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Strong Insertion of a Contra-$\gamma-$Continuous Function

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Abstract: Necessary and sufficient conditions in terms of lower cut sets are given for the insertion of a contra-$\gamma-$continuous function between two comparable real-valued functions.

Keywords: Insertion, Strong binary relation, Semi-open set, Preopen set, $\gamma-$open set, Lower cut set

1 Introduction

The concept of a preopen set in a topological space was introduced by H.H. Corson and E. Michael in 1964 [6]. A subset A of a topological space (X,τ) is called *preopen* or *locally dense* or *nearly open* if $A \subseteq Int(Cl(A))$. A set A is called *preclosed* if its complement is preopen or equivalently if $Cl(Int(A)) \subseteq A$. The term ,preopen, was used for the first time by A.S. Mashhour, M.E. Abd El-Monsef and S.N. El-Deeb [26], while the concept of a , locally dense, set was introduced by H.H. Corson and E. Michael [6].

The concept of a semi-open set in a topological space was introduced by N. Levine in 1963 [23]. A subset A of a topological space (X,τ) is called *semi-open* [16] if $A \subseteq Cl(Int(A))$. A set A is called *semi-closed* if its complement is semi-open or equivalently if $Int(Cl(A)) \subseteq A$.

Recall that a subset A of a topological space (X,τ) is called $\gamma-$*open* if $A \cap S$ is preopen, whenever S is preopen [2]. A set A is called $\gamma-$closed if its complement is $\gamma-$open or equivalently if $A \cup S$ is preclosed, whenever S is preclosed.

we have that if a set is $\gamma-$open then it is semi-open and preopen.

A generalized class of closed sets was considered by Maki in [25]. He investigated the sets that can be represented as union of closed sets and called them $V-$sets. Complements of $V-$sets, i.e., sets that are intersection of open sets are called $\Lambda-$sets [25].

Recall that a real-valued function f defined on a topological space X is called $A-$continuous [32] if the preimage of every open subset of $\mathbb{R}$ belongs to A, where A is a collection of subsets of X. Most of the definitions of function used throughout this paper are consequences of the definition of $A-$continuity. However, for unknown concepts the reader may refer to [7, 17]. In the recent literature many topologists had focused their research in the direction of investigating different types of generalized continuity.

J. Dontchev in [8] introduced a new class of mappings called contra-continuity. A good number of researchers have also initiated different types of contra-continuous like mappings in the papers [1, 4, 5, 10, 11, 12, 13, 14, 15, 16, 18, 19, 31]. So, these papers are related to the present paper directly. Because, these are special examples for the paper as contra-continuity.

Hence, a real-valued function f defined on a topological space X is called *contra-$\gamma-$continuous* (resp. *contra-semi$-$continuous* , *contra-precontinuous*) if the preimage of every open subset of $\mathbb{R}$ is $\gamma-$closed (resp. *semi*$-$closed , preclosed) in X[8].

Results of Katětov [20, 21] concerning binary relations and the concept of an indefinite lower cut set for a real-valued function, which is due to Brooks [3], are used in order to give a necessary and sufficient conditions for the insertion of a contra-$\gamma-$continuous function between two comparable real-valued functions.

If g and f are real-valued functions defined on a space X, we write $g \leq f$ in case $g(x) \leq f(x)$ for all x in X.

The following definitions are modifications of conditions considered in [22].

A property P defined relative to a real-valued function on a topological space is a $c\gamma-$*property* provided that any constant function has property P and provided that the

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sum of a function with property P and any contra-$\gamma-$continuous function also has property P. If P_1 and P_2 are $c\gamma-$property, the following terminology is used:(i) A space X has the *weak* $c\gamma-$*insertion property* for (P_1,P_2) if and only if for any functions g and f on X such that $g \leq f, g$ has property P_1 and f has property P_2, then there exists a contra-$\gamma-$continuous function h such that $g \leq h \leq f$.(ii) A space X has the *strong* $c\gamma-$*insertion property* for (P_1,P_2) if and only if for any functions g and f on X such that $g \leq f, g$ has property P_1 and f has property P_2, then there exists a contra-$\gamma-$continuous function h such that $g \leq h \leq f$ and if $g(x) < f(x)$ for any x in X, then $g(x) < h(x) < f(x)$.

