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Some Integral Inequalities for Harmonically logarithmic h-convex functions

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Abstract: In this paper, we introduce and consider a new class of harmonically convex function which is called harmonically logarithmic h-convex function. Some other new concepts are also defined. Hermite-Hadamard type inequalities for harmonically logarithmic h-convex functions are derived. The ideas and techniques of this paper may stimulate further research in this area.

Keywords: Convex functions, harmonically convex functions, harmonically logarithmic h-convex functions, Hermite-Hadamard inequality.

1 Introduction

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function with $a < b$ and $a, b \in I$. Then the following double inequality is known as Hermite-Hadamard inequality in the literature

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1}$$

named after C. Hermite and J. Hadamard. Interested readers may find useful details on Hermite-Hadamard type of integral inequalities in [2,3,4,5,7,8,9,10,11,12,13,14,15,16,17,19,20,21,22,23,24]. Recently convexity has been extended in different directions using novel and innovative ideas, (see [1,2,3,4,6,7,8,9,11,12,13,17,18,19,21,22,23]. Iscan [7] investigated and studied a new class of convex functions, which are called harmonically convex functions. In [7] author has obtained some Hermite-Hadamard type inequalities via harmonically convex functions. Zhang et al. [22] have defined and investigated the concept of harmonically quasi-convex functions. Noor et al. [11] and Noor et al. [12] have defined and studied the classes of harmonically h-convex functions and harmonically log-convex functions respectively.

Inspired by the ongoing research, we introduce a new class of harmonically convex function which is called harmonically logarithmic h-convex functions. We derive some new Hermite-Hadamard type inequalities for harmonically logarithmic h-convex functions. Several new results for other classes of harmonically convex functions are obtained as special cases of our results. This illustrates the significance of our new results. It is expected that interested readers may explore novel and innovative applications of the main results in other fields. This is the main motivation of this paper.

2 Preliminaries

In this section, we recall some basic previously known concepts and define the concept of harmonically logarithmic h-convex functions. We also define some other new classes of harmonically convex functions. Throughout the paper $I \subset \mathbb{R}\setminus\{0\}$ and $[0,1] \subseteq J$ be the intervals, unless otherwise specified.

Definition 1([7]). *A function $f : I \to \mathbb{R}$ is said to be harmonically convex function, if*

$$f\left(\frac{xy}{tx+(1-t)y}\right) \leq (1-t)f(x)+tf(y), \quad \forall x,y \in I, t \in [0,1]. \tag{1}$$

Noor et al. [12] noticed that for $t = \frac{1}{2}$, we have the definition of Jensen type of harmonic convex functions,

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that is

$$f\left(\frac{2xy}{x+y}\right) \leq \frac{f(x)+f(y)}{2}, \quad \forall x,y \in I. \tag{2}$$

Definition 2([22]). *A function $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}_+$ is said to be harmonically quasi-convex function on I, if*

$$f\left(\frac{xy}{tx+(1-t)y}\right) \leq \sup\{f(x), f(y)\}, \quad \forall x,y \in I, t \in [0,1]. \tag{3}$$

In [22] authors proved that harmonically convex functions on I is a harmonically quasi-convex function, but not conversely.
Noor et al. [12] introduced the concept of harmonically log-convex functions.

Definition 3([12]). *A function $f : I \to \mathbb{R}$ is said to be harmonically* log*-convex function, if*

$$f\left(\frac{xy}{tx+(1-t)y}\right) \leq (f(x))^{1-t}(f(y))^{t}, \quad \forall x,y \in I, t \in [0,1]. \tag{4}$$

From (1), (3) and (4) it follows that

$$\begin{aligned} f\left(\frac{xy}{tx+(1-t)y}\right) &\leq (f(x))^{1-t}(f(y))^{t} \\ &\leq (1-t)f(x)+tf(y) \\ &\leq \sup\{f(x), f(y)\}. \end{aligned}$$

This implies that harmonically log-convex function implies harmonically convex function which implies harmonically quasi-convex function, but the converse may not be true.
We now recall the concept of harmonically h-convex functions, which is mainly due to Noor et al. [11].

