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# On Generalized Fractional Order Difference Sequence Spaces Defined by a Sequence of Modulus Functions

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**Abstract:** In this paper, we introduce new generalized fractional order difference sequence spaces which are defined by a sequence of modulus functions. Different algebraic and topological properties of these spaces like linearity, completeness and solidity ... etc has been studied. Furthermore, we derive necessary and sufficient conditions for the inclusion relations involving these spaces.

**Keywords:** Fractional order difference operator, Modulus function, Paranorm, Sequence space, Solid space.

## 1 Notations and Definitions

In this section, we list some notations and definitions which can be used in the sequel (all of them may be conveniently found in [4], [8], [12], [14], [16]): let $s$ be the space of all real or complex-valued sequences. Any subspace $X$ of $s$ is called a sequence space. Let $\ell_\infty$, $c$ and $c_0$ be the spaces of all bounded, convergent and null sequences $x = (x_k)$ with real or complex terms, respectively, normed by $\|x\|_\infty = \sup_k |x_k|$, where $k \in \mathbb{N} = \{1,2,3,...\}$.
Throughout this paper, we will use the following inequality, given $(p_n)$ a sequence of positive real numbers such that $0 < h = \inf_n p_n \leq p_n \leq \sup_n p_n = H < \infty$,

$$|a_n + b_n|^{p_n} \leq D\{|a_n|^{p_n} + |b_n|^{p_n}\},\ \textit{where}\ a_n, b_n \in \mathbb{C}\ \textit{and}\ D = \max\{1, 2^{H-1}\}. \tag{1}$$

Let $\Gamma(m)$ be the Gamma function of a real number $m$ and $m \neq 0,-1,-2,....$ By the definition, it can be expressed as an improper integral

$$\Gamma(m) = \int_0^\infty e^{-t} t^{m-1} \mathrm{d}t. \tag{2}$$

**Definition 1.1.** A Modulus function is a function $f : [0,\infty) \to [0,\infty)$ such that $f(x) = 0$ if and only if $x = 0$, $f(x+y) \leq f(x) + f(y)\ \forall\ x,y \geq 0$, $f$ is increasing and continuous from the right of zero.

**Remark 1.2.** If $f$ is a modulus function, then $g(x) = f(x^p)$ is also a modulus function for $p \leq 1$.
**Remark 1.3.** If $f$ is a modulus function, then $f(\mu x) \leq 2\mu f(x)\ \forall\ \mu \in \mathbb{R}$, $\mu > 1$.
In fact, $\forall\ \mu \in \mathbb{R}$, we have $2^n \leq \mu \leq 2^{n+1}\ \forall\ n \in \mathbb{N}$ and from conditions of modulus function, we have $f(\mu x) \leq f(2^{n+1}x) \leq 2^{n+1} f(x) \leq 2\mu f(x)$.
**Definition 1.4.** A paranormed space $X = (X,g)$ is a topological linear space in which the topology is given by a paranorm $g$ such that $g$ is a real subadditive function on $X$,
$g(x) \geq 0\ \forall\ x \in X$, $g(\theta) = 0$, where $\theta$ is the zero in the linear space $X$, $g(-x) = g(x)\ \forall\ x \in X$ and the scalar multiplication is continuous, i.e. $\forall\ (x_n) \in X$: $g(x_n - x) \to 0$ as $n \to \infty$ and $\forall\ (\eta_n) \in \mathbb{R}$: $|\eta_n - \eta| \to 0$ as $n \to \infty$, we get $g(\eta_n x_n - \eta x) \to 0$ as $n \to \infty$.
Let $X,Y \subset s$, then we shall write

$$\mathscr{M}(X,Y) = \bigcap_{x \in X} x^{-1} * Y = \{z \in s : zx \in Y\ \forall\ x \in X\}.$$

The set $X^\delta = \mathscr{M}(X,\ell_1)$ is called Köthe-Toeplitz dual of $X$.
If $X \subset Y$, then $Y^\delta \subset X^\delta$. It is clear that $X \subset (X^\delta)^\delta = X^{\delta\delta}$.
If $X = X^{\delta\delta}$, then $X$ is called a *Köthe* space or a perfect sequence space.
**Definition 1.5.** A linear sequence space $X$ is called:

1. Solid (normal), if $[y = (y_n) \in X$ whenever $|y_n| \leq |x_n|\ \forall\ n \in \mathbb{N}$ for some $x = (x_n) \in X]$. Solidity

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(normality) of $X$ is equivalent to $[(\beta_n x_n) \in X$ whenever $x = (x_n) \in X$ for all $\beta = (\beta_n) \in \mathbb{R}$ with $|\beta_n| \leq 1 \, \forall \, n \in \mathbb{N}]$.