In this paper, for a topological space whose $\gamma-$kernel of sets are $\gamma-$open, is given a sufficient condition for the weak $c\gamma-$insertion property. Also for a space with the weak $c\gamma-$insertion property, we give necessary and sufficient conditions for the space to have the strong $c\gamma-$insertion property. Several insertion theorems are obtained as corollaries of these results. In addition, the weak insertion of a contra-continuous function and insertion of a contra-$\alpha-$continuous function between two comparable real-valued functions have recently considered by the authors in [28, 29].

2 The Main Result

Before giving a sufficient condition for insertability of a contra-$\gamma-$continuous function, the necessary definitions and terminology are stated.

Let (X,τ) be a topological space, the family of all $\gamma-$open, $\gamma-$closed, semi-open, semi-closed, preopen and preclosed will be denoted by $\gamma O(X,\tau)$, $\gamma C(X,\tau)$, $sO(X,\tau)$, $sC(X,\tau)$, $pO(X,\tau)$ and $pC(X,\tau)$, respectively.

Definition 2.1. Let A be a subset of a topological space (X,τ). We define the subsets A^Λ and A^V as follows:
$A^\Lambda = \cap\{O : O \supseteq A, O \in (X,\tau)\}$ and
$A^V = \cup\{F : F \subseteq A, F^c \in (X,\tau)\}$.
In [9, 24, 30], A^Λ is called the *kernel* of A.

We define the subsets $\gamma(A^\Lambda), \gamma(A^V), p(A^\Lambda), p(A^V), s(A^\Lambda)$ and $s(A^V)$ as follows:
$\gamma(A^\Lambda) = \cap\{O : O \supseteq A, O \in \gamma O(X,\tau)\}$
$\gamma(A^V) = \cup\{F : F \subseteq A, F \in \gamma C(X,\tau)\}$,
$p(A^\Lambda) = \cap\{O : O \supseteq A, O \in pO(X,\tau)\}$,
$p(A^V) = \cup\{F : F \subseteq A, F \in pC(X,\tau)\}$,
$s(A^\Lambda) = \cap\{O : O \supseteq A, O \in sO(X,\tau)\}$ and
$s(A^V) = \cup\{F : F \subseteq A, F \in sC(X,\tau)\}$.
$\gamma(A^\Lambda)$ (resp. $p(A^\Lambda)$, $s(A^\Lambda)$) is called the $\gamma-$ *kernel* (resp. *prekernel*, *semi* $-$ *kernel*) of A.

The following first two definitions are modifications of conditions considered in [20, 21].

Definition 2.2. If ρ is a binary relation in a set S then $\bar{\rho}$ is defined as follows: $x\ \bar{\rho}\ y$ if and only if $y\ \rho\ v$ implies $x\ \rho\ v$ and $u\ \rho\ x$ implies $u\ \rho\ y$ for any u and v in S.

Definition 2.3. A binary relation ρ in the power set $P(X)$ of a topological space X is called a *strong binary relation* in $P(X)$ in case ρ satisfies each of the following conditions:

1) If $A_i\ \rho\ B_j$ for any $i \in \{1,\ldots,m\}$ and for any $j \in \{1,\ldots,n\}$, then there exists a set C in $P(X)$ such that $A_i\ \rho\ C$ and $C\ \rho\ B_j$ for any $i \in \{1,\ldots,m\}$ and any $j \in \{1,\ldots,n\}$.

2) If $A \subseteq B$, then $A\ \bar{\rho}\ B$.

3) If $A\ \rho\ B$, then $\gamma(A^\Lambda) \subseteq B$ and $A \subseteq \gamma(B^V)$.

The concept of a lower indefinite cut set for a real-valued function was defined by Brooks [3] as follows:

Definition 2.4. If f is a real-valued function defined on a space X and if $\{x \in X : f(x) < \ell\} \subseteq A(f,\ell) \subseteq \{x \in X : f(x) \leq \ell\}$ for a real number ℓ, then $A(f,\ell)$ is called a *lower indefinite cut set* in the domain of f at the level ℓ.

We now give the following main result:

Theorem 2.1. Let g and f be real-valued functions on the topological space X, in which $\gamma-$kernel sets are $\gamma-$open, with $g \leq f$. If there exists a strong binary relation ρ on the power set of X and if there exist lower indefinite cut sets $A(f,t)$ and $A(g,t)$ in the domain of f and g at the level t for each rational number t such that if $t_1 < t_2$ then $A(f,t_1)\ \rho\ A(g,t_2)$, then there exists a contra-$\gamma-$continuous function h defined on X such that $g \leq h \leq f$.

