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Computational Method for Singularly Perturbed Delay Differential Equations with Layer or Oscillatory Behaviour

G. BSL Soujanya [1,*] *and Y. N. Reddy*[2]

[1] Department of Mathematics, Kakatiya University, Warangal, 506009, India
[2] Department of Mathematics, National Institute of Technology, Warangal, 506004, India

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Abstract: In this paper, we describe a computational method for singularly perturbed delay differential equations with layer or oscillatory behaviour. In general, the numerical solution of a second order boundary value problem will be more difficult than the numerical solution of the first order differential equation. Hence, it is preferable to convert the second order problem into first order problems. In this method, we first convert the second order singularly perturbed delay differential equation to first order neutral type delay differential equation and employ the Simpson rule. Then, we use the linear interpolation to get tridiagonal system which is solved easily by discrete invariant imbedding algorithm. Several model examples for various values of the delay parameter and perturbation parameter are solved and the computational results are presented. We also discuss the convergence of the method.

Keywords: Singular perturbations, Delay term, Boundary layer, Simpson's rule

1 Introduction

A singularly perturbed delay differential equation is an ordinary differential equation in which the highest derivative is multiplied by a small parameter and containing delay term. In these problems, typically there are thin transition layers where the solution varies rapidly or jumps abruptly, while away from the layers the solution behaves regularly and varies slowly. In the recent years, there has been a growing interest in the numerical treatment of such differential equations. This is due to the versatility of such type of differential equations in the mathematical modeling of processes in various application fields, for e.g., the first exit time problem in the modeling of the activation of neuronal variability [11], in the study of bistable devices [2], and variational problems in control theory [8] where they provide the best and in many cases the only realistic simulation of the observed.

In [5], the authors Gabil M. Amiraliyev, Erkan Cimen had given an exponentially fitted difference scheme on a uniform mesh for singularly perturbed boundary value problem for a linear second order delay differential equation with a large delay in the reaction term. Gemechis File and Y. N. Reddy [6] presented a numerical integration of a class of singularly perturbed delay differential equations with small shift, where delay is in differentiated term. In [8], the authors Jugal Mohapatra, Srinivasan Natesan constructed a numerical method for a class of singularly perturbed differential-difference equations with small delay. The numerical method comprises of upwind finite difference operator on an adaptive grid, which is formed by equidistributing the arc-length monitor function. Kadalbajoo and Sharma [10] presented a numerical approach to solve singularly perturbed differential-difference equation, which contains negative shift in the function but not in the derivative term. Lange and Miura [11]-[12] gave an asymptotic approach for a class of boundary-value problems for linear second-order singularly perturbed differential-difference equations. In [13], the authors Pramod Chakravarthy and Rao presented a finite difference method for singularly perturbed linear second order differential-difference equation of the convection-diffusion type with small delay parameter. Taylor series is used to tackle the delay term. The

* Corresponding author e-mail: gbslsoujanya@gmail.com

exponentially fitted technique is employed to solve the problem.

In this paper, we extend the method given in [6] for the numerical solution of singularly perturbed delay differential equations with layer or oscillatory behaviour. Here, the delay term is not present in the differentiated term. In this method, we first convert the second order singularly perturbed delay differential equation to first order neutral type delay differential equation and employ the Simpson rule. Then, linear interpolation is used to get three term recurrence relation which is solved easily by discrete invariant imbedding algorithm. Several model examples for various values of the delay parameter and perturbation parameter are solved and computational results are presented. We also discuss the convergence of the method.

2 Numerical Method

Consider singularly perturbed delay differential equation of the form

$$Ly \equiv \varepsilon y''(x) + a(x)y(x-\delta) + b(x)y(x) = f(x),\ 0 < \mathrm{x} < 1, \tag{1}$$

with boundary conditions

$$y(0) = \phi(x)\ ,\ \text{-}\delta \leq x \leq 0 \quad \text{and} \quad y(1) = \beta \tag{2}$$

where ε is small parameter, $0 < \varepsilon << 1$ and δ is also small shifting parameter, $0 < \delta < 1$; $a(x)$, *b(x)*, *f(x)* and $\phi(x)$are bounded continuous functions in (0, 1) and β is finite constant. For δ= 0, the solution of the boundary value problem (1), (2)exhibits layer or oscillatory behaviour depending on the sign of $(a(x)+b(x))$. If $(a(x)+b(x)) \leq -M < 0$, where $M$is a positive constant, the solution of the problem (1), (2) exhibits layer behaviour and if $(a(x)+b(x)) \geq M > 0$, it exhibits oscillatory behaviour. The boundary value problem considered here is of the reaction-diffusion type, so there will be twin boundary layers which will be at both the end points i.e., at x = 0 and x = 1. In this paper, we present both the cases, i.e., when the solution of the problem exhibits layer at both ends as well as oscillatory behaviour and show the effect of the delay on the layer and oscillatory behaviour.