2. Monotone, if $X$ contains the canonical preimages of all its step spaces.
3. Sequence algebra, if $x = (x_n) \in X$ and $y = (y_n) \in X$ implies $(x_n y_n) \in X$.
4. Convergence free, if $x = (x_n) \in X$ implies $y = (y_n) \in X$ and $x_n = 0$ implies $y_n = 0$.
5. Symmetric, if $x = (x_n) \in X$ implies $(x_{\pi(n)}) \in X$, where $\pi(n)$ is a permutation of $\mathbb{N}$.
6. Perfect, if $X = X^{\delta\delta}$.

## 2 Motivation and Introduction

On 1981, Kizmaz [9] defined the difference sequence spaces:
$X(\Delta) = \{x = (x_k) \in s : (\Delta x_k) \in X\}, \; for \; X \in \{\ell_\infty, c, c_0\}$, where $\Delta x = (x_k - x_{k+1})$. They are Banach spaces with the norm $\|x\|_\Delta = |x_1| + \|\Delta x\|_\infty$. During the last 35 years, a lot of results have been found by many mathematicians satisfying more various generalizations of difference sequence spaces defined by Kizmaz.

First generalization was introduced by Colak and Et [3] who defined the sequence spaces: $X(\Delta^m) = \{x = (x_k) \in s : (\Delta^m x_k) \in X\}, \; for \; X \in \{\ell_\infty, c, c_0\}$, where $m \in \mathbb{N}$, $\Delta^0 x = (x_k)$, $\Delta x = (x_k - x_{k+1})$, $\Delta^m x = (\Delta^{m-1} x_k - \Delta^{m-1} x_{k+1})$. So
$\Delta^m x_k = \sum_{i=0}^{m} (-1)^i \begin{pmatrix} m \\ i \end{pmatrix} x_{k+i}$. They are Banach spaces with the norm
$\|x\|_{\Delta^m} = \sum_{i=1}^{m} |x_i| + \|\Delta^m x\|_\infty$.

Later, Et and Esi [5] defined the following sequence spaces:
$X(\Delta_\nu^m) = \{x = (x_k) \in s : (\Delta_\nu^m x_k) \in X\}, \; for \; X \in \{\ell_\infty, c, c_0\}$, where $\nu = (\nu_k)$ is any fixed sequence of non-zero complex numbers, $m \in \mathbb{N}$ is a fixed number, $\Delta_\nu^0 x_k = (\nu_k x_k)$, $\Delta_\nu x_k = (\nu_k x_k - \nu_{k+1} x_{k+1})$ and $\Delta_\nu^m x_k = (\Delta_\nu^{m-1} x_k - \Delta_\nu^{m-1} x_{k+1})$. So
$\Delta_\nu^m x_k = \sum_{i=0}^{m} (-1)^i \begin{pmatrix} m \\ i \end{pmatrix} \nu_{k+i} x_{k+i}$.

After that, Baliarsingh and Dutta [2], [4] unified most of the difference sequence spaces defined earlier and extended these results to the fractional case. For a positive proper fraction $\alpha$ and a bounded sequence of positive reals $(p_k)$, they introduced the fractional order difference sequence spaces:
$X(\Delta^\alpha, (p_k)) = \{x = (x_k) \in s : (\Delta^\alpha x_k) \in X((p_k))\}, \; for \; X \in \{\ell_\infty, c, c_0\}$, where $\Delta^\alpha$ is called the fractional order difference operator and defined by:
$\Delta^\alpha x_k = \sum_{i=0}^{\infty} (-1)^i \frac{\Gamma(\alpha+1)}{i! \Gamma(\alpha - i + 1)} x_{k+i}$.

For a modulus $f$, the following sequence spaces are defined by Maddox [13] and Ruckle [15]: $X(f) = \{x = (x_k) \in s : (f(|x_k|)) \in X\}, \; for \; X \in \{\ell_\infty, c, c_0\}$.

Kolk [10], [11] gave an extension of these sequence spaces by considering a sequence of modulus functions $(f_k)$ as follows:
$X((f_k)) = \{x = (x_k) \in s : (f_k(|x_k|)) \in X\}, \; for \; X \in \{\ell_\infty, c, c_0\}$.

After then, Gaur and Mursaleen [6] defined the following sequence spaces:
$X((f_k), \Delta) = \{x = (x_k) \in s : (f_k(|\Delta x_k|)) \in X\}, \; for \; X \in \{\ell_\infty, c, c_0\}$, for any sequence of modulus functions $(f_k)$.

Recently, Khan [7], [8] defined the following sequence spaces for any sequence of real numbers $(p_k)$ and any sequence of modulus functions $(f_k)$:
$X((f_k), (p_k)) = \{x = (x_k) \in s : (f_k(|x_k|)) \in X((p_k))\}, \; for \; X \in \{\ell_\infty, c, c_0\}$
and $X((f_k), (p_k), \Delta_\nu^m) = \{x = (x_k) \in s : (\Delta_\nu^m x_k) \in X((f_k), (p_k))\}, \; for \; X \in \{\ell_\infty, c, c_0\}$. The following two lemmas will be used in the sequel:

**Lemma 2.1.** The condition $\sup_k f_k(t) < \infty$, $t > 0$ holds if and only if there exists a point $t_0 > 0$ such that $\sup_k f_k(t_0) < \infty$.