Proof. Let g and f be real-valued functions defined on the X such that $g \leq f$. By hypothesis there exists a strong binary relation ρ on the power set of X and there exist lower indefinite cut sets $A(f,t)$ and $A(g,t)$ in the domain of f and g at the level t for each rational number t such that if $t_1 < t_2$ then $A(f,t_1)\ \rho\ A(g,t_2)$.

Define functions F and G mapping the rational numbers $\mathbb{Q}$ into the power set of X by $F(t) = A(f,t)$ and $G(t) = A(g,t)$. If t_1 and t_2 are any elements of $\mathbb{Q}$ with $t_1 < t_2$, then $F(t_1)\ \bar{\rho}\ F(t_2), G(t_1)\ \bar{\rho}\ G(t_2)$, and $F(t_1)\ \rho\ G(t_2)$. By Lemmas 1 and 2 of [21] it follows that there exists a function H mapping $\mathbb{Q}$ into the power set of X such that if t_1 and t_2 are any rational numbers with $t_1 < t_2$, then $F(t_1)\ \rho\ H(t_2), H(t_1)\ \rho\ H(t_2)$ and $H(t_1)\ \rho\ G(t_2)$.

For any x in X, let $h(x) = \inf\{t \in \mathbb{Q} : x \in H(t)\}$.

We first verify that $g \leq h \leq f$: If x is in $H(t)$ then x is in $G(t')$ for any $t' > t$; since x is in $G(t') = A(g,t')$ implies that $g(x) \leq t'$, it follows that $g(x) \leq t$. Hence $g \leq h$. If x is not in $H(t)$, then x is not in $F(t')$ for any $t' < t$; since x is not in $F(t') = A(f,t')$ implies that $f(x) > t'$, it follows that $f(x) \geq t$. Hence $h \leq f$.

Also, for any rational numbers t_1 and t_2 with $t_1 < t_2$, we have $h^{-1}(t_1,t_2) = \gamma(H(t_2)^V) \setminus \gamma(H(t_1)^\Lambda)$. Hence $h^{-1}(t_1,t_2)$ is $\gamma-$closed in X, i.e., h is a contra-$\gamma-$continuous function on X. ■

The above proof used the technique of theorem 1 in [20].

Theorem 2.2. Let P_1 and P_2 be $c\gamma-$property and X be a space that satisfies the weak $c\gamma-$insertion property for (P_1,P_2). Also assume that g and f are functions on X such that $g \leq f, g$ has property P_1 and f has property P_2. The space X has the strong $c\gamma-$insertion property for (P_1,P_2) if and only if there exist lower cut sets $A(f-g,2^{-n})$ and there exists a sequence $\{H_n\}$ of subsets of X such that (i) for each n, H_n and $A(f-g,2^{-n})$ are completely separated by contra-$\gamma-$continuous functions, and (ii)$\{x \in X : (f-g)(x) > 0\} = \bigcup_{n=1}^{\infty} H_n$.
Proof. Theorem 3.1, of [27].■

Theorem 2.3. Let P_1 and P_2 be $c\alpha-$properties and assume that the space X satisfied the weak $c\gamma-$insertion property for (P_1,P_2). The space X satisfies the strong $c\gamma-$insertion property for (P_1,P_2) if and only if X satisfies the strong $c\gamma-$insertion property for $(P_1, c\gamma c)$ and for $(c\gamma c, P_2)$.
Proof. Theorem 3.2, of [27]. ■

3 Applications

The abbreviations $c\gamma c$, cpc and csc are used for contra-$\gamma-$continuous, contra-precontinuous and contra-*semi*$-$continuous, respectively.

Before stating the consequences of theorems 2.1, 2.2 and 2.3 we suppose that X is a topological space whose $\gamma-$kernel sets are $\gamma-$open.