We divide the interval [0, 1] into N equal parts with mesh size h. Let $0 = x_0, x_1, ..., x_N = 1$ be the mesh points. Then we have $x_i = ih$ for $i = 0, 1, \ldots, N$. Since the problem exhibits two boundary layers or oscillatory behaviour across the interval, we divide the interval [0, 1] into two sub intervals $\left[0,\ \frac{1}{2}\right]and\left[\frac{1}{2},\ 1\right]$. We choose n such that $x_n = \frac{1}{2}$. In the interval $\left[0,\ \frac{1}{2}\right]$ the boundary layer will be in the left hand side i.e., at $x = 0$ and in the interval $\left[\frac{1}{2},\ 1\right]$ the boundary layer will be in the right hand side i.e., $x = 1$. Hence, we derive the numerical method for both left-end layer and right-end layer cases. In the interval$\left[0,\ \frac{1}{2}\right]$, using Taylor series expansion to approximate $y'(x-\varepsilon)$, we get

$$y'(x-\varepsilon) \approx y'(x) - \varepsilon y''(x) \tag{3}$$

using this approximation, equation (1) is replaced by the following approximate first order differential equation with a small deviating argument:

$$y'(x) = y'(x-\varepsilon) - a(x)y(x-\delta) - b(x)y(x) + f(x) \tag{4}$$

This replacement is significant from the computational point of view [4]. Integrating (4) with respect to x fromx_i to x_{i+1}, we get

$$y_{i+1} - y_i = \int_{x_i}^{x_{i+1}} y'(x-\varepsilon)dx - \int_{x_i}^{x_{i+1}} a(x)y(x-\delta)dx + \int_{x_i}^{x_{i+1}} b(x)ydx - \int_{x_i}^{x_{i+1}} f(x)dx$$

$$y_{i+1} - y_i = y(x_{i+1}-\varepsilon) - y(x_i-\varepsilon) + \int_{x_i}^{x_{i+1}} a(x)y(x-\delta)dx - \int_{x_i}^{x_{i+1}} b(x)ydx + \int_{x_i}^{x_{i+1}} f(x)dx$$

By using Simpson's rule to evaluate the integrals in the above equation, we get

$$\begin{aligned} y_{i+1} - y_i &= y(x_{i+1}-\varepsilon) - y(x_i-\varepsilon) - \tfrac{h}{6}\left(a(x_i)y(x_i-\delta)\right) \\ &- \tfrac{h}{6}\left(4a(x_{i+1/2})y(x_{i+1/2}-\delta) + a(x_{i+1})y(x_{i+1}-\delta)\right) \\ &- \tfrac{h}{6}\left(b(x_i)y_i + 4b(x_{i+1/2})y_{i+1/2} + b(x_{i+1})y_{i+1}\right) \\ &- \tfrac{h}{6}\left(f(x_{i+1}) + f(x_i) + 4f(x_{i+1/2})\right) \end{aligned} \tag{5}$$

Again, by using Taylor series expansion and using linear interpolation, we get

$$y(x_i-\delta) \approx \left(1-\frac{\delta}{h}\right)y_i + \frac{\delta}{h}y_{i-1} \tag{6}$$

$$y(x_{i+1}-\delta) \approx \left(1-\frac{\delta}{h}\right)y_{i+1} + \frac{\delta}{h}y_i \tag{7}$$

$$\begin{aligned} y(x_{i+1/2}-\delta) &\approx y(x_{i+1/2}) - \delta\, y'(x_{i+1/2}) \\ &= y_{i+1/2} - \delta\left(\frac{y_{i+1}-y_i}{h}\right) \end{aligned} \tag{8}$$

By making use of the above equations (6), (7) and (8) in (5), rearranging the terms we get

$$\begin{aligned}\frac{\varepsilon}{h}(y_{i+1}-2y_i+y_{i-1}) = &\\ &\left(\frac{2}{3}\delta a_{i+1/2}-\frac{h}{6}a_{i+1}(1-\frac{\delta}{h})-\frac{h}{6}b_{i+1}\right)y_{i+1}\\ &+\left(-\frac{2}{3}\delta a_{i+1/2}-\frac{\delta}{6}a_{i+1}-\frac{h}{6}a_i(1-\frac{\delta}{h})-\frac{h}{6}b_i\right)y_i\\ &-\left(\frac{\delta}{6}a_i\right)y_{i-1}+\left(-\frac{2}{3}ha_{i+1/2}-\frac{2}{3}ha_{i+1/2}\right)y_{i+1/2}\\ &+\frac{h}{6}(f_i+4f_{i+1/2}+f_{i+1}) \quad (9)\end{aligned}$$

Approximating $y_{i+1/2}$ using Hermite interpolation, we have

$$y_{i+1/2}\approx\frac{y_i+y_{i+1}}{2}+\frac{h}{8}\left(y'_i-y'_{i+1}\right)+O\left(h^4\right) \quad (10)$$

In view of (4), the above (10) becomes

$$\begin{aligned}y_{i+1/2} = &\frac{y_i+y_{i+1}}{2}\\ &+\frac{h}{8}\left(y'(x_i-\varepsilon)-a_iy(x_i-\delta)-b_iy(x_i)+f(x_i)\right)\\ &-\frac{h}{8}\left(y'(x_{i+1}-\varepsilon)-a_{i+1}y(x_{i+1}-\delta)\right)\\ &-\frac{h}{8}\left(b_{i+1}y(x_{i+1})+f(x_{i+1})\right) \quad (11)\end{aligned}$$