**Lemma 2.2.** The condition $\inf_k f_k(t) > 0$, $t > 0$ holds if and only if there exists a point $t_0 > 0$ such that $\inf_k f_k(t_0) > 0$.

## 3 Main Results

In this section, we construct newly fractional order difference sequence spaces using a sequence of modulus functions. It splits into two major sections; the first one has been devoted for studying their different algebraic and topological properties and the other has been devoted for studying necessary and sufficient conditions for the inclusion relations between them.

### *3.1 Algebraic and topological properties*

In this section, we define these spaces and study some algebraic and topological properties of them like linearity, completeness and solidity, etc.

**Definition 3.1.** Let $(f_k)$ be a sequence of modulus functions and $(p_k)$ be a bounded sequence of real numbers such that $1 \leq p_k < \infty$, define

$$X((f_k), (p_k), \Delta_\nu^\alpha) = \{x = (x_k) \in s : (\Delta_\nu^\alpha x_k) \in X((f_k), (p_k))\}, \; for \; X \in \{\ell_\infty, c, c_0\},$$

where $\Delta_\nu^\alpha x_k = \sum_{i=0}^{\infty} (-1)^i \frac{\Gamma(\alpha+1)}{i! \Gamma(\alpha - i + 1)} \nu_{k+i} x_{k+i}$, $\alpha$ is a proper fraction and $\nu = (\nu_k)$ is any fixed sequence of non-zero complex numbers.

**Remark 3.2.** The fractional order difference operator generalizes the following operators:

1. Operator considered by Khan [8] (for $\alpha = m \in \mathbb{N}$).
2. Operator defined by Et and Colak [3] (for $\alpha = m \in \mathbb{N}$ and $\nu_k = 1$).
3. Operator considered by Ahmad and Mursaleen [1] (for $\alpha = 1$).

4. Operator introduced by Kizmaz [9] (for $\alpha = 1$ and $v_k = 1$).

**Theorem 3.3.** The sequence spaces $\ell_\infty((f_k),(p_k),\Delta_v^\alpha)$, $c((f_k),(p_k),\Delta_v^\alpha)$ and $c_0((f_k),(p_k),\Delta_v^\alpha)$, where $(f_k)$, $(p_k)$, $\Delta_v^\alpha x_k$, $\alpha$ and $v$ are as mentioned above in Definition (3.1), are linear spaces over the field of real numbers $\mathbb{R}$.

**Proof.** We establish it for the case $c_0((f_k),(p_k),\Delta_v^\alpha)$ and rest of the cases will follow similarly.

1. Let $x,y \in c_0((f_k),(p_k),\Delta_v^\alpha)$, then $f_k(|\Delta_v^\alpha x_k|^{p_k}) \to 0 \; as \; k \to \infty$ and $f_k(|\Delta_v^\alpha y_k|^{p_k}) \to 0 \; as \; k \to \infty$. By using Inequality (1), we have: $f_k(|\Delta_v^\alpha(x_k + y_k)|^{p_k}) \leq f_k(2^{H-1}\{|\Delta_v^\alpha x_k|^{p_k} + |\Delta_v^\alpha y_k|^{p_k}\})$. By using Remark (1.3), we have: $f_k(|\Delta_v^\alpha(x_k + y_k)|^{p_k}) \leq 2^H\{f_k(|\Delta_v^\alpha x_k|^{p_k}) + f_k(|\Delta_v^\alpha y_k|^{p_k})\}$, where $H = \sup_k p_k$ and $1 \leq p_k < \infty$. Therefore $f_k(|\Delta_v^\alpha(x_k + y_k)|^{p_k}) \to 0 \; as \; k \to \infty$. Hence $x+y \in c_0((f_k),(p_k),\Delta_v^\alpha)$.
2. Let $x \in c_0(F,p,\Delta_v^\alpha)$ and $\lambda \in \mathbb{R}$, then we have $f_k(|\Delta_v^\alpha x_k|^{p_k}) \to 0 \; as \; k \to \infty$. Since $(p_k)$ is a bounded sequence, then $f_k(|\Delta_v^\alpha(\lambda x_k)|^{p_k}) \leq f_k(\sup_k |\lambda|^{p_k} \Delta_v^\alpha x_k|^{p_k})$. By using Remark (1.3), we have: $f_k(|\Delta_v^\alpha(\lambda x_k)|^{p_k}) \leq 2\sup_k |\lambda|^{p_k} f_k(|\Delta_v^\alpha x_k|^{p_k})$. Therefore $f_k(|\Delta_v^\alpha \lambda x_k|^{p_k}) \to 0 \; as \; k \to \infty$. Hence $\lambda x \in c_0((f_k),(p_k),\Delta_v^\alpha)$.

**Corollary 3.4.** The sequence spaces $\ell_\infty((f_k),(p_k),\Delta_v^j)$, $c((f_k),(p_k),\Delta_v^j)$ and $c_0((f_k),(p_k),\Delta_v^j)$, where $j : 0,1,2,...,m$ and $m \in \mathbb{N}$, are linear spaces over $\mathbb{R}$.