Definition 4([11]). *Let $h : J \to \mathbb{R}$ be a non-negative function, $h \neq 0$. A function $f : I \to \mathbb{R}$ is said to be harmonically h-convex function, if f is non-negative function, and*

$$f\left(\frac{xy}{tx+(1-t)y}\right) \leq h(1-t)f(x)+h(t)f(y), \ \forall x,y \in I, \ t \in (0,1). \tag{5}$$

For other properties and special cases of harmonically h-convex functions see [11].
Now we define the concept of harmonically logarithmic h-convex functions, which appears to be new one.

Definition 5.*Let $h : J \to \mathbb{R}$ be a non-negative function, $h \neq 0$. A function $f : I \to \mathbb{R}$ is said to be harmonically logarithmic h-convex function, if*

$$f\left(\frac{xy}{tx+(1-t)y}\right) \leq (f(x))^{h(1-t)}(f(y))^{h(t)}, \quad \forall x,y \in I, t \in (0,1). \tag{6}$$

Note that

$$\log f\left(\frac{xy}{tx+(1-t)y}\right) \leq h(1-t)\log f(x)+h(t)\log f(y). \tag{7}$$

Note that if f be harmonically logarithmic h-convex function. Then $h(t)+h(1-t) \geq 1$ implies $f(x) \geq 1$. And similarly $h(t)+h(1-t) \leq 1$ implies $f(x) \leq 1$.

For suitable and appropriate choice of the h-convex function, we now obtain several new classes of harmonically convex functions. For example, for $h(t) = t^s$, $h(t) = 1$ and $h(t) = \frac{1}{t}$, we obtain the new definitions of harmonically logarithmic s-convex functions, harmonically logarithmic P-functions and harmonically logarithmic Q-functions respectively. For the sake of completeness and to convey the main ideas, we state these results.

Definition 6.*A function $f : I \to \mathbb{R}$ is said to be harmonically logarithmic s-convex function, if*

$$f\left(\frac{xy}{tx+(1-t)y}\right) \leq (f(x))^{(1-t)^s}(f(y))^{t^s}, \quad \forall x,y \in I, t \in [0,1]. \tag{8}$$

From this, we have

$$\log f\left(\frac{xy}{tx+(1-t)y}\right) \leq (1-t)^s \log f(x)+t^s \log f(y).$$

Definition 7.*A function $f : I \to \mathbb{R}$ is said to be harmonically logarithmic P-convex function, if*

$$f\left(\frac{xy}{tx+(1-t)y}\right) \leq f(x)f(y), \quad \forall x,y \in I, t \in [0,1]. \tag{9}$$

Thus

$$\log f\left(\frac{xy}{tx+(1-t)y}\right) \leq \log f(x)+\log f(y).$$

Definition 8.*A function $f : I \to \mathbb{R}$ is said to be harmonically logarithmic Q-convex function, if*

$$f\left(\frac{xy}{tx+(1-t)y}\right) \leq (f(x))^{\frac{1}{1-t}}(f(y))^{\frac{1}{t}}, \quad \forall x,y \in I, t \in [0,1]. \tag{10}$$

From which, we have

$$\log f\left(\frac{xy}{tx+(1-t)y}\right) \leq \frac{1}{1-t}\log f(x)+\frac{1}{t}\log f(y).$$

In [5] Dragomir et al. established a very novel result. Since then number of authors have utilized this result directly and indirectly to derive several new Hermite-Hadamard type of inequalities.

Lemma 1([5]). *Let $f:(a,b)\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable mapping with $a<b$. If $f'\in L[a,b]$, then the following equality holds:*

$$\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx=\frac{b-a}{2}\int_0^1(1-2t)f'(ta+(1-t)b)dt.$$

Lemma 2. *For $a,b\in I$, $t\in[0,1]$ with $a<b$, we have*

$$\frac{ab}{(1-t)a+tb}\le ta+(1-t)b.$$

3 Main Results

In this section, we prove our main results for the new class of harmonically logarithmic h-convex functions.