Using (11) in (9), we get

$$\begin{aligned}&\begin{bmatrix}\frac{\varepsilon}{h}-\frac{2}{3}\delta a_{i+1/2}+\frac{h}{6}b_{i+1}+\frac{h}{6}a_{i+1}\left(1-\frac{d}{h}\right)+\\ \left(a_{i+1/2}+b_{i+1/2}\right)\left(\frac{h}{3}+\frac{h^2}{12}b_{i+1}\right)\end{bmatrix}y_{i+1}\\ &+\begin{bmatrix}-\frac{2\varepsilon}{h}+\frac{h}{6}a_i\left(1-\frac{d}{h}\right)+\frac{2}{3}\delta a_{i+1/2}+\frac{\delta}{6}a_{i+1}\\ +\frac{h}{6}b_i+\left(a_{i+1/2}+b_{i+1/2}\right)\left(\frac{h}{3}-\frac{h^2}{12}b_{i+1}\right)\end{bmatrix}y_i\\ &+\left[\frac{\varepsilon}{h}+\frac{\delta}{6}a_i\right]y_{i-1}=\frac{h}{6}\left[f_i+4f_{i+1/2}+f_{i+1}\right]\\ &+\frac{h^2}{12}\left(f_{i+1}-f_i\right)\left(a_{i+1/2}+b_{i+1/2}\right)\\ &-\frac{2h}{3}\left(a_{i+1/2}+b_{i+1/2}\right)\left[\frac{h}{8}y'\left(x_i-\varepsilon\right)-\frac{h}{8}y'\left(x_{i+1}-\varepsilon\right)\right]\\ &-\frac{2h}{3}\left(a_{i+1/2}+b_{i+1/2}\right)\frac{h}{8}\begin{pmatrix}a_{i+1}\left(\left(1-\frac{d}{h}\right)y_{i+1}+\frac{\delta}{h}y_i\right)\\ -a_i\left(\left(1-\frac{d}{h}\right)y_i+\frac{\delta}{h}y_{i-1}\right)\end{pmatrix}\end{aligned} \quad (12)$$

$$y'\left(x_{i+1}-\varepsilon\right)-y'\left(x_i-\varepsilon\right)=\frac{\left(y\left(x_{i+1}-\varepsilon\right)-2y\left(x_i-\varepsilon\right)+y\left(x_{i-1}-\varepsilon\right)\right)}{h} \quad (13)$$

Again by Taylor series and linear interpolation, we have

$$y(x_i-\varepsilon)\approx\left(1-\frac{\varepsilon}{h}\right)y_i+\frac{\varepsilon}{h}y_{i-1} \quad (14)$$

$$y(x_{i+1}-\varepsilon)\approx\left(1-\frac{\varepsilon}{h}\right)y_{i+1}+\frac{\varepsilon}{h}y_i \quad (15)$$

$$y(x_{i-1}-\varepsilon)\approx\left(1+\frac{\varepsilon}{h}\right)y_{i-1}-\frac{\varepsilon}{h}y_i \quad (16)$$

Substituting (13) - (16) in (12), we get the following tridiagonal system

$$E_iy_{i-1}-F_iy_i+G_iy_{i+1}=H_i,\ \text{ for }\ i=1,2,\ldots,n-1. \quad (17)$$

where

$$E_i=\frac{\varepsilon}{h}+\frac{\delta}{6}a_i-\frac{h}{12}\left(a_{i+1/2}+b_{i+1/2}\right)\left(\delta a_i+\left(1-\frac{\varepsilon}{h}\right)\right)$$

$$\begin{aligned}F_i=&\frac{2\varepsilon}{h}-\frac{(h-\delta)}{6}a_i-\frac{2}{3}\delta a_{i+1/2}-\frac{\delta}{6}a_{i+1}-\frac{h}{6}b_i\\ &-\frac{h}{3}\left(a_{i+1/2}+b_{i+1/2}\right)\left(\frac{3}{2}+\frac{\delta}{4}a_{i+1}-\frac{h}{4}b_i-\frac{(h-\delta)}{4}a_i+\frac{\varepsilon}{2h}\right)\end{aligned}$$

$$\begin{aligned}G_i=&\frac{\varepsilon}{h}+\frac{(h-\delta)}{6}a_i-\frac{2}{3}\delta a_{i+1/2}+\frac{h}{6}b_i\\ &+\frac{h}{3}\left(a_{i+1/2}+b_{i+1/2}\right)\left(\frac{3}{4}+\frac{h}{4}b_{i+1}+\frac{(h-\delta)}{4}a_{i+1}+\frac{\varepsilon}{4h}\right)\end{aligned}$$

$$H_i=\frac{h}{6}\left[f_i+4f_{i+1/2}+f_{i+1}\right]+\frac{h^2}{12}\left(f_{i+1}-f_i\right)\left(a_{i+1/2}+b_{i+1/2}\right)$$

Now in the interval $\left[\frac{1}{2},\ 1\right]$, we use Taylor series expansion to approximate $y'\left(x+\varepsilon\right)$, we get

$$y'\left(x+\varepsilon\right)\approx y'(x)+\varepsilon y''(x) \quad (18)$$

and consequently, equation (2) is replaced by the following approximate first order differential equation:

$$y'(x)=y'(x+\varepsilon)+a(x)y(x-\delta)+b(x)y-f(x), \quad (19)$$

Integrating (21) with respect to x from x_{i-1} to x_i, we get

$$\begin{aligned}y_i-y_{i-1}=&\int_{x_{i-1}}^{x_i}y'(x+\varepsilon)dx+\int_{x_{i-1}}^{x_i}a(x)y(x-\delta)dx\\ &+\int_{x_{i-1}}^{x_i}b(x)ydx-\int_{x_{i-1}}^{x_i}f(x)dx\end{aligned}$$

$$\begin{aligned}y_i-y_{i-1}=&y(x_i+\varepsilon)-y(x_{i-1}+\varepsilon)+\int_{x_{i-1}}^{x_i}a(x)y(x-\delta)dx\\ &+\int_{x_{i-1}}^{x_i}b(x)ydx-\int_{x_{i-1}}^{x_i}f(x)dx\end{aligned}$$