**Theorem 3.5.** The sequence spaces $\ell_\infty((f_k),(p_k),\Delta_v^\alpha)$, $c((f_k),(p_k),\Delta_v^\alpha)$ and $c_0((f_k),(p_k),\Delta_v^\alpha)$, where $(f_k)$, $(p_k)$, $\Delta_v^\alpha x_k$, $\alpha$ and $v$ are as mentioned above in Definition (3.1), are Banach spaces with the norm

$$\|x\|_{\Delta_v^\alpha} = \sum_{i=1}^{\lceil\alpha\rceil} |x_i| + \|\Delta_v^\alpha x\|_\infty = \sum_{i=1}^{\lceil\alpha\rceil} |x_i| + \sup_k |\Delta_v^\alpha x_k|,$$

where $\lceil\alpha\rceil = [\alpha + 1]$ is the integral part of $\alpha + 1$ (the ceiling of $\alpha$).

**Proof.** We establish it for the case $c_0((f_k),(p_k),\Delta_v^\alpha)$ and rest of the cases will follow similarly. Let $(x^n)$ be any Cauchy sequence in $c_0((f_k),(p_k),\Delta_v^\alpha)$, where $x^n = (x_k^n) = (x_1^n, x_2^n, x_3^n, ...) \in c_0((f_k),(p_k),\Delta_v^\alpha) \; \forall \, n \in \mathbb{N}$.
Then $\forall \; \varepsilon > 0$, $\exists \; n_0 = n_0(\varepsilon) \in \mathbb{N}$ $s.t.$ $\|x^n - x^m\|_{\Delta_v^\alpha} < \varepsilon \; \forall \, n,m \geq n_0$.
Using the definition of the norm, we get:

$$\|x^n - x^m\|_{\Delta_v^\alpha} = \sum_{i=1}^{\lceil\alpha\rceil} |x_i^n - x_i^m| + \sup_k |\Delta_v^\alpha(x_k^n - x_k^m)| < \varepsilon \; \forall \, n,m \geq n_0. \quad (3)$$

For fixed $k$, $|x_k^n - x_k^m| < \varepsilon \; \forall \, n,m \geq n_0$. Then $(x_k^n)_{n=1}^\infty$ is a Cauchy sequence in $\mathbb{R} \; \forall \, k$. By completeness of $\mathbb{R}$, we have $(x_k^n)$ is convergent to a point in $\mathbb{R}$,
i.e. $\lim_{n\to\infty} x_k^n = x_k^0 \; (say) \; \in \mathbb{R} \; \forall \, k$.
Taking limit as $m \to \infty$ in (3), we get:

$$\sum_{i=1}^{\lceil\alpha\rceil} |x_i^n - x_i^0| + \sup_k |\Delta_v^\alpha(x_k^n - x_k^0)| < \varepsilon \; \forall \, n \geq n_0.$$

Hence $(x_k^n - x_k^0) \in c_0((f_k),(p_k),\Delta_v^\alpha)$. It is known from Theorem (3.3) that $c_0((f_k),(p_k),\Delta_v^\alpha)$ is a linear space and since $(x_k^n) \in c_0((f_k),(p_k),\Delta_v^\alpha)$ and $(x_k^n - x_k^0) \in c_0((f_k),(p_k),\Delta_v^\alpha)$, then $(x_k^0) = (x_k^n) - (x_k^n - x_k^0) \in c_0((f_k),(p_k),\Delta_v^\alpha)$. This completes the proof.

**Corollary 3.6.** The sequence spaces $\ell_\infty((f_k),(p_k),\Delta_v^j)$, $c((f_k),(p_k),\Delta_v^j)$ and $c_0((f_k),(p_k),\Delta_v^j)$ are Banach spaces with the norm

$$\|x\|_{\Delta_v^j} = \sum_{i=1}^{j} |x_i| + \sup_k |\Delta_v^j x_k|, \; where \; j : 0,1,2,...,m \; and \; m \in \mathbb{N}.$$

**Theorem 3.7.** The sequence spaces $\ell_\infty((f_k),(p_k),\Delta_v^\alpha)$, $c((f_k),(p_k),\Delta_v^\alpha)$ and $c_0((f_k),(p_k),\Delta_v^\alpha)$, where $(f_k)$, $(p_k)$, $\Delta_v^\alpha x_k$, $\alpha$ and $v$ are as mentioned above in Definition (3.1), are quasi paranormed spaces with the quasi paranorm

$$g(x) = \sup_k [f_k(|\Delta_v^\alpha x_k|^{p_k})]^{\frac{1}{H}}, \; where \; H = \sup_k p_k \; and \; 1 \leq p_k < \infty.$$

