

Some Inequalities for the Class of Hyponormal Operators

MOHAMMED Shehu Shagari* and IBRAHIM Aliyu Fulatan

Department of Mathematics, Faculty of Physical Sciences, Ahmadu Bello University, Zaria, Kaduna, Nigeria

Received: 2 Nov. 2017, Revised: 22 Dec. 2017, Accepted: 26 Dec. 2017

Published online: 1 Jan. 2018

Abstract: This work studies some characterizations of hyponormal operators. It presents certain operator inequalities by using a modified Aluthge transform $\tilde{T}$ given as $|T|^{\frac{k}{2m}}U|T|^{\frac{k}{2m}}$, for $k \leq m$, where k, m are natural numbers and U is a partial isometry. Under certain assumptions, together with the properties of the operator U , the paper concludes that the transform $\tilde{T}$ is hyponormal, p -hyponormal, $(p + \frac{k}{2m})$ -hyponormal, and $(\frac{p+q}{p+n})$ -hyponormal given that p, q are positive reals.

Keywords: Aluthge transformation, partial isometry, polar decomposition, p -hyponormal and log-hyponormal operators.

1 Introduction

Let $B(H)$ be the space of all bounded linear operators on H , the Hilbert space. It is known that every operator T on H can be factorized into $T = U|T|$, where U is a partial isometry. If U is uniquely determined under the condition $N(U) = N(|T|)$, then this factorization is called the *polar decomposition* of T . A bounded linear operator $T : H \rightarrow H$ on a Hilbert space H is said to be unitary if $T^*T = TT^* = I$, an isometry if $\|Tx\| = \|x\|$, $\forall x \in H$ and a partial isometry if it is isometric on $N(T)^\perp$. It is familiar that $|T|^2 = T^*T$ is the square root of T and $[T^*T] = T^*T - TT^* = |T|^2 - |T^*|^2$ is known as the self commutator of T . The operator T is said to be normal if its self commutator equals zero, p -hyponormal for $p > 0$ if $(T^*T)^p \geq (TT^*)^p$. An invertible p -hyponormal operator T is said to be log-hyponormal if it satisfies $\log(T^*T) \geq \log(TT^*)$, see [4]. If $p = 1$, T is called hyponormal and semi-hyponormal for $p = \frac{1}{2}$. We shall denote the set of all p -hyponormal operators by $P-H$.

The expression $\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$, where U is unitary, is known as the Aluthge transform. This was introduced by Aluthge [1]. For results on generalized Aluthge transform, see [7]. In this paper, we shall use the modified form $\tilde{T} = |T|^{\frac{k}{2m}}U|T|^{\frac{k}{2m}}$, $k \leq m$, where U is a

partial isometry to obtain some inequalities for the class of hyponormal operators.

2 Preliminaries

Under this section, some of the well known related results on inequalities for hyponormal operators are briefly revisited. In this article, we will work exclusively with Hilbert spaces over the field of complex numbers.

The first tool to study the elements of $P-H$ is the Aluthge transform. In [1], Aluthge introduced the transform $\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ of a $T \in P-H$ with the polar decomposition $T = U|T|$, where $0 < p \leq 1$ and U is a unitary operator. He showed that the transform $\tilde{T}$ is hyponormal for $\frac{1}{2} \leq p \leq 1$ and $(p + \frac{1}{2})$ -hyponormal when $0 < p \leq \frac{1}{2}$. This result was extended, among others, by T. Furuta [7], T. Huruya [9], T. Furuta *et al.* [10], and K. Tanahashi [4]. In all the above generalizations of Aluthge's work, the operator U is a partial isometry. For similar work, we refer to ([5], [6], [2]). Our work is mainly in the extension of Aluthge's work [1].

In the sequel, we require the applications of the following results:

Theorem 1.[10](Löwner-Heinz Inequality)

Let $A, B \in B(H)$, where H is a complex Hilbert space. If

* Corresponding author e-mail: ssmohammed@abu.edu.ng

$A \geq B \geq 0$ and $\alpha \in [0, 1]$, then

$$A^\alpha \geq B^\alpha \geq 0. \quad (1)$$

Lemma 1.[5] Let $T \in B(H)$, where H is a complex Hilbert space and $T = U|T|$ be the polar decomposition of T . Then, the following hold:

- (i) $N(|T|) = N(T)$,
- (ii) $|T^*|^n = U|T|^n U^*$,
- (iii) $|T|^n = U^*|T|^n U$, for any positive integer n .