By using Simpson's rule to evaluate the integral, we get

$$
\begin{aligned}
y_i - y_{i-1} &= y(x_i+\varepsilon) - y(x_{i-1}+\varepsilon) \\
&+ \frac{h}{6}\begin{pmatrix} a(x_i)y(x_i-\delta) + 4a(x_{i-1/2})y(x_{i-1/2}-\delta) \\ +a(x_{i-1})y(x_{i-1}-\delta) \end{pmatrix} \\
&+ \frac{h}{6}\begin{pmatrix} b(x_i)y_i + 4b(x_{i-1/2})y_{i-1/2} + b(x_{i-1})y_{i-1} \\ -f(x_{i-1}) - 4f(x_{i-1/2}) - f(x_i) \end{pmatrix}
\end{aligned} \tag{20}
$$

Again, by means of Taylor series expansion and then by approximating $y'(x)$by linear interpolation, we get

$$
\begin{aligned}
y(x_i-\delta) \approx y(x_i) - \delta y'(x_i) &= y_i - \delta\left(\frac{y_{i+1}-y_i}{h}\right) \\
&= \left(1+\frac{\delta}{h}\right)y_i - \frac{\delta}{h}y_{i+1}
\end{aligned} \tag{21}
$$

$$
\begin{aligned}
y(x_{i-1}-\delta) \approx y(x_{i-1}) - \delta\, y'(x_{i-1}) &= y_{i-1} - \delta\left(\frac{y_i-y_{i-1}}{h}\right) \\
&= \left(1+\frac{\delta}{h}\right)y_{i-1} - \frac{\delta}{h}y_i
\end{aligned} \tag{22}
$$

$$
\begin{aligned}
y(x_{i-1/2}-\delta) &\approx y(x_{i-1/2}) - \delta y'(x_{i-1/2}) \\
&= y_{i-1/2} - \delta\left(\frac{y_i-y_{i-1}}{h}\right)
\end{aligned} \tag{23}
$$

By making use of the above equations in (2) and rearranging the terms, we get

$$
\begin{aligned}
&\frac{\varepsilon}{h}(y_{i+1}-2y_i+y_{i-1}) = \\
&\left(\frac{2}{3}\delta a_{i-1/2} + \frac{h}{6}a_{i-1}\left(1+\frac{\delta}{h}\right) + \frac{h}{6}b_{i+1}\right)y_{i-1} \\
&+\left(-\frac{2}{3}\delta a_{i-1/2} - \frac{\delta}{6}a_{i-1} + \frac{h}{6}a_i\left(1+\frac{\delta}{h}\right) + \frac{h}{6}b_i\right)y_i \text{-} \frac{\delta}{6}a_i y_{i+1} \\
&+\left(\frac{2}{3}ha_{i-1/2} + \frac{2}{3}hb_{i-1/2}\right)y_{i-1/2} - \frac{h}{6}\left(f_i + 4f_{i-1/2} + f_{i-1}\right)
\end{aligned} \tag{24}
$$

By using Hermite interpolation, we have

$$
y_{i-1/2} = \frac{y_i + y_{i-1}}{2} + \frac{h}{8}\left(y'_{i-1} - y'_i\right) + O\left(h^4\right) \tag{25}
$$

In view of (4), the above equation (25) becomes

$$
\begin{aligned}
y_{i-1/2} &= \frac{y_i+y_{i-1}}{2} \\
&+ \frac{h}{8}\left(y'(x_{i-1}+\varepsilon) + a_{i-1}y(x_{i-1}-\delta) + b_{i-1}y(x_{i-1}) - f(x_{i-1})\right) \\
&\qquad - \frac{h}{8}\left(y'(x_i+\varepsilon) + a_i y(x_i-\delta)\right) \\
&\qquad\qquad - \frac{h}{8}\left(b_i y(x_i) + f(x_i)\right)
\end{aligned} \tag{26}
$$

