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Bayesian Inference on Progressive-Stress Accelerated Life Testing for the Exponentiated Weibull Distribution under Progressive Type-II Censoring

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Abstract: In this paper, a progressive-stress accelerated life test (ALT) under progressive type-II censoring is considered. The cumulative exposure model is used when the lifetime of test units follows the exponentiated weibull distribution (EW). Moreover, the maximum likelihood estimates (MLEs) and Bayes estimates (BEs) of the model parameters are obtained. Furthermore, the approximate and credible confidence intervals (CIs) of the estimators are derived. The accuracy of the MLEs and BEs for the model parameters is investigated through the simulation studies. Finally, the simulation studies are used to compare between two different designs of the progressive-stress test (simple ramp-stress test and multiple ramp-stress test).

Keywords: accelerated life testing; progressive stress; progressive type-II censoring; Bayes estimation; exponentiated weibull distribution; cumulative exposure model; simulation study.
Mathematics Subject Classification: 62N05.

1 Introduction

In traditional life testing and reliability experiments, it is not easy to collect lifetimes on highly reliable products with very long lifetimes, because very few or even no failures may occur within a limited testing time under normal conditions. ALT is one of the most modern approaches that used to obtain enough failure data, in a short period. In such testing, products are tested at higher than usual levels of stress (e.g., temperature, voltage, humidity, vibration or pressure) to induce early failures. The stress loading in ALT can be applied in different ways, commonly used methods are constant-stress, step-stress and progressive-stress. For further information on these accelerated models; see Nelson [1].

In constant-stress ALT, each unit is run at constant high stress until either all units fail or the test terminates. Constant-stress models were studied by several authors; see Kim and Bai [2], Watkins and John [3], Jaheen et al. [4], Guan et al. [5] and Mohie El-Din et al. [6,7]

In step-stress ALT, the stress on every unit is not fixed but is increasing step by step at prespecified times or simultaneous

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the occurrence of a fixed number of failures. The step-stress models were studied extensively in the literature; see Miller and Nelson [8], Bai et al. [9], Gouno et al. [10], Balakrishnan et al. [11] and Mohie El-Din et al. [12,13,14].
In progressive-stress ALT, the stress on each test unit is continuously increasing in time. If an ALT includes linearly increasing stress, this test referred to as a ramp-stress test. Many authors discussed progressive-stress models, Yin and Sheng [15] obtained the MLEs of parameters of the exponential progressive-stress model. Bai et al. [16] considered the problem of the optimal simple ramp test for Weibull distribution under type-I censoring. Ronghua and Heliang [17] studied the progressive-stress ALT for Weibull distribution under tampered failure rate model. Abdel-Hamid and AL-Hussaini [18] investigated the progressive-stress ALT for a finite mixture of distributions. Abdel-Hamid and AL-Hussaini [19] applied the progressive stress ALT under progressive censoring for Weibull distribution. AL-Hussaini et al. [20] derived Bayesian prediction intervals based on progressively type-II censored data from the half-logistic distribution under progressive-stress model. Abdel-Hamid and Abushul [21] obtained the Bayes estimates of exponentiated exponential distribution under type-II progressive hybrid censoring. Mohie El-Din et al. [22] considered the Bayesian estimation of the extension of the exponential distribution under progressive type-II censoring.
Lifetime distributions under censored sampling have been attracting great interest due to their wide applications in natural science, engineering, social sciences, public health and medicine. The scheme of progressive type-II censoring is important in life testing experiments. It allows the experimenter to remove units from a life test at various stages during the experiment. A saving of costs and of time may be the consequence of such a sampling scheme for sufficiently large samples, censoring may be progressive through several stages. Such samples arise naturally when certain specimens must be withdrawn from a life test prior to failure for use as test objects in related experimentation.
This type of censoring scheme can be described as follows: Suppose n identical items are put on a life test, the integer $m \leq n$ is a pre-specified number of failures and $R_1,R_2,...,R_m$ are m pre-fixed integers satisfying $R_1+R_2+...+R_m+m=n$. At the time of the first failure $t_{1:m:n}$, R_1 of the surviving units are randomly withdrawn. Likewise, at the time of the second failure $t_{2:m:n}$, R_2 of the surviving units are randomly withdrawn and so on. At the time of the m-th failure $t_{m:m:n}$, the test is stopped and all surviving $R_m = n-m-(R_1+...+R_{m-1})$ units are withdrawn. For more details about progressive type-II censoring; see Balakrishnan and Aggarwala [23], and Balakrishnan and Cramer [24].
The paper is drafted as follows: In Section 2, a description of the lifetime model and test assumptions are presented. In Section 3, the MLEs of model parameters under a ramp-stress ALT are derived. The BEs of model parameters using MCMC method are obtained in Section 4. In Section 5, the asymptotic and credible confidence bounds for the model parameters are constructed. Section 6, contains the simulation studies. The conclusion is made in Section 7.

2 Model description and test assumptions

2.1 Exponentiated Weibull distribution

The exponentiated Weibull distribution with three parameters $EW(\alpha,\beta,\sigma)$ was introduced by Mudholkar and Srivastava [25] as an extension of Weibull family of distribution by adding an additional shape parameter. Jiang and Murthy [26] obtained the parameter estimation of EW model by graphical approaches. New statistical measures on the EW distribution were provided by Nassar and Eissa [27]). The probability density function (PDF) of the EW distribution and the cumulative distribution function (CDF) are given respectively by

$$f(t) = \frac{\beta\alpha}{\sigma^{\alpha}} t^{\alpha-1} e^{-\left(\frac{t}{\sigma}\right)^{\alpha}} \left[1-e^{-\left(\frac{t}{\sigma}\right)^{\alpha}}\right]^{\beta-1}, \quad t>0,\ \alpha>0,\ \beta>0,\ \sigma>0, \tag{2.1}$$

$$F(t) = \left[1-e^{-\left(\frac{t}{\sigma}\right)^{\alpha}}\right]^{\beta}, \quad t>0,\ \alpha>0,\ \beta>0,\ \sigma>0. \tag{2.2}$$

2.2 Test assumptions

The following assumptions are used during the paper in the framework of progressive-stress ALT:

1.Under use condition stress, the lifetime of an unit follows $EW(\alpha,\beta,\sigma)$.
2.The progressive-stress S(t) is directly proportional to the time with constant rate ν, where $S(t)=\nu t,\ \nu>0$.
3.The relationship between the life characteristic σ and the stress loading $S(t)$ satisfies the inverse power law,

$$\sigma(t)=\frac{1}{a[s(t)]^b}, \tag{2.3}$$

where a, b are positive parameters should be estimated. For further information on this accelerated model; see Chapter 2 of Nelson [1].
4.The linear cumulative exposure model holds to illustrate the effect of changing the stress from one level to another level; see Nelson [1].

Suppose n be the total number of units under the test, $S_0<S_1(t)<...<S_k(t)$ be the stress levels in the test and S_0 be the use-stress. Under each progressive-stress level $S_i(t)=\nu_i t,\ i=1,2,...,k,\ n_i$ identical units are tested and the progressive type-II CS is performed as follows: At the time of the first failure $t_{i1:m_i:n_i}$, R_{i1} units are randomly withdrawn from the remaining n_i-1 surviving units. At the time of the second failure $t_{i2:m_i:n_i}$, R_{i2} items from the remaining n_i-2-R_{i1} units are randomly withdrawn. The test is terminated at the time of m_i-th failure occurs $t_{im_i:m_i:n_i}$, at this time all remaining $R_{im_i}=n_i-m_i-\sum_{j=1}^{m_i-1}R_{ij}$ items are withdrawn. It is clear that the complete samples and type-II censored samples are special cases of this scheme. With these notations the observed progressive censored data under the progressive-stress $S_i(t)$ are $t_{i1:m_i:n_i}<t_{i2:m_i:n_i}<...<t_{im_i:m_i:n_i},\ i=1,2,...,k$.