Remark. This result is also true for any positive real.

Lemma 2.[9](Furuta Inequality)

Let H be a complex Hilbert space and $A, B \in B(H)$. If $A \geq B \geq 0$, then the following inequalities:

- (i) $(B^r A^p B^r)^{\frac{1}{q}} \geq B^{\frac{p+2r}{q}}$
 - (ii) $A^{\frac{p+2r}{q}} \geq (A^r B^p A^r)^{\frac{1}{q}}$
- hold for $p \geq 0, q \geq 1, r \geq 0$ with $(1+2r)q \geq p+2r$.

Lemma 3.[8](Hansen Inequality) Let H be a complex Hilbert space and $A, B \in B(H)$. If $A \geq 0, 0 \leq \alpha \leq 1$ and $\|B\| \leq 1$, then

$$(B^* A B)^\alpha \geq B^* A^\alpha B. \quad (2)$$

Corollary 1.[3].

If $A, B > 0$, then $\log A > \log B$ if and only if there exists an $\alpha \in (0, 1]$ such that $A^\alpha > B^\alpha$.

3 Results and Discussion

In this section, by introducing the operator $\tilde{T} = |T|^{\frac{k}{2m}} U |T|^{\frac{k}{2m}}$, where k, m are natural numbers, with $k \leq m$, and U is a partial isometry, some operator inequalities are obtained. Under particular values of k, m , some existing results are obtained as corollaries.

Theorem 2. Let $T \in P-H$ and $T = U|T|$ be the polar decomposition of T . If $0 < \frac{k}{2m} \leq p$ and U is a partial isometry, then the operator $\tilde{T} = |T|^{\frac{k}{2m}} U |T|^{\frac{k}{2m}}$ satisfies $\tilde{T}^* \tilde{T} \geq \tilde{T} \tilde{T}^*$.

Proof. Let $T \in P-H$ for $0 < \frac{k}{2m} \leq p$. For any $q \leq p$, by Löwner-Heinz Theorem, p -hyponormality of T implies that of q . Therefore, T is $\frac{k}{2m}$ -hyponormal. This means

$$(T^* T)^{\frac{k}{2m}} \geq (T T^*)^{\frac{k}{2m}}. \quad (3)$$

Using $T^* T = |T|^2$ and $T T^* = |T^*|^2$ in (3), we have

$$|T|^{\frac{k}{m}} \geq |T^*|^{\frac{k}{m}}. \quad (4)$$

Using $|T|^n = U^* |T|^n U$ and $|T^*|^n = U |T|^n U^*$ in (4) to have

$$U^* |T|^{\frac{k}{m}} U \geq |T|^{\frac{k}{m}} \geq U |T|^{\frac{k}{m}} U^*.$$

That is

$$U^* |T|^{\frac{k}{m}} U - U |T|^{\frac{k}{m}} U^* \geq 0. \quad (5)$$

Now, consider

$$\begin{aligned} \tilde{T}_n^* \tilde{T}_n - \tilde{T}_n \tilde{T}_n^* &= \left(|T|^{\frac{k}{2m}} U^* |T|^{\frac{k}{2m}} |T|^{\frac{k}{2m}} U |T|^{\frac{k}{2m}} \right) - \left(|T|^{\frac{k}{2m}} U |T|^{\frac{k}{2m}} |T|^{\frac{k}{2m}} U^* |T|^{\frac{k}{2m}} \right) \\ &= \left(|T|^{\frac{k}{2m}} U^* |T|^{\frac{k}{m}} U |T|^{\frac{k}{2m}} \right) - \left(|T|^{\frac{k}{2m}} U |T|^{\frac{k}{m}} U^* |T|^{\frac{k}{2m}} \right) \\ &= |T|^{\frac{k}{2m}} \left(U^* |T|^{\frac{k}{m}} U - U |T|^{\frac{k}{m}} U^* \right) |T|^{\frac{k}{2m}} \geq 0. \end{aligned}$$

That is, $\tilde{T}_n^* \tilde{T}_n \geq \tilde{T}_n \tilde{T}_n^*$.

In particular, by setting $k = m = 1$ in Theorem 2, yields the following corollary.

Corollary 2.(see [1]) Let $T \in P-H$ and $T = U|T|$ be the polar decomposition of T . If $0 < \frac{1}{2} \leq p$ and U is a partial isometry, then the operator $\tilde{T} = |T|^{\frac{1}{2}} U |T|^{\frac{1}{2}}$ satisfies $\tilde{T}^* \tilde{T} \geq \tilde{T} \tilde{T}^*$.