**Proof.** We establish it for the case $c_0((f_k),(p_k),\Delta_v^\alpha)$ and rest of the cases will follow similarly.

1. Let $x,y \in c_0((f_k),(p_k),\Delta_v^\alpha)$.
By using Inequality (1), we have: $g(x+y) \leq \sup_k [f_k(2^{H-1}\{|\Delta_v^\alpha x_k|^{p_k} + |\Delta_v^\alpha y_k|^{p_k}\})]^{\frac{1}{H}}$, where $H = \sup_k p_k$ and $1 \leq p_k < \infty$.
By using Remark (1.3), we have:
$$g(x+y) \leq 2\sup_k [f_k(|\Delta_v^\alpha x_k|^{p_k}) + f_k(|\Delta_v^\alpha y_k|^{p_k})]^{\frac{1}{H}}$$
$$\leq 2[\sup_k \{f_k(|\Delta_v^\alpha x_k|^{p_k})\}^{\frac{1}{H}} + \sup_k \{f_k(|\Delta_v^\alpha y_k|^{p_k})\}^{\frac{1}{H}}].$$
Then
$g(x+y) \leq 2[g(x) + g(y)] \; \forall \, x,y \in c_0((f_k),(p_k),\Delta_v^\alpha)$.
2. $\forall \; x \in c_0((f_k),(p_k),\Delta_v^\alpha)$, we have: $g(x) = \sup_k [f_k(|\Delta_v^\alpha x_k|^{p_k})]^{\frac{1}{H}} \geq 0$, and it is trivial that $\Delta_v^\alpha x_k = 0$ for $x = \theta$, then $g(\theta) = 0$, where $\theta$ is the zero in the linear space $c_0((f_k),(p_k),\Delta_v^\alpha)$.
3. $\forall \, x \in c_0((f_k),(p_k),\Delta_v^\alpha)$, we have:
$$g(-x) = \sup_k [f_k(|\Delta_v^\alpha(-x_k)|^{p_k})]^{\frac{1}{H}} = \sup_k [f_k(|\Delta_v^\alpha x_k|^{p_k})]^{\frac{1}{H}} = g(x).$$
Then $g(-x) = g(x) \; \forall \, x \in c_0((f_k),(p_k),\Delta_v^\alpha)$.
4. We want to prove that the scalar multiplication is continuous, let $(x_n) \in c_0((f_k),(p_k),\Delta_v^\alpha)$ s.t.

$g(x_n - x) \to 0$ as $n \to \infty$ and let $(\eta_n) \in \mathbb{R}$ with $|\eta_n - \eta| \to 0$ as $n \to \infty$,

$$\begin{aligned} g(\eta_n x_n - \eta x) &= g(\eta_n x_n - \eta x + \eta_n x - \eta_n x) \\ &\le 2\{g(\eta_n(x_n - x)) + g((\eta_n - \eta)x)\}. \end{aligned}$$

Since $(\eta_n)$ is convergent, then it is bounded. So

$$g(\eta_n x_n - \eta x) \le 2CKg(x_n - x) + 2C(\eta_n - \eta)g(x),$$

where $C$ and $K$ are constants.
Then $g(\eta_n x_n - \eta x) \to 0$ as $n \to \infty$.

**Corollary 3.8.** The sequence spaces $\ell_\infty((f_k),(p_k),\Delta_\nu^j)$, $c((f_k),(p_k),\Delta_\nu^j)$ and $c_0((f_k),(p_k),\Delta_\nu^j)$ are quasi paranormed spaces with the quasi paranorm

$$g(x) = \sup_k [f_k(|\Delta_\nu^j x_k|^{p_k})]^{\frac{1}{H}}, \text{ where } H = \sup_k p_k, 1 \le p_k < \infty, j : 0,1,2,...,m \text{ and } m \in \mathbb{N}.$$