Theorem 1. *Let $f:I\to\mathbb{R}$ be harmonically logarithmic h-convex function, where $a,b\in I$ and $a<b$. Then*

$$f\left(\tfrac{2ab}{a+b}\right)^{\frac{1}{2h(\frac{1}{2})}}\le\exp\left[\tfrac{ab}{b-a}\int_a^b\log\left(\tfrac{f(x)}{x^2}\right)dx\right]\le(f(a)f(b))^{\int_0^1 h(t)dt}. \tag{11}$$

Proof. Let f be harmonically logarithmic h-convex function. For $t=\frac{1}{2}$ in (6), we have

$$f\left(\frac{2xy}{x+y}\right)\le[(f(x))(f(y))]^{h(\frac{1}{2})}.$$

This implies that

$$f\left(\frac{2ab}{a+b}\right)\le\left[\left\{f\left(\frac{ab}{(1-t)a+tb}\right)\right\}\left\{f\left(\frac{ab}{ta+(1-t)b}\right)\right\}\right]^{h(\frac{1}{2})}.$$

From this it follows that

$$\log f\left(\frac{2ab}{a+b}\right)\le h(\tfrac{1}{2})\left[\log f\left(\frac{ab}{(1-t)a+tb}\right)+\log f\left(\frac{ab}{ta+(1-t)b}\right)\right].$$

Integrating above inequality with respect to t on $[0,1]$, we have

$$\log f\left(\frac{2ab}{a+b}\right)\le h(\tfrac{1}{2})\left[\int_0^1\log f\left(\frac{ab}{(1-t)a+tb}\right)dt+\int_0^1\log f\left(\frac{ab}{ta+(1-t)b}\right)dt\right].$$

This implies that

$$\frac{1}{2h(\frac{1}{2})}\log f\left(\frac{2ab}{a+b}\right)\le\frac{ab}{b-a}\int_a^b\log\left(\frac{f(x)}{x^2}\right)dx. \tag{12}$$

Also integrating inequality (7) with respect to t on $[0,1]$, we have

$$\tfrac{ab}{b-a}\int_a^b\log\left(\tfrac{f(x)}{x^2}\right)dx\le(\log(f(a))+\log(f(b)))\int_0^1 h(t)dt=\log(f(a)f(b))^{\int_0^1 h(t)dt}. \tag{13}$$

Combining (12) and (13), we have

$$\log f\left(\tfrac{2ab}{a+b}\right)^{\frac{1}{2h(\frac{1}{2})}}\le\tfrac{ab}{b-a}\int_a^b\log\left(\tfrac{f(x)}{x^2}\right)dx\le\log(f(a)f(b))^{\int_0^1 h(t)dt}.$$

From which, we obtain

$$f\left(\tfrac{2ab}{a+b}\right)^{\frac{1}{2h(\frac{1}{2})}}\le\exp\left[\tfrac{ab}{b-a}\int_a^b\log\left(\tfrac{f(x)}{x^2}\right)dx\right]\le(f(a)f(b))^{\int_0^1 h(t)dt}.$$

This completes the proof.

For $h(t)=t^s$, $h(t)=1$ and $h(t)=\frac{1}{t}$, we obtain the new results for harmonically logarithmic s-convex functions, harmonically logarithmic P-functions and harmonically logarithmic Q-functions respectively.

Corollary 1. *Let $f:I\to\mathbb{R}$ be harmonically logarithmic s-convex function, where $a,b\in I$ with $a<b$ and $s\in(0,1)$. Then*

$$f\left(\tfrac{2ab}{a+b}\right)^{2^{s-1}}\le\exp\left[\tfrac{ab}{b-a}\int_a^b\log\left(\tfrac{f(x)}{x^2}\right)dx\right]\le(f(a)f(b))^{\frac{1}{s+1}}.$$

Corollary 2. *Let $f:I\to\mathbb{R}$ be harmonically logarithmic P-function, where $a,b\in I$ with $a<b$. Then*

$$f\left(\tfrac{2ab}{a+b}\right)^{\frac{1}{2}}\le\exp\left[\tfrac{ab}{b-a}\int_a^b\log\left(\tfrac{f(x)}{x^2}\right)dx\right]\le(f(a)f(b)).$$

Corollary 3. *Let $f:I\to\mathbb{R}$ be harmonically logarithmic Q-convex function, where $a,b\in I$ with $a<b$. Then*

$$f\left(\frac{2ab}{a+b}\right)^{\frac{1}{4}}\le\exp\left[\frac{ab}{b-a}\int_a^b\log\left(\frac{f(x)}{x^2}\right)dx\right].$$