To study the next result, the following Lemma is necessary which we shall state and prove for completeness.

Lemma 4. Let $T \in P-H$ for $p > 0$ and $U : H \rightarrow H$ be a partial isometry. If $N(T)$ and $N(T^*)$ coincide, then the equation

$$(U^* |T| U)^n = U^* |T|^n U, \quad (6)$$

holds for $n > 0$

Proof. The hypothesis that $N(T)$ and $N(T^*)$ coincide implies $N(T)^\perp = N(T^*)^\perp$. This means the initial and final projection coincide. That is $U^* U$ equals $U U^*$. Therefore,

$$\begin{aligned} (U^* |T| U)^2 &= U^* |T| U U^* |T| U \\ &= U^* |T| U^* U |T| U \\ &= U^* |T| |T| U \\ &= U^* |T|^2 U. \end{aligned}$$

Similarly, by induction on $x, y \in \mathbf{N}$, we have $(U^* |T| U)^{\frac{x}{y}} = U^* |T|^{\frac{x}{y}} U$.

Letting $\frac{x}{y} \rightarrow n$, the result $(U^* |T| U)^n = U^* |T|^n U$ holds for any $n > 0$, as required.

Theorem 3. Let $T \in P-H$ with the polar decomposition $T = U|T|$, where U is a partial isometry. If $N(T)$ and $N(T^*)$ coincide, with the operator $\tilde{T} = |T|^{\frac{k}{2m}} U |T|^{\frac{k}{2m}}$, then the inequality $(\tilde{T}^* \tilde{T})^p \geq (\tilde{T} \tilde{T}^*)^p$ holds for $p > 0$, and $\frac{k}{2m} < p$.

Proof. By p -hyponormality of T , we have $|T|^{2p} \geq |T^*|^{2p}$. From which we get

$$U^* |T|^{2p} U \geq |T|^{2p} \geq U |T|^{2p} U^*. \quad (7)$$

Since $A := |T|^{2p} \geq |T^*|^{2p} =: B \geq 0$, then, we can apply Löwner-Heinz Theorem to the above expression for $\frac{k}{2m} \in (0, 1]$ to obtain

$$(|T|^{2p})^{\frac{k}{2m}} \geq (|T^*|^{2p})^{\frac{k}{2m}}.$$

This gives

$$\left(U^* |T|^{\frac{kp}{m}} U - U |T|^{\frac{kp}{m}} U^* \right) \geq 0. \quad (8)$$

Now,

$$\begin{aligned} \left(\widetilde{T^*T} \right)^p - \left(\widetilde{TT^*} \right)^p &= \left(|T|^{\frac{k}{2m}} U^* |T|^{\frac{k}{2m}} \cdot |T|^{\frac{k}{2m}} U |T|^{\frac{k}{2m}} \right)^p \\ &\quad - \left(|T|^{\frac{k}{2m}} U |T|^{\frac{k}{2m}} \cdot |T|^{\frac{k}{2m}} U^* |T|^{\frac{k}{2m}} \right)^p \\ &= \left(|T|^{\frac{k}{2m}} U^* |T|^{\frac{k}{m}} U |T|^{\frac{k}{2m}} \right)^p \\ &\quad - \left(|T|^{\frac{k}{2m}} U |T|^{\frac{k}{m}} U^* |T|^{\frac{k}{2m}} \right)^p \\ &= |T|^{\frac{kp}{2m}} \left(U^* |T|^{\frac{kp}{m}} U - U |T|^{\frac{kp}{m}} U^* \right) |T|^{\frac{kp}{2m}} \\ &\geq 0. \end{aligned}$$

$$\text{Hence, } \left(\widetilde{T^*T} \right)^p \geq \left(\widetilde{TT^*} \right)^p.$$

By re-writing $\frac{k}{2m} < p$ as $p < \frac{k}{2m}$, we refine the last two results as follows:

Theorem 4. Let $T \in P - H$ with the polar decomposition $T = U|T|$ and U be a partial isometry. Then, for $0 < p < \frac{k}{2m}$, the operator $\widetilde{T} = |T|^{\frac{k}{2m}} U |T|^{\frac{k}{2m}}$ satisfies

$$\left(\widetilde{T^*T} \right)^{p+\frac{k}{2m}} \geq \left(\widetilde{TT^*} \right)^{p+\frac{k}{2m}}.$$