From the assumption of the linear cumulative exposure model and the CDF given in (2.2), the CDF of a test unit under progressive-stress $S_i(t)$ is

$$G_i(t)=F_i(\triangle(t)),\ i=1,2,...,k, \tag{2.4}$$

where $\triangle(t)=\int_0^t \frac{du}{\sigma_i(u)}=\frac{a\nu_i^b t^{b+1}}{b+1}$, and $F_i(.)$ is the CDF as defined by (2.2) under stress level $S_i(t)$ with scale parameter equal one. Therefore,

$$G_i(t)=\left[1-e^{-\left(\frac{a\nu_i^b t^{b+1}}{b+1}\right)^{\alpha}}\right]^{\beta},\ t>0. \tag{2.5}$$

The PDF of (2.5) is given by:

$$g_i(t)=a\nu_i^b\beta\alpha t^b e^{-\left(\frac{a\nu_i^b t^{b+1}}{b+1}\right)^{\alpha}}\left[1-e^{-\left(\frac{a\nu_i^b t^{b+1}}{b+1}\right)^{\alpha}}\right]^{\beta-1}\left(\frac{a\nu_i^b t^{b+1}}{b+1}\right)^{\alpha-1},\ t>0. \tag{2.6}$$

3 Maximum likelihood estimation

In this section, the MLEs of the model parameters are obtained. Let $t_{ij:m_i:n_i}=t_{ij}$ be the observed values of the lifetime T obtained from a progressive type-II censoring under progressive-stress level $S_i(t)$, $i=1,2,...,k$ and $j=1,2,...,m_i$. From the CDF in (2.5) and the corresponding PDF in (2.6), the likelihood function of the four parameters α, a, b and β based on the progressive type-II censoring sample is obtained as follows:

$$L(\alpha,\beta,a,b)=\prod_{i=1}^{k}C_i\prod_{j=1}^{m_i}g_i(t_{ij})\left[1-G_i(t_{ij})\right]^{R_{ij}}, \tag{3.1}$$

where $C_i = n_i(n_i - 1 - R_{i1})(n_i - 2 - R_{i1} - R_{i2})\ldots\left(n_i - m_i + 1 - \sum_{j=1}^{m_i-1} R_{ij}\right)$.
From (2.5) and (2.6) in (3.1), we get

$$L(\alpha,\beta,a,b) = \prod_{i=1}^{k} C_i \prod_{j=1}^{m_i} a\beta\alpha v_i^b t_{ij}^b \left[1 - \left(1 - e^{-\left(\frac{a v_i^b (t_{ij})^{b+1}}{b+1}\right)^{\alpha}}\right)^{\beta}\right]^{R_{ij}} e^{-\left(\frac{a v_i^b (t_{ij})^{b+1}}{b+1}\right)^{\alpha}} \times \left[1 - e^{-\left(\frac{a v_i^b (t_{ij})^{b+1}}{b+1}\right)^{\alpha}}\right]^{\beta-1} \left(fraca v_i^b (t_{ij})^{b+1} b+1\right)^{\alpha-1}. \tag{3.2}$$

Therefore, the log-likelihood function can be written as

$$\begin{aligned} \ell(\alpha,\beta,a,b) = & \sum_{i=1}^{k} \log C_i + (\log\alpha + \log a + \log\beta) \sum_{i=1}^{k} m_i + b\sum_{i=1}^{k} m_i \log v_i + b\sum_{i=1}^{k}\sum_{j=1}^{m_i} \log t_{ij} \\ & + (\beta-1)\sum_{i=1}^{k}\sum_{j=1}^{m_i} \log\left[1 - e^{-\left(\frac{a v_i^b (t_{ij})^{b+1}}{b+1}\right)^{\alpha}}\right] - \sum_{i=1}^{k}\sum_{j=1}^{m_i}\left(\frac{a v_i^b (t_{ij})^{b+1}}{b+1}\right)^{\alpha} \\ & + \sum_{i=1}^{k}\sum_{j=1}^{m_i} \log\left[1 - \left[1 - e^{-\left(\frac{a v_i^b (t_{ij})^{b+1}}{b+1}\right)^{\alpha}}\right]^{\beta}\right](R_{ij}) \\ & + (\alpha-1)\sum_{i=1}^{k}\sum_{j=1}^{m_i} \log\left(\frac{a v_i^b (t_{ij})^{b+1}}{b+1}\right). \end{aligned} \tag{3.3}$$

The likelihood equations of α, a, b and β are respectively,

$$\begin{aligned} \frac{\partial\ell}{\partial\alpha} = & -\sum_{i=1}^{k}\sum_{j=1}^{m_i} \frac{(R_{ij})\left[1 - e^{-(\omega(t_{ij}))^{\alpha}}\right]^{\beta-1} \beta\log(\omega(t_{ij}))(\omega(t_{ij}))^{\alpha} e^{-(\omega(t_{ij}))^{\alpha}}}{1 - \left(1 - e^{-(\omega(t_{ij}))^{\alpha}}\right)^{\beta}} \\ & + (\beta-1)\sum_{i=1}^{k}\sum_{j=1}^{m_i} \frac{\log((\omega(t_{ij})))(\omega(t_{ij}))^{\alpha} e^{-((\omega(t_{ij})))^{\alpha}}}{1 - e^{-(\omega(t_{ij}))^{\alpha}}} \\ & + \frac{\sum_{i=1} m_i}{\alpha} + \sum_{i=1}^{k}\sum_{j=1}^{m_i} \log(\omega(t_{ij})) - \sum_{i=1}^{k}\sum_{j=1}^{m_i} (\omega(t_{ij}))^{\alpha} \log(\omega(t_{ij})), \end{aligned} \tag{3.4}$$

$$\begin{aligned} \frac{\partial\ell}{\partial a} = & \sum_{i=1}^{k}\sum_{j=1}^{m_i} \frac{1}{a} + \sum_{i=1}^{k}\sum_{j=1}^{m_i} \frac{\alpha(\beta-1)a(\omega(t_{ij}))^{\alpha-1}(t_{ij})^{b+1}(v_i)^b e^{-(\omega(t_{ij}))^{\alpha}}}{1 - \left(e^{-(\omega(t_{ij}))^{\alpha}}\right)(b+1)} \\ & - \frac{a}{b+1}\sum_{i=1}^{k}\sum_{j=1}^{m_i} (\omega(t_{ij}))^{\alpha}(t_{ij})^{b+1}(v_i)^b + \frac{\alpha-1}{b+1}\sum_{i=1}^{k}\sum_{j=1}^{m_i} \frac{(t_{ij})^{b+1}(v_i)^b}{(\omega(t_{ij}))} \\ & - \sum_{i=1}^{k}\sum_{j=1}^{m_i} \frac{(t_{ij})^{b+1}(v_i)^b \alpha\beta(R_{ij})\left(1 - e^{-((\omega(t_{ij}))^{\alpha})}\right)^{\beta-1}(\omega(t_{ij}))^{\alpha-1} e^{-((\omega(t_{ij})))^{\alpha}}}{1 - \left(1 - e^{-(\omega(t_{ij}))^{\alpha}}\right)^{\beta}(b+1)}, \end{aligned} \tag{3.5}$$

$$
\begin{aligned}
\frac{\partial \ell}{\partial b} =& \frac{\alpha a(\beta-1)}{(b+1)^2}\sum_{i=1}^{k}\sum_{j=1}^{m_i}\frac{(\omega(t_{ij}))^{\alpha-1}(t_{ij})\left[(b+1)(v_i t_{ij})^b \log v_i t_{ij}-(v_i(t_{ij}))^b\right]e^{-(\omega(t_{ij}))^{\alpha}}}{1-e^{-(\omega(t_{ij}))^{\alpha}}} \\
&-\sum_{i=1}^{k}\sum_{j=1}^{m_i}\frac{\alpha a(t_{ij})}{(b+1)^2}(\omega(t_{ij}))^{\alpha-1}\left[(b+1)(v_i t_{ij})^b \log v_i t_{ij}-(v_i(t_{ij}))^b\right]+\sum_{i=1}^{k} m_i \log t_{ij} \\
&+\sum_{i=1}^{k}\sum_{j=1}^{m_i}\frac{(\alpha-1)a}{(b+1)^2}\frac{t_{ij}\left[(b+1)(v_i t_{ij})^b \log v_i t_{ij}-(v_i(t_{ij}))^b\right]}{\omega(t_{ij})}+\sum_{i=1}^{k} m_i \log v_i \\
&-\sum_{i=1}^{k}\sum_{j=1}^{m_i}\frac{R_{ij}\alpha\beta a(\omega(t_{ij}))^{\alpha-1}e^{-(\omega(t_{ij}))^{\alpha}}(t_{ij})\left[(b+1)(v_i t_{ij})^b \log v_i t_{ij}-(v_i(t_{ij}))^b\right]}{1-\left(1-e^{-(\omega(t_{ij}))^{\alpha}}\right)^{\beta}(b+1)^2} \\
&\times\left(1-e^{-(\omega(t_{ij}))^{\alpha}}\right)^{\beta-1},
\end{aligned} \tag{3.6}
$$

$$
\frac{\partial \ell}{\partial \beta} = \sum_{i=1}^{k}\frac{m_i}{\beta}+\sum_{i=1}^{k}\sum_{j=1}^{m_i}\log\left(1-e^{-(\omega(t_{ij}))^{\alpha}}\right)+\sum_{i=1}^{k}\sum_{j=1}^{m_i}\frac{R_{ij}\log\left(1-e^{-(\omega(t_{ij}))^{\alpha}}\right)\left[1-e^{-(\omega(t_{ij}))^{\alpha}}\right]^{\beta}}{1-\left[1-e^{-(\omega(t_{ij}))^{\alpha}}\right]^{\beta}} \tag{3.7}
$$