Using equation (28) in equation (26), w get

$$
\begin{aligned}
&\begin{bmatrix} \frac{\varepsilon}{h} + \frac{2}{3}\delta a_{i-1/2} + \frac{h}{6}b_{i-1} + \frac{h}{6}a_{i-1}\left(1+\frac{d}{h}\right) \\ +\left(a_{i-1/2}+b_{i-1/2}\right)\left(\frac{h}{3} + \frac{h^2}{12}b_{i-1} + \frac{h^2}{12}a_{i-1}\left(1+\frac{d}{h}\right)\right) \end{bmatrix} y_{i-1} \\
&+ \begin{bmatrix} -\frac{2\varepsilon}{h} + \frac{h}{6}a_i\left(1+\frac{d}{h}\right) - \frac{2}{3}\delta a_{i-1/2} - \frac{\delta}{6}a_{i-1} \\ +\frac{h}{6}b_i + \left(a_{i-1/2}+b_{i-1/2}\right)\begin{pmatrix} \frac{h}{3} - \frac{\delta h^2}{12}a_{i-1} \\ -\frac{h^2}{12}a_i\left(1+\frac{d}{h}\right) - \frac{h}{8}b_i \end{pmatrix} \end{bmatrix} y_i \\
&+ \left[\frac{\varepsilon}{h} - \frac{\delta}{6}a_i + \frac{h}{12}\delta\left(a_{i-1/2}+b_{i-1/2}\right)\right]y_{i+1} \\
&= \frac{h}{6}\left[f_i + 4f_{i-1/2} + f_{i-1}\right] \\
&+ \frac{h^2}{12}\left(f_i - f_{i-1}\right)\left(a_{i-1/2}+b_{i-1/2}\right) \\
&- \frac{2h}{3}\left(a_{i-1/2}+b_{i-1/2}\right)\left[\frac{h}{8}y'\left(x_i+\varepsilon\right) - \frac{h}{8}y'\left(x_{i+1}+\varepsilon\right)\right]
\end{aligned} \tag{27}
$$

using the finite difference approximations, we have

$$
y'(x_{i-1}+\varepsilon) - y'(x_i+\varepsilon) = \frac{\left(-y(x_{i+1}+\varepsilon) + 2y(x_i+\varepsilon) - y(x_{i-1}+\varepsilon)\right)}{h} \tag{28}
$$

Again by Taylor series, we have

$$
y(x_i+\varepsilon) \approx \left(1+\frac{\varepsilon}{h}\right)y_i - \frac{\varepsilon}{h}y_{i-1} \tag{29}
$$

$$
y(x_{i+1}+\varepsilon) \approx \left(1+\frac{\varepsilon}{h}\right)y_{i+1} - \frac{\varepsilon}{h}y_i \tag{30}
$$

$$
y(x_{i-1}-\varepsilon) \approx \left(1-\frac{\varepsilon}{h}\right)y_{i-1} + \frac{\varepsilon}{h}y_i \tag{31}
$$

Substituting (28)-(31) in equation (27), we get the three term recurrence relation

$$
E_i y_{i-1} - F_i y_i + G_i y_{i+1} = H_i,\ for\quad i = n+1, n+2, \ldots, N-1. \tag{32}
$$

where

$$
\begin{aligned}
E_i = \frac{\varepsilon}{h} &+ \frac{(h+\delta)}{6}a_{i-1} + \frac{2}{3}\delta a_{i-1/2} + \frac{h}{6}b_{i-1} \\
&+ \frac{h}{3}\left(a_{i-1/2}+b_{i-1/2}\right)\left(\frac{3}{4} + \frac{h}{4}b_{i-1} + \frac{(h+\delta)}{4}a_{i-1} - \frac{\varepsilon}{4h}\right)
\end{aligned}
$$

$$
\begin{aligned}
F_i = \frac{2\varepsilon}{h} &- \frac{(h+\delta)}{6}a_i + \frac{2}{3}\delta a_{i-1/2} + \frac{\delta}{6}a_{i-1} - \frac{h}{6}b_i \\
&- \frac{h}{3}\left(a_{i-1/2}+b_{i-1/2}\right)\left(\frac{3}{2} - \frac{\delta}{4}a_{i-1} - \frac{h}{4}b_i - \frac{(h+\delta)}{4}a_i + \frac{\varepsilon}{2h}\right)
\end{aligned}
$$

$$
G_i = \frac{\varepsilon}{h} - \frac{\delta}{6}a_i + \frac{h}{12}\left(a_{i-1/2}+b_{i-1/2}\right)\left(\delta a_i - \left(1+\frac{\varepsilon}{h}\right)\right)
$$

$$
H_i = \frac{h}{6}\left[f_i + 4f_{i-1/2} + f_{i-1}\right] - \frac{h^2}{12}\left(f_i - f_{i-1}\right)\left(a_{i-1/2}+b_{i-1/2}\right)
$$

Now we have, from (17) in $\left[0,\frac{1}{2}\right]$ for i= 1, 2, . . . , n-1; and (32) in $\left[\frac{1}{2},1\right]$for $i = n$+1, n+2, . . . ,N-1; a system of (N-2) equations with (N+1) unknowns. From the given boundary conditions (2) we get two equations. We need one more equation to solve for the unknowns $y_0, y_1, ...y_n$. To get this equation we consider the reduced problem of equation (1) by setting $\varepsilon = 0$ i.e.,

$$a(x)y(x-\delta) + b(x)y(x) = f(x) \tag{33}$$

which does not satisfy both the boundary conditions. At $x = x_n = \frac{1}{2}$, equation (33) becomes

$$a(x_n)y(x_n-\delta) + b(x_n)y(x_n) = f(x_n) \tag{34}$$

Using Taylor series expansion, we have

$$y(x_n-\delta) \approx y(x_n) - \delta y'(x_n) = y_n - \delta\left(\frac{y_{n+1}-y_{n-1}}{2h}\right)$$

Substituting this in (34) and by simplification, we get

$$\frac{a_n\,\delta}{2h}y_{n-1} - (-a_n - b_n)y_n - \frac{a_n\;\delta}{2h}y_{n+1} = f_n \tag{35}$$

With this equation, we now have (N+1) equations to solve for the (N+1) unknowns $y_0, y_1, ..., y_n$. We solve this tridiagonal algebraic system by using an efficient and stable method of invariant imbedding [1].