Proof. Let $T \in P - H$. This means $(T^*T)^p \geq (TT^*)^p$. From which we have $|T|^{2p} \geq |T^*|^{2p}$. This gives $U^*|T|^{2p}U \geq |T|^{2p} \geq U|T|^{2p}U^*$. Put

$$A' := U^*|T|^{2p}U, \quad B' := |T|^{2p}, \quad C' := U|T|^{2p}U^*. \quad (9)$$

Then,

$$\begin{aligned} \left(\widetilde{T^*T} \right)^{p+\frac{k}{2m}} &= \left(|T|^{\frac{k}{2m}} U^* |T|^{\frac{k}{2m}} \cdot |T|^{\frac{k}{2m}} U |T|^{\frac{k}{2m}} \right)^{p+\frac{k}{2m}} \\ &= \left(|T|^{\frac{k}{2m}} U^* |T|^{\frac{k}{m}} U |T|^{\frac{k}{2m}} \right)^{p+\frac{k}{2m}} \\ &= \left(B'^{\frac{k}{4mp}} A'^{\frac{k}{2mp}} B'^{\frac{k}{4mp}} \right)^{p+\frac{k}{2m}}. \end{aligned}$$

To apply Lemma 2 (Furuta Inequality)(i), we let $r' = \frac{k}{4mp}$, $p' = \frac{k}{2mp}$, $q' = \frac{2m}{2mp+k}$. Then,

$$(1+2r')q' = \left(1+2\frac{k}{4mp}\right)\frac{2m}{2mp+k} = \frac{1}{p}$$

and

$$p' + 2r' = \frac{k}{2mp} + 2\frac{k}{4mp} = \frac{k}{mp}.$$

This shows that

$$(1+2r')q' = \frac{1}{p} \geq p' + 2r' = \frac{k}{mp}.$$

Hence,

$$\begin{aligned} \left(\widetilde{T^*T} \right)^{p+\frac{k}{2m}} &= \left(B'^{\frac{k}{4mp}} A'^{\frac{k}{2mp}} B'^{\frac{k}{4mp}} \right)^{p+\frac{k}{2m}} \\ &\geq \left(B'^{\frac{k}{4mp} + \frac{k}{2mp} + \frac{k}{4mp}} \right)^{p+\frac{k}{2m}} \\ &= B'^{\frac{k}{m}(1+\frac{k}{2mp})}. \end{aligned} \quad (10)$$

Similarly, by Lemma 2(ii), we have

$$\begin{aligned} \left(\widetilde{TT^*} \right)^{p+\frac{k}{2m}} &= \left(|T|^{\frac{k}{2m}} U |T|^{\frac{k}{2m}} U^* |T|^{\frac{k}{2m}} \right)^{p+\frac{k}{2m}} \\ &= \left(B'^{\frac{k}{4mp}} C'^{\frac{k}{2mp}} B'^{\frac{k}{4mp}} \right)^{p+\frac{k}{2m}} \\ &\leq \left(B'^{\frac{k}{4mp} + \frac{k}{2mp} + \frac{k}{4mp}} \right)^{p+\frac{k}{2m}} = B'^{\frac{k}{m}(1+\frac{k}{2mp})}. \end{aligned} \quad (11)$$

From (10) and (11), it follows that

$$\left(\widetilde{T^*T} \right)^{p+\frac{k}{2m}} \geq \left(\widetilde{TT^*} \right)^{p+\frac{k}{2m}}.$$

Putting $k = m = 1$ in Theorem 3, gives the following corollary.

Corollary 3. (see [1]) Let $T \in P - H$ with the polar decomposition $T = U|T|$, where U is a partial isometry. If $N(T)$ and $N(T^*)$ coincide, with the operator $\widetilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$, then the inequality $\left(\widetilde{T^*T} \right)^p \geq \left(\widetilde{TT^*} \right)^p$ holds for $p > 0$, and $\frac{1}{2} < p$.