Where, $\omega(t_{ij}) = \frac{a v_i^b t^{b+1}}{b+1}$.
Now, we have a system of four nonlinear equations in four variables α, a, b and β, it is very difficult to obtain solution for them. Therefore, an iterative procedure such as Newton-Raphson can be used to find numerical solutions of the nonlinear system in (3.4), (3.5) , (3.6) and (3.7) .

4 Bayesian estimation

This section deals with obtaining the Bayesian estimation for the EW distribution parameters under different loss functions. In practical works the parameters cannot be treated as a constant during the life testing time. Therefore, it would be a fact to assume the parameters used in the lifetime model as random variables. Based on square error (SE) loss function and linear exponential (LINEX) loss function, BEs of the model parameters α, a, b and β under progressive type-II censoring are obtained. Assume that the model parameters α and β have gamma priors, and the parameters a and b have non informative prior (NIP) , thus

$$\pi_1(\alpha) \propto \alpha^{\mu-1} e^{\frac{-\alpha}{\lambda}}, \quad \alpha,\ \mu, \lambda > 0,$$

$$\pi_2(\beta) \propto \beta^{\mu_1-1} e^{\frac{-\beta}{\lambda_1}}, \quad \beta,\ \mu_1, \lambda_1 > 0,$$

$$\pi_3(a) \propto \frac{1}{a}, \quad a > 0,$$

and

$$\pi_4(b) \propto \frac{1}{b}, \quad b > 0.$$

In case of NIPs, we take $\mu \to 0$, $\mu_1 \to 0$, $\lambda_1 \to \infty$ and $\lambda \to \infty$.

Assume that the model parameters α, a, b and β are independent, then the joint prior PDF of α, β, a and b is given by

$$\pi(\alpha,a,b,\beta) \propto \frac{\beta^{\mu_1-1}\alpha^{\mu-1}}{a\,b} e^{-(\frac{\alpha}{\lambda}+\frac{\beta}{\lambda_1})}, \quad \alpha,\beta,a,b > 0. \tag{4.1}$$

The joint posterior density function of the parameters α,β, a and b can be written from (3.2) and (4.1) as follows:

$$\begin{aligned}\pi^*(\alpha,a,b,\beta) &\propto L(\alpha,a,b,\beta)\ \pi(\alpha,a,b,\beta)\\ &\propto \frac{\beta^{\mu_1-1+\sum_{i=1}^k m_i}\alpha^{\mu-1+\sum_{i=1}^k m_i}a^{-1+\sum_{i=1}^k m_i}}{b}e^{-(\frac{\alpha}{\lambda}+\frac{\beta}{\lambda_1})}\\ &\times\prod_{i=1}^k\prod_{j=1}^{m_i}v_i^b t_{ij}^b\left[1-\left(1-e^{-\left(\frac{av_i^b(t_{ij})^{b+1}}{b+1}\right)^\alpha}\right)^\beta\right]^{R_{ij}} e^{-\left(\frac{av_i^b(t_{ij})^{b+1}}{b+1}\right)^\alpha}\\ &\times\left[1-e^{-\left(\frac{av_i^b(t_{ij})^{b+1}}{b+1}\right)^\alpha}\right]^{\beta-1}\left(\frac{av_i^b(t_{ij})^{b+1}}{b+1}\right)^{\alpha-1},\end{aligned} \tag{4.2}$$

based on SE loss function, the Bayes estimator of the function of parameters $U=U(\Theta)$, $\Theta=(\alpha,a,b,\beta)$ is given by

$$\hat{U}_{SE}=\int_\Theta U\ \pi^*(\Theta)\ d\Theta, \tag{4.3}$$

where $\pi^*(\Theta)$ is given by equation (4.2).

Based on LINEX loss function, the BE of $U=U(\Theta)$ is given by

$$\hat{U}_{LINEX}=-\frac{1}{c}\log[\int_\Theta e^{-cU}\ \pi^*(\Theta)\ d\Theta], \tag{4.4}$$

where $c\neq 0$ is the shape parameter of the LINEX loss function.

It is clear that, integrals in equations (4.3) and (4.4) cannot be computed analytically. Consequently, MCMC method is applied to approximate these integrals.

4.1 Bayesian estimation using MCMC method

In this subsection, MCMC method is considered to generate samples from the posterior distribution and then compute the BEs of α, a, b and β under progressive-stress ALT.
From the joint posterior density function in (4.2), the conditional posterior distributions of α, a, b and β are given respectively by:

$$\begin{aligned}\pi^*(\alpha|a,b,\beta)\propto&\alpha^{\mu-1+\sum_{i=1}^k m_i}e^{\frac{-\alpha}{\lambda}}\prod_{i=1}^k\prod_{j=1}^{m_i}\left[1-\left(1-e^{-\left(\frac{av_i^b(t_{ij})^{b+1}}{b+1}\right)^\alpha}\right)^\beta\right]^{R_{ij}} e^{-\left(\frac{av_i^b(t_{ij})^{b+1}}{b+1}\right)^\alpha}\\ &\times\left[1-e^{-\left(\frac{av_i^b(t_{ij})^{b+1}}{b+1}\right)^\alpha}\right]^{\beta-1}\left(\frac{av_i^b(t_{ij})^{b+1}}{b+1}\right)^{\alpha-1},\end{aligned} \tag{4.5}$$

$$\begin{aligned}\pi^*(a|\alpha,b.\beta)\propto&a^{-1+\sum_{i=1}^k m_i}\prod_{i=1}^k\prod_{j=1}^{m_i}\left[1-\left(1-e^{-\left(\frac{av_i^b(t_{ij})^{b+1}}{b+1}\right)^\alpha}\right)^\beta\right]^{R_{ij}} e^{-\left(\frac{av_i^b(t_{ij})_t^{b+1}}{b+1}\right)^\alpha}\\ &\times\left[1-e^{-\left(\frac{av_i^b(t_{ij})^{b+1}}{b+1}\right)^\alpha}\right]^{\beta-1}\left(\frac{av_i^b(t_{ij})^{b+1}}{b+1}\right)^{\alpha-1},\end{aligned} \tag{4.6}$$

$$\pi^*(b|\alpha,a,\beta) \propto \frac{1}{b}\prod_{i=1}^{k}\prod_{j=1}^{m_i} v_i^b t_{ij}^b \left[1-\left(1-e^{-\left(\frac{av_i^b(t_{ij})^{b+1}}{b+1}\right)^{\alpha}}\right)^{\beta}\right]^{R_{ij}} e^{-\left(\frac{av_i^b(t_{ij})^{b+1}}{b+1}\right)^{\alpha}} \times \left[1-e^{-\left(\frac{av_i^b(t_{ij})^{b+1}}{b+1}\right)^{\alpha}}\right]^{\beta-1} \left(\frac{av_i^b(t_{ij})^{b+1}}{b+1}\right)^{\alpha-1}, \tag{4.7}$$

$$\pi^*(\beta|\alpha,a,b,) \propto e^{\frac{-\beta}{\lambda_1}} \beta^{\mu_1-1+\sum_{i=1}^{k} m_i} \prod_{i=1}^{k}\prod_{j=1}^{m_i} \left[1-\left(1-e^{-\left(\frac{av_i^b(t_{ij})^{b+1}}{b+1}\right)^{\alpha}}\right)^{\beta}\right]^{R_{ij}} \times \left[1-e^{-\left(\frac{av_i^b(t_{ij})^{b+1}}{b+1}\right)^{\alpha}}\right]^{\beta-1}. \tag{4.8}$$

The conditional posterior distributions of α, a , b and β in (4.5), (4.6) , (4.7) and (4.8) cannot be reduced analytically to well known distribution. Therefore, Metropolis-Hasting algorithm is used to generate random samples from these distributions; see Upadhyay and Gupta [28]. The following algorithm is proposed to compute BE of $U = U(\alpha,a,b,\beta)$ under SE and LINEX loss functions.

