2v^2\kappa + \kappa - \tau_s v}{2(1+\tau^2v^2)^{3/2}}. \qquad (44)$$

From the above equations, the binormal surface $\mathbf{x}(s,v,t)$ is developable ($\kappa_g = 0$) iff

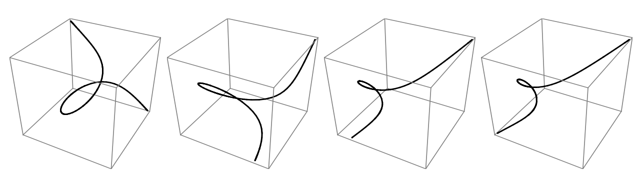
$$\tau = 0, \qquad (45)$$

and minimal ($\kappa_m = 0$) iff

$$\tau^2v^2\kappa + \kappa - \tau_s v = 0 \qquad (46)$$

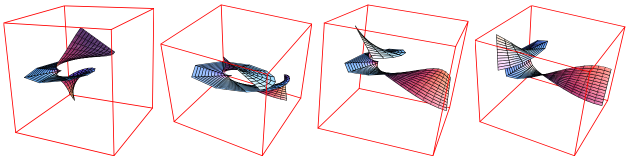
4 Geometric visualization of the normal and binormal surfaces and its generator

In this section, we shall display the evolution of Normal and Binormal Surfaces depends on the evolution of their directrices. The surfaces below represent snapshots of the evolving normal and binormal surfaces obtained by solving the Gauss-Weingarten Eqs. (16) for specified first and second fundamental forms using Mathematica [38]. Any moving surface can be studied from two perspectives, namely, the shape of the surface and the evolution of the surface. At every fixed t, we clearly have a representation of the corresponding (static) surface at that instant. The program [38] as it stands generates static surfaces. It was extended slightly to generate the evolution of the surfaces with t. We used the curvature and torsion of the curves which obtained in section (1) which represents the base curves of the normal and binormal surfaces.



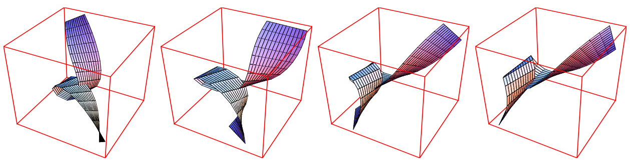
(a) t=0 (b) t=1 (c) t=2 (d) t=3

Fig. 1: Time evolution of the curve $\kappa = \text{sech}(0.4t + 0.5s), \tau = 0.2$