Corollary 3.1. If for each pair of disjoint preopen (resp. *semi*$-$open) sets G_1, G_2 of X , there exist $\gamma-$closed sets F_1 and F_2 of X such that $G_1 \subseteq F_1$, $G_2 \subseteq F_2$ and $F_1 \cap F_2 = \varnothing$ then X has the weak $c\gamma-$insertion property for (cpc, cpc) (resp. (csc, csc)).
Proof. Let g and f be real-valued functions defined on X, such that f and g are cpc (resp. csc), and $g \leq f$.If a binary relation ρ is defined by $A \ \rho \ B$ in case $p(A^{\Lambda}) \subseteq p(B^{V})$ (resp. $s(A^{\Lambda}) \subseteq s(B^{V})$), then by hypothesis ρ is a strong binary relation in the power set of X. If t_1 and t_2 are any elements of $\mathbb{Q}$ with $t_1 < t_2$, then

$$A(f,t_1) \subseteq \{x \in X : f(x) \leq t_1\} \subseteq \{x \in X : g(x) < t_2\} \subseteq A(g,t_2);$$

since $\{x \in X : f(x) \leq t_1\}$ is a preopen (resp. *semi*$-$open) set and since $\{x \in X : g(x) < t_2\}$ is a preclosed (resp. *semi*$-$closed) set, it follows that $p(A(f,t_1)^{\Lambda}) \subseteq p(A(g,t_2)^{V})$ (resp. $s(A(f,t_1)^{\Lambda}) \subseteq s(A(g,t_2)^{V})$). Hence $t_1 < t_2$ implies that $A(f,t_1) \ \rho \ A(g,t_2)$. The proof follows from Theorem 2.1. ■

Corollary 3.2. If for each pair of disjoint preopen (resp. *semi*$-$open) sets G_1, G_2, there exist $\gamma-$closed sets F_1 and F_2 such that $G_1 \subseteq F_1$, $G_2 \subseteq F_2$ and $F_1 \cap F_2 = \varnothing$ then every contra-precontinuous (resp. contra-*semi*$-$continuous) function is contra-$\gamma-$continuous.
Proof. Let f be a real-valued contra-precontinuous (resp. contra-*semi*$-$continuous) function defined on X. Set $g = f$, then by Corollary 3.1, there exists a contra-$\gamma-$continuous function h such that $g = h = f$.■

Corollary 3.3. If for each pair of disjoint preopen (resp. *semi*$-$open) sets G_1, G_2 of X , there exist $\gamma-$closed sets F_1 and F_2 of X such that $G_1 \subseteq F_1$, $G_2 \subseteq F_2$ and $F_1 \cap F_2 = \varnothing$ then X has the strong $c\gamma-$insertion property for (cpc, cpc) (resp. (csc, csc)).
Proof. Let g and f be real-valued functions defined on the X, such that f and g are cpc (resp. csc), and $g \leq f$. Set $h = (f+g)/2$, thus $g \leq h \leq f$ and if $g(x) < f(x)$ for any x in X, then $g(x) < h(x) < f(x)$. Also, by Corollary 3.2, since g and f are contra-$\gamma-$continuous functions hence h is a contra-$\gamma-$continuous function.■

Corollary 3.4. If for each pair of disjoint subsets G_1, G_2 of X , such that G_1 is preopen and G_2 is *semi*$-$open, there exist $\gamma-$closed subsets F_1 and F_2 of X such that $G_1 \subseteq F_1$, $G_2 \subseteq F_2$ and $F_1 \cap F_2 = \varnothing$ then X have the weak $c\gamma-$insertion property for (cpc, csc) and (csc, cpc).
Proof. Let g and f be real-valued functions defined on X, such that g is cpc (resp. csc) and f is csc (resp. cpc), with $g \leq f$.If a binary relation ρ is defined by $A \ \rho \ B$ in case $s(A^{\Lambda}) \subseteq p(B^{V})$ (resp. $p(A^{\Lambda}) \subseteq s(B^{V})$), then by hypothesis ρ is a strong binary relation in the power set of X. If t_1 and t_2 are any elements of $\mathbb{Q}$ with $t_1 < t_2$, then

$$A(f,t_1) \subseteq \{x \in X : f(x) \leq t_1\} \subseteq \{x \in X : g(x) < t_2\} \subseteq A(g,t_2);$$

since $\{x \in X : f(x) \leq t_1\}$ is a *semi*$-$open (resp. preopen) set and since $\{x \in X : g(x) < t_2\}$ is a preclosed (resp. *semi*$-$closed) set, it follows that $s(A(f,t_1)^{\Lambda}) \subseteq p(A(g,t_2)^{V})$ (resp. $p(A(f,t_1)^{\Lambda}) \subseteq s(A(g,t_2)^{V})$). Hence $t_1 < t_2$ implies that $A(f,t_1) \ \rho \ A(g,t_2)$. The proof follows from Theorem 2.1. ■

Before stating consequences of Theorem 2.2 and 2.3 we state and prove the necessary lemmas.
Lemma 3.1. The following conditions on the space X are equivalent:

(i) For each pair of disjoint subsets G_1, G_2 of X, such that G_1 is preopen and G_2 is *semi*$-$open, there exist $\gamma-$closed subsets F_1, F_2 of X such that $G_1 \subseteq F_1, G_2 \subseteq F_2$ and $F_1 \cap F_2 = \varnothing$.