Algorithm(1)

1.Start with $\alpha^{(0)} = \hat{\alpha}_{MLE}$, $a^{(0)} = \hat{a}_{MLE}$, $b^{(0)} = \hat{b}_{MLE}$,$\beta^{(0)} = \hat{\beta}_{MLE}$.
2.Set $i = 1$.
3.Generate $\alpha^{(i)}$, $a^{(i)}$,$b^{(i)}$ and $\beta^{(i)}$ from equations (4.5), (4.6), (4.7) and (4.8) respectively by using the algorithm of Upadhyay and Gupta [27].
4.Set $i = i+1$.
5.Repeat steps ((3)-(4)) N times.
6.The approximate means of U and e^{-cU} are given respectively by

$$E(U) = \frac{1}{N-M}\sum_{i=M+1}^{N} U(\alpha^{(i)},a^{(i)},b^{(i)},\beta^{(i)}), \tag{4.9}$$

$$E(e^{-cU}) = \frac{1}{N-M}\sum_{i=M+1}^{N} \exp\{-cU(\alpha^{(i)},a^{(i)},b^{(i)},\beta^{(i)})\} \tag{4.10}$$

where M is the burn-in period.

5 Interval estimation

In this section, the approximate and credible (CIs) of the parameters α, a, b and β are derived.

5.1 Approximate confidence intervals

In this subsection, the approximate CIs of the parameters are obtained based on the asymptotic distributions of the MLEs of the elements of the vector of unknown parameters $\Theta = (\alpha,a,b,\beta)$. It is known that the asymptotic distribution of the MLEs of Θ is given by Miller [29].

$$\left((\hat{\alpha}-\alpha),(\hat{a}-a),(\hat{b}-b),(\hat{\beta}-\beta)\right) \sim \mathbf{N}(0,\Sigma),$$

where $\Sigma = (\sigma_{ij}),\ i,\ j = 1,2,3,4$ is the variance-covariance matrix of the unknown parameters $\Theta = (\alpha, a, b, \beta)$.
The approximate $100\ (1-\gamma)\%$ two sided CIs for θ_i is given by

$$\left(\hat{\theta}_{iL},\ \hat{\theta}_{iU}\right) = \hat{\theta}_i \pm Z_{1-\gamma/2}\sqrt{\sigma_{ii}}, \quad i = 1,2,3,4 \tag{5.1}$$

where $(\hat{\theta}_1 \equiv \hat{\alpha},\ \hat{\theta}_2 \equiv \hat{a},\ \hat{\theta}_3 \equiv \hat{b},\ \hat{\theta}_4 \equiv \hat{\beta})$ and Z_q is the $100q-th$ percentile of a standard normal distribution.

5.2 Credible confidence intervals

A $100(1-\gamma)\%$ Bayesian credible or posterior interval for a random quantity θ is the interval that has the posterior probability $(1-\gamma)$ that θ lies in the interval such that

$$p(L \le \theta \le U) = \int_L^U \pi^*(\theta|\mathbf{t})d\theta = 1-\gamma.$$

There are several kinds of credible interval, containing a central interval of posterior probability which is the range of values between the $\gamma/2$ and $(1-\gamma)/2$ percentiles.
The following algorithm is performed to obtain credible CIs of α, a , b and β, see Mohie El-Din et al. [30] .
Algorithm (2)

1.Do steps $((1)-(6))$ in algorithm (1).

2.Repeat step (1), K times and arrange each estimate in ascending order as $\{\hat{\theta}_{iSE}^{[1]}, \hat{\theta}_{iSE}^{[2]}, ..., \hat{\theta}_{iSE}^{[K]}\}$, $i = 1,2,3,4$, where $\hat{\theta}_{1SE} \equiv \hat{\alpha}_{SE}$, $\hat{\theta}_{2SE} \equiv \hat{a}_{SE}$, $\hat{\theta}_{3SE} \equiv \hat{b}_{SE}$ and $\hat{\theta}_{4SE} \equiv \hat{\beta}_{SE}$.

Then, the $100\ (1-\gamma)\%$ credible CIs for θ_i is given by

$$\left(\hat{\theta}_{iSE}^{[\gamma K/2]},\ \hat{\theta}_{iSE}^{[(1-\gamma/2)K]}\right), \quad i = 1,2,3,4. \tag{5.2}$$

6 Simulation studies

In this section, simulation studies are conducted to investigate the performances of the MLEs and BEs (under square error loss function (SEL) and LINEX loss function (LINEXL)) in terms of their mean square errors (MSEs) and relative absolute biases (RABs) for various choices of sample sizes (n_i, m_i, $i = 1,2,...,k$) and censoring schemes (R_{ij}, $j = 1,2,...,m_i$). Moreover, the simulation studies are used to compare between two different designs of the progressive-stress ALT. The first design (simple ramp-stress test) consists of two stress levels $(k = 2)$ with ramp rates $v_1 = 4$ and $v_2 = 16$. The second design (multiple ramp-stress test) consists of four stress levels $(k = 4)$ with ramp rates $v_1 = 4$, $v_2 = 8$, $v_3 = 12$ and $v_4 = 16$. The comparison between the two designs is conducted through the accuracy of the estimators of the model parameters. the simulation results are presented in the following Tables. Tables (1 ,2) introduces MSEs and RABs of The MLEs and BEs of the model parameters. Tables (3 - 5) include the lengths and the coverage probabilities of 95% approximate and credible CIs of the model parameters. In our study, we use three different SCs, namely CS 1, CS 2 and CS 3: For $i = 1,2,...,k$,

$$\text{CS 1:} \quad R_{ij} = \begin{cases} n_i - m_i, & j = 1, \\ 0, & other\ wise. \end{cases}$$

$$\text{CS 2:} \quad R_{ij} = \begin{cases} 1, & j = 1,2,...,n_i - m_i, \\ 0, & other\ wise. \end{cases}$$

$$\text{CS 3:} \quad \text{If } m_i \text{ even,} \quad R_{ij} = \begin{cases} n_i - m_i, & j = \frac{m_i}{2}, \\ 0, & other\ wise. \end{cases}$$

$$\text{If } m_i \text{ odd,} \quad R_{ij} = \begin{cases} n_i - m_i, & j = \frac{m_i+1}{2}, \\ 0, & other\ wise. \end{cases}$$

The estimation procedure is performed according to the following algorithm.
Algorithm (3)

1.Specify the values of k, n_i, m_i, , c, a, b and ν_i, $i=1,2,...,k$.
2.For given values of the prior parameters μ and λ generate α from $\pi_1(\alpha)$.
3.For given values of the prior parameters μ_1 and λ_1 generate β from $\pi_2(\beta)$.
4.Generate k simple random samples of size m_i from Uniform$(0,1)$ distribution, $(U_{i1},U_{i2},...,U_{im_i})$, $i=1,2,...,k$.
5.Determine the values of the censored schemes, R_{ij}, $i=1,2,...,k$, and $j=1,2,...,m_i$ such that $\sum_{j=1}^{m_i} R_{ij}=n_i-m_i$.
6.Set $E_{ij}=U_{ij}^{1/(j+\sum_{d=m_i-j+1}^{m_i} R_{id})}$, $j=1,2,...,m_i$, and $i=1,2,...,k$.
7.Obtain the progressive type-II censored samples $(U^*_{i1},U^*_{i2},...,U^*_{im_i})$, where $U^*_{ij}=1-\prod_{d=m_i-j+1}^{m_i} E_{id}$, $j=1,2,...,m_i$, $i=1,2,...,k$.
8.Use step 7, to generate random samples $(t_{i1},t_{i2},...,t_{im_i})$, $i=1,2,...,k$, from (2.5) as follows:

$$t_{ij}=\left(\frac{b+1}{a\nu_i^b}\left(-\log(1-U^{*\frac{1}{\beta}}_{ij})\right)^{\frac{1}{\alpha}}\right)^{\frac{1}{b+1}}, \quad j=1,2,...,m_i,\ i=1,2,...,k.$$