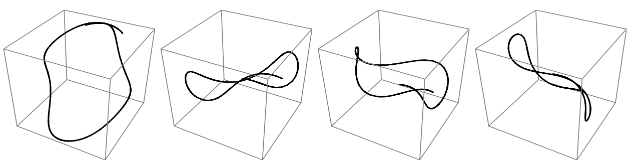
(a) t=0 (b) t=1 (c) t=2 (d) t=3

Fig. 2: Time evolution of the normal surfaces



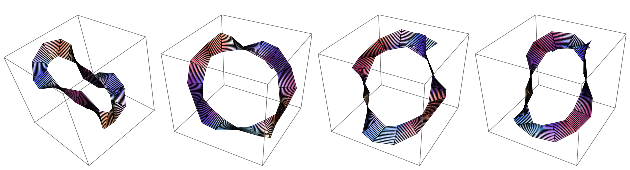
(a) t=0 (b) t=1 (c) t=2 (d) t=3

Fig. 3: Time evolution of the binormal surfaces



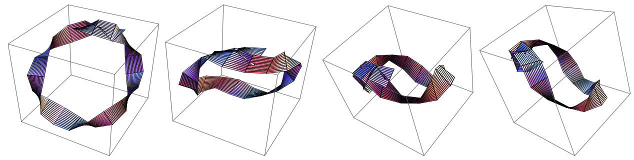
(a) t=0 (b) t=1 (c) t=2 (d) t=3

Fig. 4: Time evolution of the curve $\kappa = 0.5, \tau = \sin(t+s)$



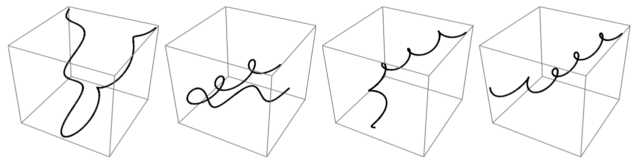
(a) t=0 (b) t=1 (c) t=2 (d) t=3

Fig. 5: Time evolution of the normal surfaces



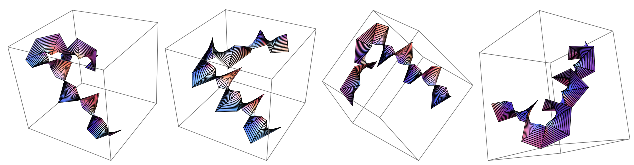
(a) t=0 (b) t=1 (c) t=2 (d) t=3

Fig. 6: Time evolution of the binormal surfaces



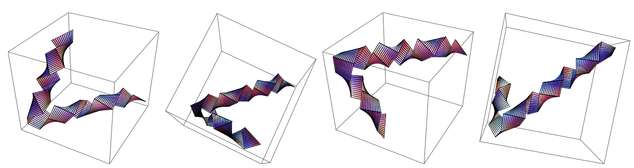
(a) t=0 (b) t=1 (c) t=2 (d) t=3

Fig. 7: Time evolution of the curve $\kappa = 1, \tau = \tanh(t+s)$



(a) t=0 (b) t=1 (c) t=2 (d) t=3

Fig. 8: Time evolution of the normal surfaces

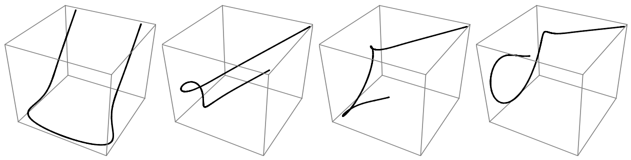


(a) t=0 (b) t=1 (c) t=2 (d) t=3

Fig. 9: Time evolution of the binormal surfaces

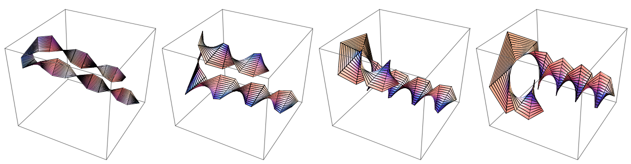
5 Discussion

1. In Figure 1, we obtained the time evolution of the curve $\kappa = \mathrm{sech}(0.4t + 0.5s), \tau = 0.2$. We found that $\kappa \to 0$ as $t \to \pm\infty$ and $\tau \to \pm 1$ as $t \to \pm\infty$. Thus, for large values of t, the curve straightens out at both ends.
2. In Figure 4, we obtained the time evolution of the curve $\kappa = 0.5, \tau = \sin(t+s)$. We found that that $\tau \to \pm 1$ as $t \to \pm\infty$. Thus, for large values of t, the curve collapses into helix.
3. In Figure 7, we obtained the time evolution of the curve $\kappa = 1, \tau = \tanh(t+s)$. We found that that $\kappa \to 1$ as $t \to \pm\infty$ and $\tau \to \pm 1$ as $t \to \pm\infty$. Thus, for large values of t, the curve collapses into helix.
4. In Figure 10, we obtained the time evolution of the curve $\kappa = \sqrt{2}\mathrm{sech}(t+s), \tau = \tanh(s+t)$. We found that $\kappa \to 0$ as $t \to \pm\infty$ and $\tau \to \pm 1$ as $t \to \pm\infty$. Thus,



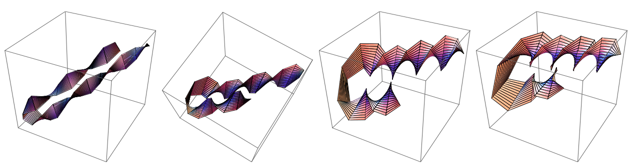
(a) t=0 (b) t=1 (c) t=2 (d) t=3

Fig. 10: Time evolution of the curve $\kappa = \sqrt{2}\text{sech}(t+s), \tau = \tanh(s+t)$



(a) t=0 (b) t=1 (c) t=2 (d) t=3

Fig. 11: Time evolution of the normal surfaces



(a) t=0 (b) t=1 (c) t=2 (d) t=3

Fig. 12: Time evolution of the binormal surfaces

for large values of t, the curve straightens out at both ends.

6 Conclusions

In this paper, we conclude that the component γ in Eqs. (2) does not affect the final shape of the evolving curve. For ruled surfaces, we present an evolution of the ruled surfaces generated by the normal and binormal of space curve. This evolution depends on the evolution of the directrix. The results that we obtained are:

1. Normal surfaces are developable if $\tau = 0$, minimal if $-u^2\kappa_s\tau + u\tau_s(u\kappa - 1) = 0$, and inextensible if $-2u\kappa_t(1-u\kappa) + 2u^2\tau\tau_t = 0$.
2. Binormal surfaces are developable if $\tau = 0$, minimal if $\tau^2 v^2\kappa + \kappa - \tau_s v = 0$, $\tau = 0$, and inextensible if $\frac{\partial \tau}{\partial t} = 0$, so $\tau = \tau_0 =$ constant (we chose $\tau_0 = 0$).

In Fig.(10), (11) and (12), we found that $\kappa \to 0$ as $s \to \pm\infty$, $\tau \to \pm 1$ as $s \to \pm\infty$. When we solve the system of Frenet-Serret, the torsion does not affect on the final shape of the evolving curve as shown in Fig.(10). But when we solve the system of Gauss-Weingarten, the curvature does not affect on the final shape of the evolving surface as shown in Figs. (11) and (12).

We linked the motion of surfaces with the motion of curves i.e., if the curve moves, then the normal and binormal surfaces moves. Geometric visualization of these surfaces are displayed.

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R. A. Hussien is a lecturer in Mathematics Department, Faculty of Science, Assiut University, Egypt. Her research interests are in differential geometry, variational problems and evolution of curves and surfaces.

Taha Youssef is Ph. D. student in Mathematics Department, Faculty of Science, Assiut University, Egypt. His Master is on ”Geometric Visualization of Evolution of Curves and Surfaces Prescribed by their Curvatures”. His research interests are in differential geometry and evolution of curves and surfaces.