Theorem 2. *Let $f,g:I\to\mathbb{R}$ be harmonically logarithmic h-convex function, where $a,b\in I$ with $a<b$. Then*

$$\begin{aligned}&\frac{ab}{b-a}\int_a^b\left(\frac{f(x)g(x)}{x^2}\right)dx\\&\le\frac{1}{4}\left[\int_0^1\left\{(f(a))^{2h(1-t)}(f(b))^{2h(t)}\right\}dt\right.\\&+\int_0^1\left\{(g(a))^{2h(1-t)}(g(b))^{2h(t)}\right\}dt\\&\left.+2\int_0^1\left\{(f(a)g(a))^{h(1-t)}(f(b)g(b))^{h(t)}\right\}dt\right].\end{aligned}$$

Proof. Let f be harmonically logarithmic h-convex function. Then using inequality

$ab \leq \frac{1}{4}(a+b)^2,\ \forall a,b \in \mathbb{R}$, we have

$$\begin{aligned}
&\frac{ab}{b-a}\int_a^b \left(\frac{f(x)g(x)}{x^2}\right)dx \\
&= \int_0^1 f\left(\frac{ab}{ta+(1-t)b}\right) g\left(\frac{ab}{ta+(1-t)b}\right)dt \\
&\leq \frac{1}{4}\int_0^1 \left[f\left(\frac{ab}{ta+(1-t)b}\right)+g\left(\frac{ab}{ta+(1-t)b}\right)\right]^2 dt \\
&\leq \frac{1}{4}\int_0^1 \left[\left\{(f(a))^{h(1-t)}(f(b))^{h(t)}\right\}+\left\{(g(a))^{h(1-t)}(g(b))^{h(t)}\right\}\right]^2 dt \\
&= \frac{1}{4}\int_0^1 \left\{(f(a))^{2h(1-t)}(f(b))^{2h(t)}\right\}dt \\
&+ \frac{1}{4}\int_0^1 \left\{(g(a))^{2h(1-t)}(g(b))^{2h(t)}\right\}dt + \\
&\quad \frac{1}{2}\int_0^1 \left\{(f(a)g(a))^{h(1-t)}(f(b)g(b))^{h(t)}\right\}dt.
\end{aligned}$$

This completes the proof.

Theorem 3. *Let $f,g: I \to \mathbb{R}$ be harmonically logarithmic h-convex function, where $a,b \in I$ with $a<b$. If $\alpha+\beta=1$, then*

$$\frac{ab}{b-a}\int_a^b \left(\frac{f(x)g(x)}{x^2}\right)dx \leq \alpha\int_0^1 \left[(f(a))^{\frac{h(1-t)}{\alpha}}(f(b))^{\frac{h(t)}{\alpha}}\right]dt + \beta\int_0^1 \left[(g(a))^{\frac{h(1-t)}{\beta}}(g(b))^{\frac{h(t)}{\beta}}\right]dt.$$

Proof. Let $f,g: I \to \mathbb{R}$ be harmonically logarithmic h-convex function. Then using inequality $ab \leq \alpha a^{\frac{1}{\alpha}}+\beta b^{\frac{1}{\beta}},\quad \alpha,\beta>0, \alpha+\beta=1$, we have

$$\begin{aligned}
&\frac{ab}{b-a}\int_a^b \left(\frac{f(x)g(x)}{x^2}\right)dx \\
&= \int_0^1 f\left(\frac{ab}{ta+(1-t)b}\right) g\left(\frac{ab}{ta+(1-t)b}\right)dt \\
&\leq \int_0^1 \left[\alpha\left\{f\left(\frac{ab}{ta+(1-t)b}\right)\right\}^{\frac{1}{\alpha}}+\beta\left\{g\left(\frac{ab}{ta+(1-t)b}\right)\right\}^{\frac{1}{\beta}}\right]dt \\
&\leq \int_0^1 \left\{\alpha[(f(a))^{h(1-t)}(f(b))^{h(t)}]^{\frac{1}{\alpha}}+\beta[(g(a))^{h(1-t)}(g(b))^{h(t)}]^{\frac{1}{\beta}}\right\}dt \\
&= \alpha\int_0^1 \left[(f(a))^{\frac{h(1-t)}{\alpha}}(f(b))^{\frac{h(t)}{\alpha}}\right]dt+\beta\int_0^1 \left[(g(a))^{\frac{h(1-t)}{\beta}}(g(b))^{\frac{h(t)}{\beta}}\right]dt.
\end{aligned}$$

This completes the proof.