**Theorem 3.9.** The sequence spaces $\ell_\infty((f_k),(p_k),\Delta_\nu^\alpha)$, $c((f_k),(p_k),\Delta_\nu^\alpha)$ and $c_0((f_k),(p_k),\Delta_\nu^\alpha)$, where $(f_k)$, $(p_k)$, $\Delta_\nu^\alpha x_k$, $\alpha$ and $\nu$ are as mentioned above in Definition (3.1), are not solid for $\alpha \ge 1$.
**Proof.** To show that these spaces are not solid in general, consider the following example: for $\ell_\infty((f_k),(p_k),\Delta_\nu^\alpha)$, let $\nu = (1,1,1,...)$, $p_k = 1$, $f_k(x) = x$, $x = (x_k) = (k^\alpha)$ and $\beta_k = (-1)^k$, $\forall\, k \in \mathbb{N}$. Then $x = (k^\alpha) \in \ell_\infty((f_k),(p_k),\Delta_\nu^\alpha)$ but $(\beta_k x_k) \notin \ell_\infty((f_k),(p_k),\Delta_\nu^\alpha)$. Hence $\ell_\infty((f_k),(p_k),\Delta_\nu^\alpha)$ is not solid in general. The other cases can be proved on considering similar examples.
**Remark 3.10.** The sequence spaces $\ell_\infty((f_k),(p_k))$, $c((f_k),(p_k))$ and $c_0((f_k),(p_k))$, where $(f_k)$ and $(p_k)$ are as mentioned above in Definition (3.1), are solid.
**Proposition 3.11.** The sequence spaces $\ell_\infty((f_k),(p_k),\Delta_\nu^\alpha)$, $c((f_k),(p_k),\Delta_\nu^\alpha)$ and $c_0((f_k),(p_k),\Delta_\nu^\alpha)$, where $(f_k)$, $(p_k)$, $\Delta_\nu^\alpha x_k$, $\alpha$ and $\nu$ are as mentioned above in Definition (3.1), are not perfect for $\alpha \ge 1$.
**Theorem 3.12.** The sequence spaces $\ell_\infty((f_k),(p_k),\Delta_\nu^\alpha)$ and $c((f_k),(p_k),\Delta_\nu^\alpha)$, where $(f_k)$, $(p_k)$, $\Delta_\nu^\alpha x_k$, $\alpha$ and $\nu$ are as mentioned above in Definition (3.1), are not symmetric for $\alpha \ge 1$.
**Proof.** To show that these spaces are not symmetric in general, consider the following example: for $\ell_\infty((f_k),(p_k),\Delta_\nu^\alpha)$, let $\nu = (1,1,1,...)$, $p_k = 1$, $f_k(x) = x$ and $x = (x_k) = (k^\alpha)$, $\forall\ k \in \mathbb{N}$. Then $x = (k^\alpha) \in \ell_\infty((f_k),(p_k),\Delta_\nu^\alpha)$. Let $y = (y_k)$ be a rearrangement of $(x_k)$, which is defined as follows:

$$(y_k) = \{x_1,x_2,x_4,x_3,x_9,x_5,x_{16},x_6,x_{25},x_7,x_{36},x_8,x_{49},x_{10},...\}.$$

Then $y \notin \ell_\infty((f_k),(p_k),\Delta_\nu^\alpha)$. Hence $\ell_\infty((f_k),(p_k),\Delta_\nu^\alpha)$ is not symmetric in general. We can prove that $c((f_k),(p_k),\Delta_\nu^\alpha)$ is not symmetric on considering similar example.
**Remark 3.13.** The space $c_0((f_k),(p_k),\Delta_\nu^\alpha)$ is not symmetric for $\alpha \ge 2$.
**Theorem 3.14.** The sequence spaces $\ell_\infty((f_k),(p_k),\Delta_\nu^\alpha)$, $c((f_k),(p_k),\Delta_\nu^\alpha)$ and $c_0((f_k),(p_k),\Delta_\nu^\alpha)$, where $(f_k)$, $(p_k)$, $\Delta_\nu^\alpha x_k$, $\alpha$ and $\nu$ are as mentioned above in Definition (3.1), are not sequence algebras.
**Proof.** To show that these spaces are not sequence algebras in general, consider the following example: for $\ell_\infty((f_k),(p_k),\Delta_\nu^\alpha)$, let $\nu = (1,1,1,...)$, $p_k = 1$, $f_k(x) = x$, $\forall\, k \in \mathbb{N}$. Consider the sequences $x = (x_k) = (k^{\alpha-2})$ and $y = (y_k) = (k^{\alpha-2})$, then $x,y \in \ell_\infty((f_k),(p_k),\Delta_\nu^\alpha)$ but $x \cdot y \notin \ell_\infty((f_k),(p_k),\Delta_\nu^\alpha)$. Hence $\ell_\infty((f_k),(p_k),\Delta_\nu^\alpha)$ is not sequence algebra in general. The other cases can be proved on considering similar examples.

### *3.2 Inclusion theorems*

In this section, we obtain necessary and sufficient conditions for the inclusion relations between $X(\Delta_\nu^\alpha)$ and $Y((f_k),(p_k),\Delta_\nu^\alpha)$, with $X,Y = \ell_\infty \text{ or } c_0$.
**Theorem 3.15.** Under assumptions of Definition (3.1), the following are equivalent:

1. $\ell_\infty(\Delta_\nu^\alpha) \subseteq \ell_\infty((f_k),(p_k),\Delta_\nu^\alpha)$.
2. $c_0(\Delta_\nu^\alpha) \subseteq \ell_\infty((f_k),(p_k),\Delta_\nu^\alpha)$.
3. $\sup_k f_k(t) < \infty$, $t > 0$.