(ii) If G is a *semi*$-$open (resp. preopen) subset of X which is contained in a preclosed (resp. *semi*$-$closed) subset F of X, then there exists a $\gamma-$closed subset H of X such that $G \subseteq H \subseteq \gamma(H^{\Lambda}) \subseteq F$.

Proof. (i) $\Rightarrow$ (ii) Suppose that $G \subseteq F$, where G and F are *semi*$-$open (resp. preopen) and preclosed (resp. *semi*$-$closed) subsets of X, respectively. Hence, F^c is a preopen (resp. *semi*$-$open) and $G \cap F^c = \varnothing$.

By (i) there exists two disjoint $\gamma-$closed subsets F_1, F_2 such that $G \subseteq F_1$ and $F^c \subseteq F_2$. But

$$F^c \subseteq F_2 \Rightarrow F_2^c \subseteq F,$$

and

$$F_1 \cap F_2 = \varnothing \Rightarrow F_1 \subseteq F_2^c$$

hence

$$G \subseteq F_1 \subseteq F_2^c \subseteq F$$

and since F_2^c is a $\gamma-$open subset containing F_1, we conclude that $\gamma(F_1^\Lambda) \subseteq F_2^c$, i.e.,

$$G \subseteq F_1 \subseteq \gamma(F_1^\Lambda) \subseteq F.$$

By setting $H = F_1$, condition (ii) holds.

(ii) $\Rightarrow$ (i) Suppose that G_1, G_2 are two disjoint subsets of X, such that G_1 is preopen and G_2 is *semi*$-$open.

This implies that $G_2 \subseteq G_1^c$ and G_1^c is a preclosed subset of X. Hence by (ii) there exists a $\gamma-$closed set H such that $G_2 \subseteq H \subseteq \gamma(H^\Lambda) \subseteq G_1^c$.
But

$$H \subseteq \gamma(H^\Lambda) \Rightarrow H \cap \gamma((H^\Lambda)^c) = \varnothing$$

and

$$\gamma(H^\Lambda) \subseteq G_1^c \Rightarrow G_1 \subseteq \gamma((H^\Lambda)^c).$$

Furthermore, $\gamma((H^\Lambda)^c)$ is a $\gamma-$closed subset of X. Hence $G_2 \subseteq H, G_1 \subseteq \gamma((H^\Lambda)^c)$ and $H \cap \gamma((H^\Lambda)^c) = \varnothing$. This means that condition (i) holds.■

Lemma 3.2. Suppose that X is a topological space. If each pair of disjoint subsets G_1, G_2 of X, where G_1 is preopen and G_2 is *semi*$-$open, can be separated by $\gamma-$closed subsets of X then there exists a contra-$\gamma-$continuous function $h : X \to [0,1]$ such that $h(G_2) = \{0\}$ and $h(G_1) = \{1\}$.

Proof. Suppose G_1 and G_2 are two disjoint subsets of X, where G_1 is preopen and G_2 is *semi*$-$open. Since $G_1 \cap G_2 = \varnothing$, hence $G_2 \subseteq G_1^c$. In particular, since G_1^c is a preclosed subset of X containing the *semi*$-$open subset G_2 of X,by Lemma 3.1, there exists a $\gamma-$closed subset $H_{1/2}$ such that

$$G_2 \subseteq H_{1/2} \subseteq \gamma(H_{1/2}^\Lambda) \subseteq G_1^c.$$

Note that $H_{1/2}$ is also a preclosed subset of X and contains G_2, and G_1^c is a preclosed subset of X and contains the *semi*$-$open subset $\gamma(H_{1/2}^\Lambda)$ of X. Hence, by Lemma 3.1, there exists $\gamma-$closed subsets $H_{1/4}$ and $H_{3/4}$ such that

$$G_2 \subseteq H_{1/4} \subseteq \gamma(H_{1/4}^\Lambda) \subseteq H_{1/2} \subseteq \gamma(H_{1/2}^\Lambda) \subseteq H_{3/4} \subseteq \gamma(H_{3/4}^\Lambda) \subseteq G_1^c.$$

By continuing this method for every $t \in D$, where $D \subseteq [0,1]$ is the set of rational numbers that their denominators are exponents of 2, we obtain $\gamma-$closed subsets H_t with the property that if $t_1, t_2 \in D$ and $t_1 < t_2$, then $H_{t_1} \subseteq H_{t_2}$. We define the function h on X by $h(x) = \inf\{t : x \in H_t\}$ for $x \notin G_1$ and $h(x) = 1$ for $x \in G_1$.