Theorem 4. *Let $f: I \to \mathbb{R}$ be a differentiable such that $f' \in [a,b]$. If $|f'|^q$ is decreasing and harmonically logarithmic h-convex function on I, where $q>1$ and $h(t)+h(1-t)=1$. Then we have*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{(b-a)|f'(b)|}{2^{\frac{2q-1}{q}}}\left(\int_0^1 (1-2t)\left(\frac{|f'(a)|}{|f'(b)|}\right)^{qh(t)}dt\right)^{\frac{1}{q}}.$$

Proof. Using Lemma 1, well known power mean inequality and the fact that $|f'|^q$ is decreasing and harmonically logarithmic h-convex function. Then, we have

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \\
&= \left|\frac{b-a}{2}\int_0^1 (1-2t)f'(ta+(1-t)b)dt\right| \\
&\leq \frac{b-a}{2}\left(\int_0^1 |1-2t|dt\right)^{1-\frac{1}{q}}\left(\int_0^1 (1-2t)|f'(ta+(1-t)b)|^q dt\right)^{\frac{1}{q}} \\
&\leq \frac{b-a}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\int_0^1 (1-2t)|f'\left(\frac{ab}{(1-t)a+tb}\right)|^q dt\right)^{\frac{1}{q}} \\
&\leq \frac{b-a}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\int_0^1 (1-2t)|f'(a)|^{qh(t)}|f'(b)|^{q(h(1-t))}dt\right)^{\frac{1}{q}} \\
&= \frac{b-a}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\int_0^1 (1-2t)|f'(a)|^{qh(t)}|f'(b)|^{q(1-h(t))}dt\right)^{\frac{1}{q}} \\
&= \frac{(b-a)|f'(b)|}{2^{\frac{2q-1}{q}}}\left(\int_0^1 (1-2t)\left(\frac{|f'(a)|}{|f'(b)|}\right)^{qh(t)}dt\right)^{\frac{1}{q}}.
\end{aligned}$$

This completes the proof.

Remark. We would like to make a remark here that for different suitable choices of h function, we have results for harmonically logarithmic convex function, harmonically logarithmic s-convex function and harmonically logarithmic P-convex function.

Remark. Note that for $q=1$ in Theorem 4, we have

Corollary 4. *Let $f: I \to \mathbb{R}$ be a differentiable such that $f' \in [a,b]$. If $|f'|$ is decreasing and harmonically logarithmic h-convex function on I. Then we have*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{(b-a)|f'(b)|}{2}\left(\int_0^1 (1-2t)\left(\frac{|f'(a)|}{|f'(b)|}\right)^{h(t)}dt\right).$$

Theorem 5. *Let $f: I \to \mathbb{R}$ be a differentiable such that $f' \in [a,b]$. If $|f'|^q$ is decreasing and harmonically logarithmic h-convex function on I, where $q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and $h(t) + h(1-t) = 1$. Then we have*

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx \right|$$
$$\leq \frac{(b-a)|f'(b)|}{2} \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \left(\int_0^1 \left(\frac{|f'(a)|}{|f'(b)|} \right)^{qh(t)} dt \right)^{\frac{1}{q}}.$$

Proof. The proof directly follows from the proof of Theorem 4.