3 Error analysis

Writing the tridiagonal system (17) in matrix-vector form, we get

$$AY = C \tag{36}$$

in which $A = (m_{i\,j})$, $1 \le i, j \le n$-1 is a tridiagonal matrix of order N-1, with

$$m_{i\,i+1} = \tfrac{\varepsilon}{h} - \tfrac{2\delta a_{i+1/2}}{3} + \tfrac{h}{6}b_{i+1} + \left(\tfrac{h-\delta}{6}\right)a_{i+1} + \tfrac{h}{3}\left(a_{i+1/2}+b_{i+1/2}\right)\left(\tfrac{3}{4} + \tfrac{h}{4}b_{i+1} + \left(\tfrac{h-\delta}{4}\right)a_{i+1} + \tfrac{\varepsilon}{4h}\right),$$

$$m_{i\,i} = -\tfrac{2\varepsilon}{h} + \tfrac{2\delta a_{i+1/2}}{3} + \tfrac{h}{6}b_i + \left(\tfrac{h-\delta}{6}\right)a_i + \tfrac{\delta}{6}a_{i+1} + \tfrac{h}{3}\left(a_{i+1/2}+b_{i+1/2}\right)\left(\tfrac{3}{2} - \tfrac{h}{4}b_i + \tfrac{\delta a_{i+1}}{4} - \left(\tfrac{h-\delta}{4}\right)a_i - \tfrac{\varepsilon}{2h}\right),$$

$$m_{i\,i-1} = \tfrac{\varepsilon}{h} + \tfrac{\delta}{6}a_i - \tfrac{h}{12}\left(a_{i+1/2}+b_{i+1/2}\right)\left(\delta a_i + (1-\tfrac{\varepsilon}{h})\right),$$

and $C = (d_i)$ is a column vector with

$$d_i = \frac{h}{6}\left(f_i + 4f_{i+1/2} + f_{i+1}\right) + \frac{h^2}{12}\left(f_{i+1}-f_i\right)\left(a_{i+1/2}+b_{i+1/2}\right), \text{where} i = 1\ (1)\ n\text{-}1 \tag{37}$$

with local truncation error

$$T_{\rm i}(h_i) = h^2\left[\frac{1}{12}\left((\varepsilon+2\delta)a_i+\varepsilon b_i\right)y_i'' - \frac{\varepsilon}{2}y_i'''\right] + 0(h^3) \tag{38}$$

Writing the tridiagonal system (32) in matrix-vector form, we get

$$AY = C \tag{39}$$

in which $A = (m_{i\,j})$, $n+1 \le i, j \le N$-1 is a tridiagonal matrix of order N-1 , with

$$m_{i\,i+1} = \tfrac{\varepsilon}{h} - \tfrac{\delta}{6}a_i + \tfrac{h}{12}\left(a_{i-1/2}+b_{i-1/2}\right)\left(\delta a_i - (1+\tfrac{\varepsilon}{h})\right)$$

$$m_{i\,i} = -\tfrac{2\varepsilon}{h} - \frac{2\delta\, a_{i-1/2}}{3} + \frac{h}{6}b_i + \left(\tfrac{h+\delta}{6}\right)a_i - \tfrac{\delta}{6}a_{i-1} + \tfrac{h}{3}\left(a_{i-1/2}+b_{i-1/2}\right)\left(\tfrac{3}{2} - \tfrac{h}{4}b_i - \tfrac{\delta\, a_{i-1}}{4} - \left(\tfrac{h+\delta}{4}\right)a_i + \tfrac{\varepsilon}{2h}\right),$$

$$m_{i\,i\text{-}1} = \tfrac{\varepsilon}{h} + \frac{2\delta\, a_{i-1/2}}{3} + \frac{h}{6}b_{i-1} + \left(\tfrac{h+\delta}{6}\right)a_{i+1} + \tfrac{h}{3}\left(a_{i-1/2}+b_{i-1/2}\right)\left(\tfrac{3}{4} + \tfrac{h}{4}b_{i-1} + \left(\tfrac{h+\delta}{4}\right)a_{i-1} - \tfrac{\varepsilon}{4h}\right)$$

and $C = (d_i)$ is a column vector with

$$d_i = \frac{h}{6}\left(f_i + 4f_{i-1/2} + f_{i-1}\right) - \frac{h^2}{12}\left(f_i - f_{i-1}\right)\left(a_{i-1/2}+b_{i-1/2}\right) \tag{40}$$

where $i = n+1\ (1)\ N-1$ with local truncation error

$$T_{\rm i}(h_i) = h^2\left[\frac{1}{12}\left((\varepsilon+2\delta)a_i+\varepsilon b_i\right)y_i'' - \frac{\varepsilon}{2}y_i'''\right] + 0(h^3) \tag{41}$$

and $Y = (y_0, y_1, y_2, ...y_N)^t$ We also have

$$A\overline{Y} - T(h) = C \tag{42}$$

where $\overline{Y} = (\overline{y_0}, \overline{y_1}, \overline{y_2}, ..., \overline{y_N})^t$ denotes the actual solution and $T(h) = (T_0(h_0), T_1(h_1), ..., T_N(h_N))^t$ is the local truncation error.