**Proof.** $(1) \Rightarrow (2)$: is obvious, since $c_0(\Delta_\nu^\alpha) \subseteq \ell_\infty(\Delta_\nu^\alpha)$.
$(2) \Rightarrow (3)$: Let $c_0(\Delta_\nu^\alpha) \subseteq \ell_\infty((f_k),(p_k),\Delta_\nu^\alpha)$. Suppose that $(3)$ is not satisfied. Then by Lemma (2.1), $\sup_k f_k(t) = \infty$ for all $t > 0$ and therefore there exists a sequence $(k_i)$ of positive integers such that

$$f_{k_i}(i^{-1}) > i, for\ i = 1,2,.... \tag{4}$$

Define $x = (x_k)$ by

$$x_k = \begin{cases} i^{-1}, \ if\ k = k_i\ for\ i = 1,2,..., \\ 0,\ otherwise. \end{cases}$$

Then $x \in c_0(\Delta_\nu^\alpha)$. But $x \notin \ell_\infty(F,p,\Delta_\nu^\alpha)$ by (4) for $\nu_k = p_k = 1$ and $k \in \mathbb{N}$, which contradicts $(2)$. Hence $(3)$ must hold.
$(3) \Rightarrow (1)$: Let $(3)$ be satisfied and $x \in \ell_\infty(\Delta_\nu^\alpha)$. Suppose that $x \notin \ell_\infty((f_k),(p_k),\Delta_\nu^\alpha)$. Then

$$\sup_k f_k(|\Delta_\nu^\alpha x_k|^{p_k}) = \infty\ for\ \Delta_\nu^\alpha x \in \ell_\infty. \tag{5}$$

Let $t = |\Delta_\nu^\alpha x_k|^{p_k}$, then by (5) $\sup_k f_k(t) = \infty$, which contradicts $(3)$. Hence $(1)$ must hold.
**Corollary 3.16.** Under assumptions of Definition (3.1), the following are equivalent:

1. $\ell_\infty(\Delta_\nu^j) \subseteq \ell_\infty((f_k),(p_k),\Delta_\nu^j)$.
2. $c_0(\Delta_\nu^j) \subseteq \ell_\infty((f_k),(p_k),\Delta_\nu^j)$.
3. $\sup_k f_k(t) < \infty$, $t > 0$,

where $j : 1,2,3,...,m$ and $m \in \mathbb{N}$.
**Theorem 3.17.** Under assumptions of Definition (3.1), the following are equivalent:

1. $c_0((f_k),(p_k),\Delta_\nu^\alpha) \subseteq c_0(\Delta_\nu^\alpha)$.
2. $c_0((f_k),(p_k),\Delta_\nu^\alpha) \subseteq \ell_\infty(\Delta_\nu^\alpha)$.
3. $\inf_k f_k(t) > 0$, $t > 0$.

**Proof.** (1) $\Rightarrow$ (2): is obvious, since $c_0(\Delta_\nu^\alpha) \subseteq \ell_\infty(\Delta_\nu^\alpha)$.
(2) $\Rightarrow$ (3): Let $c_0((f_k),(p_k),\Delta_\nu^\alpha) \subseteq \ell_\infty(\Delta_\nu^\alpha)$. Suppose that (3) is not satisfied. Then by Lemma (2.2), $\inf_k f_k(t) = 0$ for all $t > 0$ and therefore there exists a sequence $(k_i)$ of positive integers such that

$$f_{k_i}(i^2) < i^{-1}, \; for \; i = 1,2,.... \tag{6}$$

Define $x = (x_k)$ by

$$x_k = \begin{cases} i^2, \; if \; k = k_i \; for \; i = 1,2,..., \\ 0, \; otherwise. \end{cases}$$

Then $x \in c_0((f_k),(p_k),\Delta_\nu^\alpha)$. But $x \notin \ell_\infty(\Delta_\nu^\alpha)$ by (6) for $v_k = p_k = 1$ and $k \in \mathbb{N}$, which contradicts (2). Hence (3) must hold.
(3) $\Rightarrow$ (1): Let (3) be satisfied and $x \in c_0((f_k),(p_k),\Delta_\nu^\alpha)$, i.e.

$$\lim_k f_k(|\Delta_\nu^\alpha x_k|^{p_k}) = 0 \; and \; x \notin c_0(\Delta_\nu^\alpha). \tag{7}$$

Then for some $\varepsilon_0 > 0$ and positive integer $k_0$, we have $|\Delta_\nu^\alpha x_k| \geq \varepsilon_0$ for $k \geq k_0$. Then $f_k(\varepsilon_0) \leq f_k(|\Delta_\nu^\alpha x_k|)$ for $k \geq k_0$. Then by (7) $\lim_k f_k(\varepsilon_0) = 0$, which contradicts (3). Hence (1) must hold.
**Corollary 3.18.** Under assumptions of Definition (3.1), the following are equivalent:

1. $c_0((f_k),(p_k),\Delta_\nu^j) \subseteq c_0(\Delta_\nu^j)$.
2. $c_0((f_k),(p_k),\Delta_\nu^j) \subseteq \ell_\infty(\Delta_\nu^j)$.
3. $\inf_k f_k(t) > 0$, $t > 0$,

where $j : 1,2,3,...,m$ and $m \in \mathbb{N}$.
**Theorem 3.19.** Under assumptions of Definition (3.1), $\ell_\infty((f_k),(p_k),\Delta_\nu^\alpha) \subseteq c_0(\Delta_\nu^\alpha)$ if and only if

$$\lim_k f_k(t) = \infty \; for \; t > 0. \tag{8}$$

**Proof.** Let $\ell_\infty((f_k),(p_k),\Delta_\nu^\alpha) \subseteq c_0(\Delta_\nu^\alpha)$. Suppose that (8) is not satisfied. Therefore there exist a number $t_0 > 0$ and an index sequence $k_i$ of positive integers such that

$$\lim_k f_{k_i}(t_0) \leq K < \infty. \tag{9}$$

Define the sequence $x = (x_k)$ by

$$x_k = \begin{cases} t_0, \; if \; k = k_i \; for \; i = 1,2,..., \\ 0, \; otherwise. \end{cases}$$