Note that for every $x \in X, 0 \leq h(x) \leq 1$, i.e., h maps X into [0,1]. Also, we note that for any $t \in D, G_2 \subseteq H_t$; hence $h(G_2) = \{0\}$. Furthermore, by definition, $h(G_1) = \{1\}$. It remains only to prove that h is a contra-$\gamma-$continuous function on X. For every $\alpha \in \mathbb{R}$, we have if $\alpha \leq 0$ then $\{x \in X : h(x) < \alpha\} = \varnothing$ and if $0 < \alpha$ then $\{x \in X : h(x) < \alpha\} = \cup\{H_t : t < \alpha\}$, hence, they are $\gamma-$closed subsets of X. Similarly, if $\alpha < 0$ then $\{x \in X : h(x) > \alpha\} = X$ and if $0 \leq \alpha$ then $\{x \in X : h(x) > \alpha\} = \cup\{\gamma((H_t^\Lambda)^c) : t > \alpha\}$ hence, every of them is a $\gamma-$closed subset. Consequently h is a contra-$\gamma-$continuous function. ■

Lemma 3.3. Suppose that X is a topological space. If each pair of disjoint subsets G_1, G_2 of X, where G_1 is preopen and G_2 is *semi*$-$open, can separate by $\gamma-$closed subsets of X, and G_1 (resp. G_2) is an $\alpha-$closed subsets of X, then there exists a contra-continuous function $h : X \to [0,1]$ such that, $h^{-1}(0) = G_1$ (resp. $h^{-1}(0) = G_2$) and $h(G_2) = \{1\}$ (resp. $h(G_1) = \{1\}$).

Proof. Suppose that G_1 (resp. G_2) is a $\gamma-$closed subset of X. By Lemma 3.2, there exists a contra-$\gamma-$continuous function $h : X \to [0,1]$ such that, $h(G_1) = \{0\}$ (resp. $h(G_2) = \{0\}$) and $h(X \setminus G_1) = \{1\}$ (resp. $h(X \setminus G_2) = \{1\}$). Hence, $h^{-1}(0) = G_1$ (resp. $h^{-1}(0) = G_2$) and since $G_2 \subseteq X \setminus G_1$ (resp. $G_1 \subseteq X \setminus G_2$), therefore $h(G_2) = \{1\}$ (resp. $h(G_1) = \{1\}$). ■

Lemma 3.4. Suppose that X is a topological space such that every two disjoint *semi*$-$open and preopen subsets of X can be separated by $\gamma-$closed subsets of X. The following conditions are equivalent:

(i) For every two disjoint subsets G_1 and G_2 of X, where G_1 is preopen and G_2 is *semi*$-$open, there exists a contra-$\gamma-$continuous function $h : X \to [0,1]$ such that, $h^{-1}(0) = G_1$ (resp. $h^{-1}(0) = G_2$) and $h^{-1}(1) = G_2$ (resp. $h^{-1}(1) = G_1$).

(ii) Every preopen (resp. *semi*$-$open) subset of X is a $\gamma-$closed subsets of X.

(iii) Every preclosed (resp. *semi*$-$closed) subset of X is a $\gamma-$open subsets of X.

Proof. (i) $\Rightarrow$ (ii) Suppose that G is a preopen (resp. *semi*$-$open) subset of X. Since $\varnothing$ is a *semi*$-$open (resp. preopen) subset of X, by (i) there exists a contra-$\gamma-$continuous function $h : X \to [0,1]$ such that, $h^{-1}(0) = G$. Set $F_n = \{x \in X : h(x) < \frac{1}{n}\}$. Then for every $n \in \mathbb{N}$, F_n is a $\gamma-$closed subset of X and $\bigcap_{n=1}^{\infty} F_n = \{x \in X : h(x) = 0\} = G$.