9.Use the progressive censored data to compute the MLEs of the model parameters by solving the nonlinear system ((3.4)-(3.7)).
10.Compute the BEs of the model parameters relative to SE and LINEX loss functions, using algorithm (1), with $N=11000$, $M=1000$.
11.Compute the CIs bounds with confidence level 95% for the four parameters α, a, b and β.
12.Compute 95% credible CIs using algorithm (2) of the parameters α, a, b and β.
13.Replicate the steps $((4)-(12))$, 1000 times.
14.Compute the average values of the MSEs and RABs associated with the MLEs and BEs of the parameters.
15.Compute the average values of the length and the coverage probability of CIs of the parameters.
16.Do steps ((1)-(15)) with different values of n_i, m_i, ν_i and R_{ij}, $j=1,2,...,m_i$, $i=1,2,...,k$.

Table 1: MSEs and RABs inside the parentheses for MLEs and BEs under SEL and LINEXL functions of α, a, b, and β with true values ($\alpha=1$, $a=1$, $b=0.6$ and $\beta=1$), values of the prior parameters ($\mu=4$, $\mu_1=4.0597$, $\lambda=0.2219$, and $\lambda_1=0.2$), $k=2$, $\nu_1=4$ and $\nu_2=16$.

n_i	m_i	CS	θ	ML	SEL	LINEXL $c=-2$	LINEXL $c=.001$	LINEXL $c=2$
$n_i=\begin{cases}20 & i=1\\ 10 & i=2\end{cases}$	$m_i=\begin{cases}15 & i=1\\ 8 & i=2\end{cases}$		α	0.1226	0.00223	0.00288	0.00223	0.00259
				(0.23763)	(0.05163)	(0.0517)	(0.05163)	(0.04735)
			a	0.19002	0.040594	0.044259	0.040592	0.037131
				(0.3288)	(0.11891)	(0.35945)	(0.31891)	(0.29779)
			b	0.31512	0.06057	0.06726	0.06057	0.05984
				(0.66469)	(0.32914)	(0.33891)	(0.32914)	(0.33558)
			β	0.04484	0.00573	0.005567	0.00572	0.0064
				(0.66469)	(0.072914)	(0.073891)	(0.072914)	(0.073558)
		2	α	0.14354	0.00372	0.00342	0.00372	0.00404
				(0.30957)	(0.05605)	(0.05321)	(0.05605)	(0.05888)
			a	0.19456	0.04274	0.04274	0.04273	0.02692
				(0.34509)	(0.16135)	(0.20051)	(0.16133)	(0.12703)
			b	0.14299	0.04274	0.00879	0.0062	0.00606
				(0.27682)	(0.06075)	(0.06853)	(0.06075)	(0.06551)
			β	0.09217	0.0046105	0.00472	0.00404	0.00432
				(0.26729)	(0.041453)	(0.041529)	(0.041453)	(0.041371)
		3	α	0.15067	0.00468	0.00433	0.00468	0.00505
				(0.29604)	(0.06297)	(0.0601)	(0.06298)	(0.06583)
			a	0.15469	1.68088	0.22312	0.04082	0.03309
				(0.44025)	(0.3851)	(0.51178)	(0.3851)	(0.36471)
			b	0.95665	0.09694	0.11895	0.09694	0.08747
				(1.27754)	(0.36595)	(0.39347)	(0.36595)	(0.36523)
			β	0.06182	0.0065127	0.0069	0.00627	0.0068
				(0.2893)	(0.0767)	(0.073731)	(0.073647)	(0.073549)
$n_i=\begin{cases}20 & i=1\\ 10 & i=2\end{cases}$	$m_i=\begin{cases}20 & i=1\\ 10 & i=2\end{cases}$		α	0.13383	0.001031	0.00097	0.00103	0.0011
				(0.2449)	(0.02459)	(0.02351)	(0.02459)	(0.02567)
			a	0.18284	0.06742	0.067312	0.06742	0.06768
				(0.33515)	(0.10021)	(0.09874)	(0.10021)	(0.10402)
			b	0.14299	0.09022	0.09004	0.09022	0.09052
				(0.39973)	(0.10783)	(0.10444)	(0.10783)	(0.1129)
			β	0.08348	0.00101	0.00093	0.00101	0.00109
				(0.26729)	(0.02798)	(0.02675)	(0.02798)	(0.02921)
$n_i=\begin{cases}30 & i=1\\ 15 & i=2\end{cases}$	$m_i=\begin{cases}23 & i=1\\ 12 & i=2\end{cases}$	1	α	0.11963	0.00186	0.001586	0.00186	0.00214
				(0.18545)	(0.04344)	(0.03079)	(0.03344)	(0.03607)
			a	0.17082	0.01035	0.00853	0.010468	0.01798
				(0.31486)	(0.020754)	(0.02229)	(0.020754)	(0.029748)
			b	0.10869	0.01024	0.02992	0.01024	0.01179
				(0.25462)	(0.031523)	(0.033031)	(0.031523)	(00.30927)
			β	0.12497	0.00542	0.005854	0.00622	0.00644
				(0.29092)	(0.031523)	(0.033031)	(0.031523)	(00.30927)
		2	α	0.12161	0.00359	0.00311	0.00359	0.00323
				(0.27374)	(0.05577)	(0.05318)	(0.05577)	(0.05833)
			a	0.22665	0.08684	0.12622	0.08682	0.06078
				(0.38624)	(0.16103)	(0.26103)	(0.28581)	(0.2711)
			b	0.14404	0.04846	0.0541	0.04846	0.04685
				(0.32203)	(0.04335)	(0.04235)	(0.04235)	(0.044335)
			β	0.10587	0.011803	0.011987	0.011803	0.011613
				(0.25615)	(0.2843)	(0.28884)	(0.2843)	(0.28784)

Table 1 (continued)