Theorem 5. Let $T \in P - H$ be invertible for $p > 0$ with the polar decomposition $T = U|T|$. For $p+n \geq p+q$, where $n, q > 0$, the operator $\widetilde{T} = |T|^{\frac{k}{2m}}U|T|^{\frac{k}{2m}}$ satisfies the inequalities:

$$\left(\widetilde{T^*T} \right)^{\left(\frac{p+q}{p+n}\right)} \geq |T|^{\frac{2k}{m}\left(\frac{p+q}{p+n}\right)} \geq \left(\widetilde{TT^*} \right)^{\left(\frac{p+q}{p+n}\right)}.$$

Proof. It is easy to see that T is log-hyponormal. That is, $\log(T^*T) \geq \log(TT^*)$. This clearly implies $\log|T|^2 \geq \log|T^*|^2$. Hence, $\log|T| \geq \log|T^*|$.

For $A := |T| > 0$, $B := |T^*| > 0$ and $\log|T| \geq \log|T^*|$, it follows from Cor. 1 that we can find an $\alpha \in (0, 1]$ such that

$$|T|^\alpha \geq |T^*|^\alpha = U|T|^\alpha U^*.$$

That is,

$$|T|^\alpha \geq U|T|^\alpha U^*. \quad (12)$$

By pre-multiplying and post-multiplying (12) by U^* and U respectively, we have

$$U^*|T|^\alpha U \geq |T|^\alpha. \quad (13)$$

Combining (12) and (13), we get

$$U^*|T|^\alpha U \geq |T|^\alpha \geq U|T|^\alpha U^*. \quad (14)$$

Let

$$A' := U^*|T|^\alpha U, \quad B' := |T|^\alpha, \quad C' := U|T|^\alpha U^*. \quad (15)$$

Now,

$$\begin{aligned} (\widetilde{T^*T})^{(\frac{p+q}{p+n})} &= \left(|T|^{\frac{k}{2m}} U^* |T|^{\frac{k}{2m}} \cdot |T|^{\frac{k}{2m}} U |T|^{\frac{k}{2m}} \right)^{(\frac{p+q}{p+n})} \\ &= \left(|T|^{\frac{k}{2m}} U^* |T|^{\frac{k}{2m}} U |T|^{\frac{k}{2m}} \right)^{(\frac{p+q}{p+n})} \\ &= \left(B'^{\frac{k}{2m\alpha}} A'^{\frac{k}{m\alpha}} B'^{\frac{k}{2m\alpha}} \right)^{(\frac{p+q}{p+n})}. \end{aligned} \quad (16)$$

Let $\alpha = \frac{p+q}{p+n}$. Clearly, $\alpha \in (0, 1]$. Then, from (16), we get

$$(\widetilde{T^*T})^{(\frac{p+q}{p+n})} = \left(B'^{\frac{k}{2m}(\frac{p+n}{p+q})} A'^{\frac{k}{m}(\frac{p+n}{p+q})} B'^{\frac{k}{2m}(\frac{p+n}{p+q})} \right)^{(\frac{p+q}{p+n})}. \quad (17)$$

To apply Furuta Inequality, Lemma (2)(i), we let

$$r' = \frac{k}{2m} \left(\frac{p+n}{p+q} \right) \geq 0, \quad p' = \frac{k}{m} \left(\frac{p+n}{p+q} \right) \geq 0, \quad q' = \frac{p+n}{p+q} \geq 1. \quad (18)$$

Clearly, $(1 + 2r')q' \geq p' + 2r'$. Therefore,

$$\begin{aligned} (\widetilde{T^*T})^\alpha &= \left(B'^{\frac{k}{2m\alpha}} A'^{\frac{k}{m\alpha}} B'^{\frac{k}{2m\alpha}} \right)^\alpha \\ &\geq B'^{\left(\frac{k}{2m\alpha} + \frac{k}{m\alpha} + \frac{k}{2m\alpha} \right)(\alpha)} = B'^{\frac{2k}{m}} \\ &\geq |T|^{\frac{2k\alpha}{m}} = |T|^{\frac{2k}{m}(\frac{p+q}{p+n})}. \end{aligned} \quad (19)$$

On similar steps, we obtain

$$\begin{aligned} (\widetilde{TT^*})^\alpha &= \left(B'^{\frac{k}{2m\alpha}} C'^{\frac{k}{m\alpha}} B'^{\frac{k}{2m\alpha}} \right)^\alpha \\ &\leq B'^{\left(\frac{k}{2m\alpha} + \frac{k}{m\alpha} + \frac{k}{2m\alpha} \right)(\alpha)} = B'^{\frac{2k}{m}} \\ &\leq |T|^{\frac{2k\alpha}{m}} = |T|^{\frac{2k}{m}(\frac{p+q}{p+n})}. \end{aligned} \quad (20)$$

From (19) and (20), it follows that

$$(\widetilde{T^*T})^{(\frac{p+q}{p+n})} \geq |T|^{\frac{2k}{m}(\frac{p+q}{p+n})} \geq (\widetilde{TT^*})^{(\frac{p+q}{p+n})}.$$

By setting $m = k = 1$ in Theorem 5, we have the following corollary.