From (42) and (39), we get

$$A\left(\overline{Y} - Y\right) = T(h) \tag{43}$$

Thus the error equation is

$$AE = T(h) \tag{44}$$

where $E = \overline{Y} - Y = (e_0, e_1, e_2, ...e_N)^t$.

Clearly, we have

$$\begin{aligned} S_i &= \sum_{j=1}^{N-1} m_{i\,\mathrm{j}} \\ &= h\left[\tfrac{1}{6}\left(a_i+b_i+a_{i+1}+b_{i+1}\right)+\tfrac{2}{3}\left(a_{i+1/2}+b_{i+1/2}\right)\right]+0(h^2) \\ &= h\,B_i',\ \ i=1\ (1)\ n-1 \end{aligned}$$

where $B_{\mathrm{i}}' = \left[\frac{1}{6}\left(a_i+b_i+a_{i+1}+b_{i+1}\right)+\frac{2}{3}\left(a_{i+1/2}+b_{i+1/2}\right)\right]$

$$S_i = h\,(a_n+b_n) = hB_i'',\ \ i=n \ \text{ where } B_i''=(a_n+b_n)$$

$$S_i = \sum_{j=1}^{N-1} m_{i\,\mathrm{j}}$$

$$= h\left[\tfrac{1}{6}\left(a_i+b_i+a_{i-1}+b_{i-1}\right)+\tfrac{2}{3}\left(a_{i-1/2}+b_{i-1/2}\right)\right]+0(h^2)$$

$$= h\,B_i'''\ \ i=\mathrm{n}+1\ (1)\ \mathrm{N}-1$$

where $B_i''' = \left[\frac{1}{6}\left(a_i+b_i+a_{i-1}+b_{i-1}\right)+\frac{2}{3}\left(a_{i-1/2}+b_{i-1/2}\right)\right]$

We can choose h sufficiently small so that the matrix A is irreducible and monotone. It follows that A^{-1} exists and its elements are non negative.

Hence from (44), we get

$$E = A^{-1}T(h) \tag{45}$$

Also from the theory of matrices we have

$$\sum_{i=1}^{N-1} \overline{m}_{k,i}\, S_{\mathrm{i}} = 1\ ,\ \ k=1\,(1)\,N-1 \tag{46}$$

where $\overline{m}_{k,i}$ is (k,i) element of the matrix A^{-1}. Therefore,

$$\sum_{i=1}^{N\text{-}1} \overline{m}_{k,i} \le \frac{1}{\min\limits_{1\le i\le N-1} S_i} = \frac{1}{h_i B_{i_o}} \le \frac{1}{h_i\,|B_{i_o}|} \tag{47}$$

for some i_0 between 1 and N-1. and

$$B_{i0} = \left[\begin{array}{ll} B_i', & i=1(1)n-1 \\ B_i'', & i=n \\ B_i''', & i=n+1(1)N-1 \end{array}\right.$$

From (38), (45),(46) and (47), we get

$$e_j = \sum_{i=1}^{N-1} \overline{m}_{k,i}\, T_{\mathrm{i}}(h), \qquad j=1\ (1)\ \ N-1$$

which implies

$$e_j \le \frac{kh_i}{|B_{i0}|}\ , j=1\ (1)\ \ N-1 \tag{48}$$

where k is a constant independent of h.

Therefore,

$$\|E_i\| = 0\,(h)$$

i.e., our method reduces to a first order convergent for uniform mesh.

4 Numerical Experiments

To demonstrate the proposed method, we consider four numerical experiments, two with boundary layers and two with oscillatory behaviour. We have plotted the graphs of the computed solution of the problem for different values of ε and for different values of δ of $o(\varepsilon)$, which are represented by solid and dotted lines respectively. The maximum absolute errors for the examples are calculated using the double mesh principle [3], $E^N = \max\limits_{0\le i\le N} \left|y_i^N - y_{2i}^{2N}\right|$.

Example 1. Consider an example of singularly perturbed delay differential equation with layer behaviour [10]

$$\varepsilon^2 y''(x) - 2y(x-\delta) - y(x) = 1$$

with boundary conditions $y(0)$ = 1, -$\delta \le x \le 0$ and y(1) = 0.

The maximum absolute errors are presented in Table1 and Table 2 for different values of ε and for different values of δ. We also plot the graphs of the computed solution of the problem for ε= 0.1, 0.01 and for different values of δ as shown in Figs. 1 and 2 respectively.

Example 2. Consider singularly perturbed delay differential equation with layer behaviour [10]

$$\varepsilon^2 y''(x) + 0.25y(x-\delta) - y(x) = 1$$

with boundary conditions $y(0)=1, -\delta\le x\le 0, y(1)=0$. The maximum absolute errors are presented in Table 1 and Table 2 for different values of ε and for different values of δ. We plot the graphs of the computed solution of the problem for ε= 0.01 and for different values of δ in Figs. 3 and 4 respectively.

Example 3. Consider a singularly perturbed delay differential equation with oscillatory behaviour [10]

$$\varepsilon^2 y''(x) + 0.25y(x-\delta) + y(x) = 1$$

with boundary conditions $y(0)=1, -\delta\le x\le 0, y(1)=0$. The maximum absolute errors are presented in Table 1 for different values of ε and for different values of δ. The graphs of the computed solution of the problem for ε= 0.01 and for different values of δ are presented in Fig. 5.