References

[1] G. Cristescu and L. Lupsa, Non-connected Convexities and Applications, Kluwer Academic Publishers, Dordrecht, Holland, 2002.
[2] S. S. Dragomir, B. Mond, Integral inequalities of Hadamard's type for log-convex functions, Demonstration Math. 2, 354-364, (1998).
[3] S. S. Dragomir, C. E. M. Pearce, Selected topics on Hermite-Hadamard inequalities and applications, Victoria University, Australia (2000).
[4] S. S. Dragomir, J. Pečarić and L. E. Persson, Some inequalities of Hadamard type, Soochow J. Math, 21, 335-341, (1995).
[5] S. S. Dragomir, R. P. Agarwal, Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula, Appl. Math. Lett. 11(5), 91-95, (1998).
[6] E. K. Godunova and V. I. Levin, Neravenstva dlja funkcii sirokogo klassa, soderzascego vypuklye, monotonnye i nekotorye drugie vidy funkii. Vycislitel. Mat. i. Fiz. Mezvuzov. Sb. Nauc. Trudov, MGPI, Moskva. (1985) 138-142.
[7] I. Iscan, Hermite-Hadamard type inequalities for harmonically convex functions, Hacettepe J. Math. Stat. 43(6), 935-942,(2014).
[8] I. Iscan, Hermite-Hadamard type inequalities for harmonically (α,m)-convex functions, avaliable online at http://arxiv.org/abs/1307.5402.
[9] M. A. Noor, K. I. Noor, M. U. Awan, Geometrically relative convex functions, Appl. Math. Infor. Sci., 8(2), 607-616, (2014).
[10] M. A. Noor, K. I. Noor, M. U. Awan, Hermite-Hadamard inequalities for relative semi-convex functions and applications, Filomat, 28(2), 221-230, (2014).
[11] M. A. Noor, K. I. Noor, M. U. Awan, S. Costache, Some integral inequalities for harmonically h-convex functions, U.P.B. Sci. Bull., Series A. 77(1), (2015), 5-16.
[12] M. A. Noor, K. I. Noor, M. U. Awan, Some characterizations of harmonically log-convex functions, Proc. Jangjeon Math. soc.. 17(1), 51-61, (2014).
[13] M. A. Noor, M. U. Awan, K. I. Noor, On some inequalities for relative semi-convex functions, J. Ineqal. Appl. 2013, 2013:332.
[14] C. E. M. Pearce, J. Pecaric, Inequalities for differentiable mappings with application to special means and quadrature formulae, Appl. Math. Lett. 13, 51-55, (2000).
[15] M. Z. Sarikaya, A. Saglam, H. Yildirim, On some Hadamard-type inequalities for h-convex functions. Jour. Math. Ineq. 2(3), 335-341, (2008).
[16] M. Z. Sarikaya, E. Set, M. E. Özdemir, On some new inequalities of Hadamard type involving h-convex functions. Acta Math. Univ. Comenianae. 2, 265-272, (2010).
[17] Y. Shuang, H-P. Yin and F. Qi, Hermite-Hadamard type integral inequalities for geometric-arithamatically s-convex functions, Analysis 33, 197-208, (2013).
[18] S. Varosanec, On h-convexity, J. Math. Anal. Appl. **326**(2007), 303-311.
[19] Bo-Yan Xi, R.-F Bai, Feng Qi, HermiteHadamard type inequalities for the m- and (α,m)-geometrically convex functions, Aequat. Math. DOI 10.1007/s00010-011-0114-x.
[20] Bo-Yan Xi, Feng Qi, Some inequalities of Hermite-Hadamard type for h-convex functions, Advances in Inequalities and Applications 2 (2013).
[21] B.-Y Xi, S.-H Wang, F. Qi, Properties and inequalities for the $h-$ and $(h,m)-$logarithmically convex functions, Creat. Math. Inform. 22(2), (2013).
[22] T.-Y Zhang, A.-P. Ji, F. Qi, Integral inequalities of Hermite-Hadamard type for harmonically quasi-convex functions, Proceedings of the Jangjeon Mathematical Society, to appear.
[23] T-Y. Zhang A-P. Ji and F. Qi, On integral inequalities of Hermite-Hadamard type for s-geometrically convex functions, Abstract and Applied Analysis Volume 2012, Article ID 560586, 14 pages doi:10.1155/2012/560586.
[24] T-Y. Zhang A-P. Ji and F. Qi, Some inequalities of Hermite-Hadamard type for GA-convex functions with applications to means, Le Matematiche vol. LXVIII (2013) Fasc. I, pp. 229239 doi: 10.4418/2013.68.1.17.

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