n_i	m_i	CS	θ	ML	SEL	LINEXL $c=-2$	LINEXL $c=.001$	LINEXL $c=2$
		3	α	0.18214 (0.33787)	0.00432 (0.05766)	0.00404 (0.05533)	0.00432 (0.05766)	0.00461 (0.06013)
			a	0.20867 (0.35515)	0.07644 (0.15612)	0.11676 (0.18479)	0.07643 (0.1561)	0.05027 (0.13108)
			b	0.12006 (0.26989)	0.01026 (0.06591)	0.00789 (0.06195)	0.01026 (0.06591)	0.01358 (0.07019)
			β	0.14529 (0.20287)	0.021589 (0.031449)	0.021615 (0.035549)	0.021589 (0.031449)	0.021563 (0.039635)
$n_i=\begin{cases}30 & i=1\\ 15 & i=2\end{cases}$	$m_i=\begin{cases}30 & i=1\\ 15 & i=2\end{cases}$		α	0.10976 (0.28238)	0.00115 (0.0015)	0.00191 (0.0014)	0.00115 (0.0015)	0.00141 (0.00159)
			a	0.201 (0.38726)	0.06401 (0.14825)	0.08538 (0.17093)	0.064 (0.14824)	0.04793 (0.12744)
			b	0.20587 (0.36271)	0.00639 (0.04502)	0.00523 (0.04318)	0.00639 (0.04503)	0.00812 (0.05118)
			β	0.15188 (0.32849)	0.00001 (0.0025)	0.00001 (0.0024)	0.00001 (0.0025)	0.000012 (0.0026)
$n_i=\begin{cases}60 & i=1\\ 40 & i=2\end{cases}$	$m_i=\begin{cases}45 & i=1\\ 30 & i=2\end{cases}$	1	α	0.04872 (0.17744)	0.00734 (0.0684)	0.00704 (0.06623)	0.00734 (0.0684)	0.00764 (0.07057)
			a	0.2102 (0.29073)	0.01042 (0.20134)	0.01012 (0.20376)	0.01042 (0.20134)	0.01087 (0.20591)
			b	0.05069 (0.45064)	0.0183 (0.18024)	0.01974 (0.186)	0.0183 (0.18024)	0.01751 (0.17759)
			β	0.003728 (0.55064)	0.00173 (0.27024)	0.00194 (0.287)	0.00173 (0.2824)	0.01631 (0.1359)
		2	α	0.04911 (0.1838)	0.00421 (0.05268)	0.00394 (0.05673)	0.00421 (0.05268)	0.00449 (0.04868)
			a	0.2099 (0.30704)	0.0117 (0.22125)	0.01167 (0.21583)	0.0117 (0.22125)	0.01188 (0.22755)
			b	0.23697 (0.69559)	0.01677 (0.16122)	0.01833 (0.16757)	0.01677 (0.16122)	0.01587 (0.15816)
			β	0.04024 (0.16389)	0.039797 (0.34832)	0.040733 (0.36757)	0.039796 (0.34832)	0.038548 (0.35816)
		3	α	0.11315 (0.2601)	0.02682 (0.07121)	0.02718 (0.07533)	0.02682 (0.07121)	0.02648 (0.06713)
			a	0.22075 (0.30888)	0.01689 (0.25924)	0.018 (0.2677)	0.01689 (0.25924)	0.01614 (0.25085)
			b	0.19425 (0.36194)	0.03018 (0.09821)	0.02881 (0.09591)	0.03018 (0.09821)	0.03195 (0.0923)
			β	0.1376 (0.28869)	0.14384 (0.30358)	0.14553 (0.30524)	0.14384 (0.30358)	0.1422 (0.305234)
$n_i=\begin{cases}60 & i=1\\ 40 & i=2\end{cases}$	$m_i=\begin{cases}60 & i=1\\ 40 & i=2\end{cases}$		α	0.03758 (0.25627)	0.00173 (0.03882)	0.00423 (0.03729)	0.00173 (0.03882)	0.00167 (0.03035)
			a	0.0174 (0.45949)	0.010415 (0.019776)	0.010139 (0.019668)	0.010415 (0.019776)	0.10746 (0.019767)
			b	0.11638 (0.41489)	0.01176 (0.09166)	0.011953 (0.09815)	0.01576 (0.09165)	0.01226 (0.0861)
			β	0.12702 (0.29827)	0.000129 (0.0022125)	0.000119 (0.002411)	0.000129 (0.002125)	0.00014 (0.003189)

Table 2: MSEs and RABs inside the parentheses for MLEs and BEs under SEL and LINEXL function of α, a, b and β with true values ($\alpha = 0.90108$, $a = 0.4$, $b = 0.6$ and $\beta = 1$), values of the prior parameters ($\mu = 4$, $\mu_1 = 4.0597$, $\lambda = 0.2219$, and $\lambda_1 = 0.2$), $k = 4$, $v_1 = 4$, $v_2 = 8$, $v_3 = 12$ and $v_4 = 16$.

n_i	m_i	CS	θ	ML	SEL	LINEXL		
						$c=-2$	$c=.001$	$c=2$
$n_i = \begin{cases} 12 & i=1 \\ 7 & i=2 \\ 6 & i=3 \\ 5 & i=4 \end{cases}$	$m_i = \begin{cases} 9 & i=1 \\ 5 & i=2 \\ 5 & i=3 \\ 4 & i=4 \end{cases}$	1	α	0.12524	0.0041	0.00388	0.0041	0.00432
				(0.27581)	(0.0518)	(0.04981)	(0.0518)	(0.05379)
			a	0.22476	0.05329	0.05548	0.05329	0.05494
				(0.50011)	(0.42731)	(0.55893)	(0.42731)	(0.38095)
			b	0.16539	0.03444	0.03476	0.03444	0.0344
				(0.3326)	(0.05875)	(0.06035)	(0.05875)	(0.05951)
			β	0.09678	0.29162	0.293	0.29162	0.29042
				(1.21808)	(0.38674)	(0.44573)	(0.38674)	(0.37104)
		2	α	0.16876	0.00638	0.00604	0.00638	0.00672
				(0.31609)	(0.010081)	(0.0147)	(0.010081)	(0.09696)
			a	0.1863	0.08155	0.21516	0.07093	0.06148
				(0.33448)	(0.1668)	(0.18097)	(0.1668)	(0.15305)
			b	0.1983	0.02333	0.08431	0.02228	0.0216
				(0.35791)	(0.05555)	(0.06177)	(0.05555)	(0.05234)
			β	0.11102	0.00358	0.003	0.00358	0.00388
				(0.26627)	(0.39269)	(0.39343	(0.39269)	(0.39258)
$n_i = \begin{cases} 12 & i=1 \\ 7 & i=2 \\ 6 & i=3 \\ 5 & i=4 \end{cases}$	$m_i = \begin{cases} 12 & i=1 \\ 7 & i=2 \\ 6 & i=3 \\ 5 & i=4 \end{cases}$		α	0.12828	0.0010022	0.00087	0.001	0.00115
				(0.3966)	(0.0293)	(0.02702)	(0.02932)	(0.03156)
			a	0.284858	0.0062	0.0090281	0.0062	0.00693
				(0.36898)	(0.046714)	(0.05742)	(0.05714)	(0.05841)
			b	0.18303	0.00282	0.00194	0.00282	0.0048
				(0.37435)	(0.03489))	(0.02989)	(0.0349)	(0.04604)
			β	0.0011137	0.0024336	0.0024333	0.00243344	0.002456
				(0.27611)	(0.27295)	(0.27345)	(0.27294)	(0.27244)
$n_i = \begin{cases} 16 & i=1 \\ 12 & i=2 \\ 9 & i=3 \\ 8 & i=4 \end{cases}$	$m_i = \begin{cases} 12 & i=1 \\ 9 & i=2 \\ 7 & i=3 \\ 7 & i=4 \end{cases}$	1	α	0.11391	0.00419	0.00392	0.00419	0.00447
				(0.26835)	(0.05857)	(0.05632)	(0.05857)	(0.06081)
			a	0.22417	0.54723	0.54717	0.42969	0.54717
				(0.38775)	(0.30534)	(0.34343)	(0.30532)	(0.26648)
			b	0.17449	0.02676	0.02717	0.02676	0.02554
				(0.02554)	(0.07609)	(0.07738)	(0.0760)	(0.0765)
			β	0.10368	0.2333626	0.2717	0.2676	0.2554
				(0.25898)	(0.60086)	(0.62229)	(0.60083)	(0.56952)
		2	α	0.12847	0.00603	0.00571	0.00603	0.00635
				(0.26496)	(0.06537)	(0.063053)	(0.06537)	(0.06768)
			a	0.19454	0.18001	0.25908	0.17997	0.13133
				(0.35596)	(0.2207)	(0.25664)	(0.22068)	(0.34068)
			b	0.14785	0.03708	0.03802	0.03708	0.03669
				(0.3056)	(0.08669)	(0.08712)	(0.08669)	(0.08878)
			β	0.0923	0.51654	0.53572	0.51653	0.50048
				(0.23873)	(0.4982)	(0.50162)	(0.4982)	(0.49499)
$n_i = \begin{cases} 16 & i=1 \\ 12 & i=2 \\ 9 & i=3 \\ 8 & i=4 \end{cases}$	$m_i = \begin{cases} 16 & i=1 \\ 12 & i=2 \\ 9 & i=3 \\ 8 & i=4 \end{cases}$		α	0.10267	0.00229	0.00265	0.00229	0.00197
				(0.40933)	(0.05245)	(0.05646)	(0.05245)	(0.04849)
			a	0.0213	0.001309	0.001283	0.001309	0.001364
				(0.31336)	(0.24021)	(0.23412)	(0.24021)	(0.24853)
			b	0.31559	0.002291	0.002579	0.002291	0.002108
				(0.76416)	(0.20097)	(0.21231)	(0.20097)	(0.19236)
			β	0.0749	0.0021	0.0054	0.002451	0.00284
				(0.463498)	(0.292)	(0.2259)	(0.2952)	(0.205)