Thus $x \in \ell_\infty((f_k),(p_k),\Delta_\nu^\alpha)$ by (9). But $x \notin c_0(\Delta_\nu^\alpha)$ for $v_k = p_k = 1$ and $k \in \mathbb{N}$. So (8) is true if $\ell_\infty((f_k),(p_k),\Delta_\nu^\alpha) \subseteq c_0(\Delta_\nu^\alpha)$.
Conversely, let (8) be satisfied. Suppose $x \in \ell_\infty((f_k),(p_k),\Delta_\nu^\alpha)$ and $x \notin c_0(\Delta_\nu^\alpha)$. For $k \in \mathbb{N}$, $f_k(|\Delta_\nu^\alpha x_k|^{p_k}) \leq K < \infty$. Then for some $\varepsilon_0 > 0$ and positive integer $k_0$, we have $|\Delta_\nu^\alpha x_k| \geq \varepsilon_0$ for $k \geq k_0$. Then $f_k(\varepsilon_0) \leq f_k(|\Delta_\nu^\alpha x_k|) \leq K$ for $k \geq k_0$, which contradicts (8). Hence $\ell_\infty((f_k),(p_k),\Delta_\nu^\alpha) \subseteq c_0(\Delta_\nu^\alpha)$.
**Corollary 3.20.** Under assumptions of Definition (3.1), $\ell_\infty((f_k),(p_k),\Delta_\nu^j) \subseteq c_0(\Delta_\nu^j)$ if and only if

$$\lim_k f_k(t) = \infty \; for \; t > 0, \tag{10}$$

where $j : 1,2,3,...,m$ and $m \in \mathbb{N}$.
**Theorem 3.21.** Under assumptions of Definition (3.1), $\ell_\infty(\Delta_\nu^\alpha) \subseteq c_0((f_k),(p_k),\Delta_\nu^\alpha)$ if and only if

$$\lim_k f_k(t) = 0 \; for \; t > 0. \tag{11}$$

**Proof.** Let $\ell_\infty(\Delta_\nu^\alpha) \subseteq c_0((f_k),(p_k),\Delta_\nu^\alpha)$. Suppose that (11) is not satisfied. Then

$$\lim_k f_k(t_0) = l \neq 0. \tag{12}$$

for some $t_0 > 0$.
Define the sequence $x = (x_k)$ by

$$x_k = t_0 \sum_{\nu=0}^{k-\nu} (-1)^{\lfloor\alpha\rfloor} \frac{\Gamma(\alpha+k-\nu)}{(k-\nu)!\Gamma(\alpha)}, \; for \; k \in \mathbb{N},$$

where $\lfloor\alpha\rfloor = [\alpha]$ is the integral part of $\alpha$ (the floor of $\alpha$).
Then $x \notin c_0((f_k),(p_k),\Delta_\nu^\alpha)$ for $v_k = p_k = 1$ and $k \in \mathbb{N}$ by (12). So (11) must hold.
Conversely, let (11) be satisfied. Let $x \in \ell_\infty(\Delta_\nu^\alpha)$. Then $|\Delta_\nu^\alpha x_k| \leq M < \infty$ for $k \in \mathbb{N}$. Therefore $f_k(|\Delta_\nu^\alpha x_k|) \leq f_k(M)$ and $\lim_k f_k(|\Delta_\nu^\alpha x_k|^{p_k}) \leq \lim_k f_k(M) = 0$ by (11). Thus $x \in c_0((f_k),(p_k),\Delta_\nu^\alpha)$. Hence $\ell_\infty(\Delta_\nu^\alpha) \subseteq c_0((f_k),(p_k),\Delta_\nu^\alpha)$.
**Corollary 3.22.** Under assumptions of Definition (3.1), $\ell_\infty(\Delta_\nu^j) \subseteq c_0((f_k),(p_k),\Delta_\nu^j)$ if and only if

$$\lim_k f_k(t) = 0 \; for \; t > 0, \tag{13}$$

where $j : 1,2,3,...,m$ and $m \in \mathbb{N}$.

## 4 Conclusion

Introducing a new sequence spaces by using a sequence of modulus functions only or by using fractional order difference operator only has been studied by many mathematicians in different ways. But in our study, we introduced new sequence spaces by using fractional order difference operator together with a sequence of modulus functions. These spaces are more general than some spaces and not a special case of other spaces defined earlier. These type investigations fill some gaps in the literature. The authors can introduce new sequence spaces and results by using similar techniques in this paper.

## Conflict of interest

The authors declare that they have no conflict of interest.

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