(ii) $\Rightarrow$ (i) Suppose that G_1 and G_2 are two disjoint subsets of X, where G_1 is preopen and G_2 is *semi*$-$open. By Lemma 3.3, there exists a contra-$\gamma-$continuous function $f : X \to [0,1]$ such that, $f^{-1}(0) = G_1$ and $f(G_2) = \{1\}$. Set $G = \{x \in X : f(x) < \frac{1}{2}\}$, $F = \{x \in X : f(x) = \frac{1}{2}\}$, and $H = \{x \in X : f(x) > \frac{1}{2}\}$. Then $G \cup F$ and $H \cup F$ are two $\gamma-$open subsets of X and $(G \cup F) \cap G_2 = \varnothing$. By Lemma 3.3, there exists a contra-$\gamma-$continuous function $g : X \to [\frac{1}{2}, 1]$ such that,

$g^{-1}(1) = G_2$ and $g(G \cup F) = \{\frac{1}{2}\}$. Define h by $h(x) = f(x)$ for $x \in G \cup F$, and $h(x) = g(x)$ for $x \in H \cup F$. Then h is well-defined and a contra-$\gamma-$continuous function, since $(G \cup F) \cap (H \cup F) = F$ and for every $x \in F$ we have $f(x) = g(x) = \frac{1}{2}$. Furthermore, $(G \cup F) \cup (H \cup F) = X$, hence h defined on X and maps to $[0,1]$. Also, we have $h^{-1}(0) = G_1$ and $h^{-1}(1) = G_2$.

(ii) $\Leftrightarrow$ (iii) By De Morgan law and noting that the complement of every $\gamma-$open subset of X is a $\gamma-$closed subset of X and complement of every $\gamma-$closed subset of X is a $\gamma-$open subset of X, the equivalence is hold. ■

Corollary 3.5. If for every two disjoint subsets G_1 and G_2 of X, where G_1 is preopen (resp. *semi*−open) and G_2 is *semi*−open (resp. preopen), there exists a contra-$\gamma-$continuous function $h : X \to [0,1]$ such that, $h^{-1}(0) = G_1$ and $h^{-1}(1) = G_2$ then X has the strong $c\gamma-$insertion property for (cpc, csc) (resp. (csc, cpc)).

Proof. Since for every two disjoint subsets G_1 and G_2 of X, where G_1 is preopen (resp. *semi*−open) and G_2 is *semi*−open (resp. preopen), there exists a contra-$\gamma-$continuous function $h : X \to [0,1]$ such that, $h^{-1}(0) = G_1$ and $h^{-1}(1) = G_2$, define $F_1 = \{x \in X : h(x) < \frac{1}{2}\}$ and $F_2 = \{x \in X : h(x) > \frac{1}{2}\}$. Then F_1 and F_2 are two disjoint $\gamma-$closed subsets of X that contain G_1 and G_2, respectively. Hence by Corollary 3.4, X has the weak $c\gamma-$insertion property for (cpc, csc) and (csc, cpc). Now, assume that g and f are functions on X such that $g \leq f$, g is cpc (resp. csc) and f is $c\gamma c$. Since $f - g$ is cpc (resp. csc), therefore the lower cut set $A(f - g, 2^{-n}) = \{x \in X : (f - g)(x) \leq 2^{-n}\}$ is a preopen (resp. *semi*−open) subset of X. Now setting $H_n = \{x \in X : (f - g)(x) > 2^{-n}\}$ for every $n \in \mathbb{N}$, then by Lemma 3.4, H_n is a $\gamma-$open subset of X and we have $\{x \in X : (f - g)(x) > 0\} = \bigcup_{n=1}^{\infty} H_n$ and for every $n \in \mathbb{N}, H_n$ and $A(f - g, 2^{-n})$ are disjoint subsets of X. By Lemma 3.2, H_n and $A(f - g, 2^{-n})$ can be completely separated by contra-$\gamma-$continuous functions. Hence by Theorem 2.2, X has the strong $c\gamma-$insertion property for $(cpc, c\gamma c)$ (resp. $(csc, c\gamma c)$).

By an analogous argument, we can prove that X has the strong $c\gamma-$insertion property for $(c\gamma c, csc)$ (resp. $(c\gamma c, cpc)$). Hence, by Theorem 2.3, X has the strong $c\gamma-$insertion property for (cpc, csc) (resp. (csc, cpc)).■

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