Table 2 (continued)

n_i	m_i	CS	θ	ML	SEL	LINEXL		
						$c=-2$	$c=.001$	$c=2$
$n_i=\begin{cases}25 & i=1\\ 20 & i=2\\ 15 & i=3\\ 10 & i=4\end{cases}$	$m_i=\begin{cases}19 & i=1\\ 16 & i=2\\ 12 & i=3\\ 8 & i=4\end{cases}$	1	α	0.33051	0.02086	0.02123	0.02086	0.02052
				(0.52748)	(0.06738)	(0.07145)	(0.06738)	(0.06337)
			a	0.0217	0.0299	0.03838	0.0299	0.0262
				(0.30437)	(0.36159)	(0.40126)	(0.36159)	(0.34489)
			b	0.48949	0.02451	0.02652	0.02451	0.02418
				(1.00498)	(0.21452)	(0.22159)	(0.21452)	(0.21369)
			β	0.048949	0.002451	0.002652	0.002451	0.002418
				(0.3448)	(0.0252)	(0.0259)	(0.0252)	(0.029)
		2	α	0.34401	0.00279	0.00319	0.00279	0.00243
				(0.54948)	(0.05802)	(0.06211)	(0.05802)	(0.05398)
			a	0.0299	0.02487	0.03885	0.02487	0.02122
				(0.36491)	(0.30352)	(0.32228)	(0.30352)	(0.29701)
			b	0.53519	0.03531	0.03887	0.03531	0.03331
				(1.05)	(0.24192)	(0.25012)	(0.24192)	(0.2381)
			β	0.0949	0.0051	0.0052	0.002451	0.0018
				(0.467498)	(0.392)	(0.2659)	(0.2452)	(0.239)
$n_i=\begin{cases}25 & i=1\\ 20 & i=2\\ 15 & i=3\\ 10 & i=4\end{cases}$	$m_i=\begin{cases}25 & i=1\\ 20 & i=2\\ 15 & i=3\\ 10 & i=4\end{cases}$		α	0.1071	0.00067	0.000543	0.00057	0.00597
				(0.29069)	(0.02153)	(0.05983)	(0.01961)	(0.02453)
			a	0.0013455	0.0011529	0.00144	0.0011527	0.009057
				(0.2589891)	(0.028768)	(0.021071)	(0.028767)	(0.026564)
			b	0.12666	0.0005084	0.0005034	0.0005084	0.0005174
				(0.250622)	(0.001396)	(0.001446)	(0.001396)	(0.001477)
			β	0.08861	0.001826	0.0018523	0.001826	0.0018042
				(0.25014)	(0.03567)	(0.03331)	(0.03567)	(0.03802)
$n_i=\begin{cases}38 & i=1\\ 27 & i=2\\ 20 & i=3\\ 12 & i=4\end{cases}$	$m_i=\begin{cases}28 & i=1\\ 20 & i=2\\ 15 & i=3\\ 12 & i=4\end{cases}$	1	α	0.1288	0.00313	0.00302	0.00313	0.00325
				(0.26677)	(0.03919)	(0.03798)	(0.03919)	(0.04041)
			a	0.11856	0.04559	0.05167	0.04558	0.03989
				(0.27846)	(0.10145)	(0.10867)	(0.10144)	(0.09456)
			b	0.12319	0.02416	0.02432	0.02416	0.02414
				(0.26116)	(0.04278)	(0.04506)	(0.04278)	(0.04145)
			β	0.08363	0.11249	0.11263	0.11249	0.11235
				(0.22132)	(0.24651)	(0.24666)	(0.24651)	(0.24636)
		2	α	0.20457	0.00217	0.00251	0.00217	0.00187
				(0.4388)	(0.05092)	(0.05475)	(0.05092)	(0.04723)
			a	0.0243	0.01466	0.01454	0.01466	0.01496
				(0.34552)	(0.26887)	(0.26434)	(0.26887)	(0.27401)
			b	0.32574	0.01783	0.01926	0.01783	0.01707
				(0.84772)	(0.17676)	(0.18465)	(0.17676)	(0.17339)
			β	0.3234	0.06873	0.0156	0.0167	0.345
				(0.452)	(0.145)	(0.456)	(0.456)	(0.789)
$n_i=\begin{cases}38 & i=1\\ 27 & i=2\\ 20 & i=3\\ 12 & i=4\end{cases}$	$m_i=\begin{cases}38 & i=1\\ 27 & i=2\\ 20 & i=3\\ 12 & i=4\end{cases}$		γ	0.04201	0.000216	0.000253	0.000216	0.000182
				(0.12401)	(0.0102)	(0.01416)	(0.0102)	(0.01727)
			a	0.01831	0.001057	0.00284	0.001057	0.001136
				(0.2995)	(0.022093)	(0.021034)	(00.22093)	(0.02319)
			b	0.06785	0.001308	0.001468	0.001308	0.001197
				(0.3803)	(0.014928)	(0.015746)	(0.014928)	(0.014403)
			β	0.16785	0.00108	0.00097	0.00106	0.00115
				(0.2803)	(0.024928)	(0.025746)	(0.024928)	(0.024403)

Table 3: Lengths and coverage probabilities of 95% approximate and credible CIs for α, a, b and β with true values ($\alpha = 0.90108$, $a = 0.4$, $b = 0.6$and $\beta = 1$, values of the prior parameters ($\mu = 4$, $\mu_1 = 4.0597$, $\lambda = 0.2219$, and $\lambda_1 = 0.2$), $k = 2$, $\nu_1 = 4$ and $\nu_2 = 16$.

n_i	m_i	CS	θ	Length		Coverage Probability	
				Approximate CI	Credible CI	Approximate CI	Credible CI
$n_i = \begin{cases} 20 & i=1 \\ 10 & i=2 \end{cases}$	$m_i = \begin{cases} 15 & i=1 \\ 8 & i=2 \end{cases}$	1	α	6.59019	0.58529	1	1
			a	9.94634	1.03326	1	0.95
			b	9.60216	1.13334	1	1
			β	4.48312	0.89822	1	0.95
		2	α	6.5626	0.23863	0.95	0.983
			a	9.4142	0.73541	0.75	0.967
			b	6.2256	0.97423	0.983	0.933
			β	5.26	1.27423	0.983	0.933
		3	α	6.59019	0.58495	0.975	0.988
			a	9.94634	1.03243	0.775	0.95
			b	9.60216	1.13217	1	0.912
			β	4.48312	0.89435	1	0.912
$n_i = \begin{cases} 20 & i=1 \\ 10 & i=2 \end{cases}$	$m_i = \begin{cases} 20 & i=1 \\ 10 & i=2 \end{cases}$		α	5.94747	0.52	0.95	0.983
			a	5.4769	0.65134	0.86	0.942
			b	4.75675	0.88903	0.992	0.942
			β	3.41154	0.88903	1	1
$n_i = \begin{cases} 30 & i=1 \\ 15 & i=2 \end{cases}$	$m_i = \begin{cases} 23 & i=1 \\ 12 & i=2 \end{cases}$	1	α	5.234	0.574	0.985	0.995
			a	5.235	1.03494	0.805	0.945
			b	4.870	1.05128	0.99	0.91
			β	3.234	1.05128	0.99	0.91
		2	α	5.72797	0.57499	0.95	0.985
			a	5.69182	0.94958	0.78	0.92
			b	4.1088	1.0331	0.94	0.975
			β	3.09289	1.0331	0.94	0.975
		3	α	5.631	0.22735	0.975	0.975
			a	5.6391	0.50353	0.825	0.95
			b	4.601	0.82164	0.925	1
			β	3.201	0.84164	0.95	1
$n_i = \begin{cases} 30 & i=1 \\ 15 & i=2 \end{cases}$	$m_i = \begin{cases} 30 & i=1 \\ 15 & i=2 \end{cases}$		α	3.825	0.2212	0.95	1
			a	4.6356	0.48327	0.867	0.933
			b	4.8005	0.70313	0.992	0.942
			β	3.8005	0.70313	0.992	0.942
$n_i = \begin{cases} 60 & i=1 \\ 40 & i=2 \end{cases}$	$m_i = \begin{cases} 45 & i=1 \\ 30 & i=2 \end{cases}$	1	α	2.6627	0.3452	0.97	0.90
			a	2.6686	0.3375	0.917	0.878
			b	3.6758	0.5236	1	0.925
			β	2.58	0.236	0.93	0.85
		2	γ	3.4166	0.20069	0.98	0.98
			a	3.6484	0.33787	0.812	0.887
			b	2.80102	0.49718	0.975	0.938
			β	2.80102	0.49718	0.95	0.98
		3	α	2.45561	0.56056	.95	0.95
			a	3.6720	0.66325	0.85	0.87
			b	3.3258	0.57815	1	0.85
			β	2.3258	0.7815	0.9	0.895
$n_i = \begin{cases} 60 & i=1 \\ 40 & i=2 \end{cases}$	$m_i = \begin{cases} 60 & i=1 \\ 40 & i=2 \end{cases}$		α	1.83818	0.50219	0.958	1
			a	2.5268	0.31844	0.867	0.97
			b	2.2319	0.34926	0.8	0.92
			β	1.2319	0.44926	0.87	0.92