Corollary 4. Let $T \in P-H$ be invertible for $p > 0$ with the polar decomposition $T = U|T|$. For $p+n \geq p+q$, where $n, q > 0$, the operator $\widetilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ satisfies the inequalities:

$$(\widetilde{T^*T})^{(\frac{p+q}{p+n})} \geq |T|^{2(\frac{p+q}{p+n})} \geq (\widetilde{TT^*})^{(\frac{p+q}{p+n})}.$$

By setting $p+q = p+n$ in Theorem 5, yields the following corollary.

Corollary 5. Let $T \in P-H$ be invertible for $p > 0$ with the polar decomposition $T = U|T|$. Then for $k \leq m$, the operator $\widetilde{T} = |T|^{\frac{k}{2m}}U|T|^{\frac{k}{2m}}$ satisfies the inequalities:

$$\widetilde{T^*T} \geq |T|^{\frac{2k}{m}} \geq \widetilde{TT^*}.$$

4 Conclusion

This work presents a study on the order properties of p -hyponormal and $\log$ -hyponormal operators. In theorems 2, 3, 4 and 5, it is stated that under the given conditions, the modified Aluthge transform $\widetilde{T} = |T|^{\frac{k}{2m}}U|T|^{\frac{k}{2m}}$ is respectively hyponormal, p -hyponormal, $(p + \frac{k}{2m})$ -hyponormal, and $(\frac{p+q}{p+n})$ -hyponormal.

Acknowledgement

The authors are grateful to the anonymous referee(s) for careful checking of the details and for helpful comments that improved this paper.

References

- [1] A. Aluthge, *On p -hyponormal Operators* for $0 < p < 1$, Vol. 13, PP. 307-315, . Integr. Equations Oper. Theory, 1990.
- [2] A. Aluthge and D. Wang, *Powers of p -hyponormal operators*, Vol. 3, PP. 279-284, . J.Inequalities Appl., 1990
- [3] M. Fujii, J. Jiang, and E. Kamei, *Characterization of chaotic order and its application to Furuta Inequality*, Proc. Amer. Math. Soc., vol. 125(12), 3655-3658, 1997.
- [4] K. Tanahashi, *On log-hyponormal operators*, Vol. 34, no.3, PP. 364-372. Integr. Equations Oper. Theory, , 1999.
- [5] M. S. Moslehian and S.M.S. Nabavi Sales, *Some conditions implying normality of operators*, Vol. 349, no.5, PP.251-254. *Comptes Rendus Math*, 2011
- [6] Rashid, M.H.M. *Some Conditions On Non-Normal Operators Which Imply Normality*, 185-192. Thai Journal Of Mathematics , 2012 .
- [7] T. Furuta, *Generalized Aluthge transformation on p -hyponormal operators*, PP. 3071-3075. *Proc. Am. Math. Soc.*, 1996
- [8] F. Hansen, *An operator inequality*, Vol. 246,no.3, PP.249-250. *Math. Ann.*, 1980.
- [9] T. Huruya, *A note on p -hyponormal operators*, Vol. 125, no. 12, PP. 3617-3624. *Proc. Am. Math. Soc.*, 1997.
- [10] T. Furuta and M. Yanagida *On powers of p -hyponormal and log-hyponormal operators*, Vol. 5, no. 4 PP. 367-380. *J. Inequalities Appl.*, 2000.



MOHAMMED Shehu Shagari received the MSc degree in Mathematics at COMSATS Institute of Information Technology, Islamabad, Pakistan. He is currently an assistant lecturer with the department of Mathematics, in the faculty of physical sciences, Ahmadu

Bello University, Nigeria. His research interests are in the areas of Operator Algebra, Fixed Point Theory, Nonconvex Analysis and Variational Inequalities.



IBRAHIM Aliyu Fulatan received the PhD degree in Mathematics at Ahmadu Bello University, Zaria, Nigeria. He is a senior lecturer with the department of Mathematics, Ahmadu Bello University, Nigeria. He specializes in the areas of Abstract

Algebra and Operator Theory. He has published several research articles in reputable international journals of mathematical science.