Table 4: Lengths and coverage probabilities of 95% approximate and credible CIs for α, a , b and β with true values ($\alpha = 0.90108$, $a = 0.4$, $b = 0.6$ and $\beta = 1$), values of the prior parameters ($\mu = 4$, $\mu_1 = 4.0597$, $\lambda = 0.2219$, and $\lambda_1 = 0.2$), $k = 4$, $\nu_1 = 4$, $\nu_2 = 8$, $\nu_3 = 12$ and $\nu_4 = 16$.

n_i	m_i	CS	θ	Length		Coverage Probability	
				Approximate CI	Credible CI	Approximate CI	Credible CI
$n_i = \begin{cases} 12 & i=1 \\ 7 & i=2 \\ 6 & i=3 \\ 5 & i=4 \end{cases}$	$m_i = \begin{cases} 12 & i=1 \\ 7 & i=2 \\ 6 & i=3 \\ 5 & i=4 \end{cases}$		α	3.90572	0.53804	0.95	0.992
			a	2.33021	0.59177	0.80	0.90
			b	2.9575	0.90627	0.992	0.95
			β	1.97713	0.79834	0.992	0.95
$n_i = \begin{cases} 12 & i=1 \\ 7 & i=2 \\ 6 & i=3 \\ 5 & i=4 \end{cases}$	$m_i = \begin{cases} 9 & i=1 \\ 5 & i=2 \\ 5 & i=3 \\ 4 & i=4 \end{cases}$		α	2.74909	0.55419	0.958	0.992
		1	a	2.20102	0.71004	0.842	0.75
			b	2.5613	0.8452	1	0.958
			β	2.5613	0.8199	1	0.958
			γ	3.7096	0.2384	0.925	0.95
		2	a	2.78425	0.557	0.792	0.942
			b	2.49043	0.994	0.992	0.95
			β	2.5613	0.8199	1	0.958
$n_i = \begin{cases} 16 & i=1 \\ 12 & i=2 \\ 9 & i=3 \\ 8 & i=4 \end{cases}$	$m_i = \begin{cases} 16 & i=1 \\ 12 & i=2 \\ 9 & i=3 \\ 8 & i=4 \end{cases}$		α	2.9192	0.317	0.98	0.9
			a	2.70133	0.528	0.858	0.933
			b	2.1574	0.8486	1	0.8
			β	1.1574	0.986	1	0.78
$n_i = \begin{cases} 16 & i=1 \\ 12 & i=2 \\ 9 & i=3 \\ 8 & i=4 \end{cases}$	$m_i = \begin{cases} 12 & i=1 \\ 9 & i=2 \\ 7 & i=3 \\ 7 & i=4 \end{cases}$		α	3.6765	0.3596	1	0.95
		1	a	2.9164	0.7093	0.75	0.97
			b	3.3749	0.84871	0.975	0.95
			β	2.0749	0.481	0.85	0.8
			α	1.8451	0.55451	0.95	0.9
		2	a	1.8267	0.76841	0.887	0.962
			b	1.735	0.87173	0.9	0.95
			β	1.735	0.88574	0.88	0.90
$n_i = \begin{cases} 25 & i=1 \\ 20 & i=2 \\ 15 & i=3 \\ 10 & i=4 \end{cases}$	$m_i = \begin{cases} 19 & i=1 \\ 16 & i=2 \\ 12 & i=3 \\ 8 & i=4 \end{cases}$		α	2.3934	0.67037	0.9237	0.78
		1	a	2.6476	0.3457	0.87	0.9
			b	2.3609	0.63899	1	0.958
			β	2.459	0.63439	1	0.75
			α	2.1211	0.5148	0.82	0.9
		2	a	2.6356	0.45239	0.792	0.875
			b	2.4269	0.63331	0.992	0.917
			β	2.434	0.6789	0.80	0.85
$n_i = \begin{cases} 25 & i=1 \\ 20 & i=2 \\ 15 & i=3 \\ 10 & i=4 \end{cases}$	$m_i = \begin{cases} 25 & i=1 \\ 20 & i=2 \\ 15 & i=3 \\ 10 & i=4 \end{cases}$		α	1.9081	0.1505	0.917	1
			a	1.6263	0.343	0.875	0.917
			b	1.0915	0.32145	1	0.95
			β	1.345	0.3245	1	0.95

Table 5: Lengths and coverage probabilities of 95% approximate and credible CIs for α, a , b and β with true values ($\alpha = 0.90108$, $a = 0.4$, $b = 0.6$ and $\beta = 1$), values of the prior parameters ($\mu = 4$, $\mu_1 = 4.0597$, $\lambda = 0.2219$, and $\lambda_1 = 0.2$), $k = 4$, $\nu_1 = 4$, $\nu_2 = 8$, $\nu_3 = 12$ and $\nu_4 = 16$.

n_i	m_i	CS	θ	Length		Coverage Probability	
				Approximate CI	Credible CI	Approximate CI	Credible CI
$n_i = \begin{cases} 38 & i=1 \\ 27 & i=2 \\ 20 & i=3 \\ 15 & i=4 \end{cases}$	$m_i = \begin{cases} 28 & i=1 \\ 20 & i=2 \\ 15 & i=3 \\ 12 & i=4 \end{cases}$	1	α	1.2425	0.20474	0.983	1
			a	2.57251	0.35091	0.85	0.908
			b	1.546	0.53076	1	0.975
			β	1.2346	0.76	0.8	0.975
		2	α	1.0680	0.2070	0.962	1
			a	0.6175	0.34856	0.862	0.812
			b	2.2100	0.51261	1	0.938
			β	2.2100	0.51261	1	0.938
$n_i = \begin{cases} 38 & i=1 \\ 27 & i=2 \\ 20 & i=3 \\ 15 & i=4 \end{cases}$	$m_i = \begin{cases} 38 & i=1 \\ 27 & i=2 \\ 20 & i=3 \\ 15 & i=4 \end{cases}$		α	0.72922	0.1627	0.992	1
			a	0.55402	0.3175	0.85	0.883
			b	0.935221	0.4769	1	0.958
			β	0.35221	0.2769	1	0.958

7 Conclusion

In this paper, we have studied a progressive-stress ALT model for the EW distribution under progressive type-II censored data. Based on simulation studies, point estimation of the model parameters has been investigated through maximum likelihood and Bayes methods. Moreover, approximate and credible CIs have been established for the model parameters. In addition, a comparison between simple ramp-stress test and multiple ramp-stress test has been conducted through the accuracy of the estimators of the model parameters. The calculations have been worked out based on different sample sizes and two different progressive censoring schemes.

From the results in the above Tables , we observed the following:

1. The MSEs and RABs of MLEs and BEs of the considered parameters decrease as the sample size increases, except for few cases.
2. The BEs of α, a, b and β give more accurate results through the MSEs and RABs than MLEs, except for few cases.
3. The BEs of α, a and b under LINEX loss function ($c = 2$) have the smallest MSEs and RABs as compared to estimates under SE loss function, LINEX loss function ($c = -2$), and MLEs, except for few cases.
4. The length of approximate and credible CIs decreases as the sample size increases, except for few cases.
5. The credible CIs length of α, a , b and beta give more accurate results than approximate CIs through the length of CIs.
6. The simple ramp-stress test gives more accurate results than the multiple ramp-stress test in terms of the MSEs, RABs and length of CIs, except for some cases.
7. The MSEs of MLEs of β is always smaller than BE MSEs of β.
8. The length of approximate CIs of β is smaller than α, a and b.
9. The length of credible CIs of β is smaller than α, a and b.

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