Example 4. Consider the singularly perturbed delay differential equation with oscillatory behaviour [10]

$$\varepsilon^2 y''(x) + y(x-\delta) + 2y(x) = 1$$

with boundary conditions $y(0)=1, -\delta\le x\le 0, y(1)=0$

The maximum absolute errors are presented in Table 1 for ε= 0.1 and for different values of δ. We plot the graphs of the computed solution of the problem for ε= 0.01 for different values of δ as shown in Figs. 6 and 7 respectively.

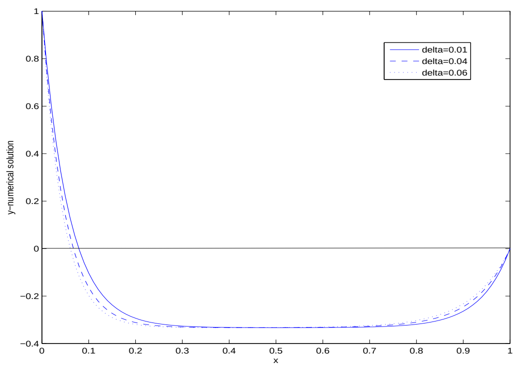

Fig. 1: The numerical solution of Ex. 1 with $\varepsilon = 0.1$

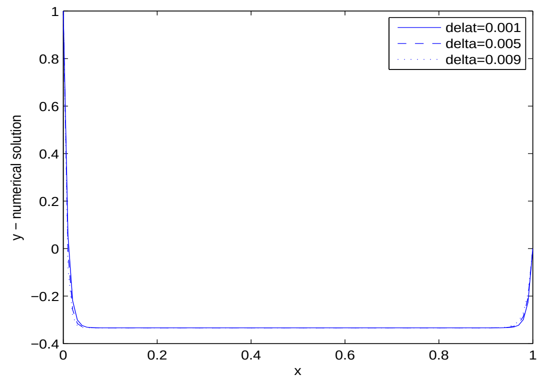

Fig. 2: The numerical solution of Ex. 1 with $\varepsilon = 0.01$

5 Discussions and Conclusion

We have presented a computational method to solve singularly perturbed delay differential equations with layer or oscillatory behaviour. In general numerical solution of second order differential equation will be more difficult than numerical solution of first order differential equation. Hence, in this method, we have reduced the second order singularly perturbed delay differential equation to first order neutral type delay differential equation and employed the Simpson rule of numerical integration. Then, linear interpolation is used to get three term recurrence relation which is solved easily by method of invariant imbedding algorithm. The method is demonstrated by implementing several model examples by taking various values for the delay parameter and perturbation parameter.

This method is very easy to implement. To show the effect of delay on the boundary layer or oscillatory behaviour of the solution, several numerical examples are carried out in section 4. To demonstrate the effect on the layer behaviour, we consider examples 1 and 2. We observe that when the order of the coefficient of the delay term is of $o(1)$, the delay affects the boundary layer solution but maintains the layer behaviour. From the Figures 1 and 2, we observe that when the delay is $o(\varepsilon)$, the solution maintains layer behaviour although the coefficient of the delay term in the equation is of $O(1)$ and as the delay increases, the thickness of the left boundary layer decreases while that of the right boundary layer increases.

To demonstrate the effect on the oscillatory behaviour, we consider the examples 3 and 4 when the solution of the problem exhibits oscillatory behaviour for delay equal to zero. We observe that if the coefficient of the delay term is of $o(1)$, the amplitude of the oscillations increases slowly as the delay increases provided the delay is of $o(\varepsilon)$. (Figure 5)

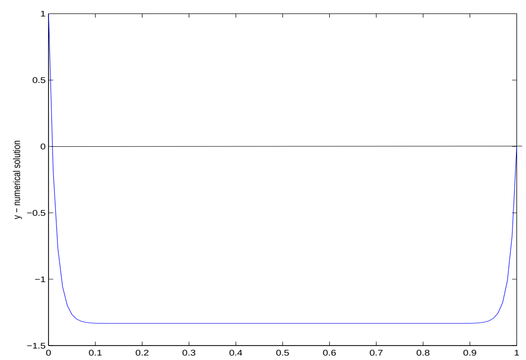

Fig. 3: The numerical solution of Ex.2 with $\varepsilon = 0.01, \delta = 0.7\varepsilon$

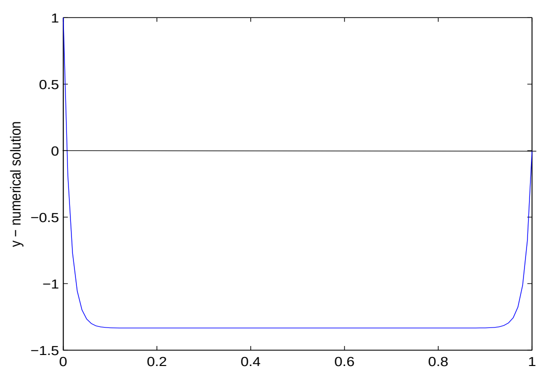

Fig. 4: The numerical solution of Ex. 2 with $\varepsilon = 0.01, \delta = \varepsilon$

Table 1: The maximum absolute error of the examples for different values of δ with $\varepsilon = 0.1$

N	100	200	300	400	500
δ	Example 1				
0.03	3.2676e-003	1.6475e-003	1.1015e-003	8.2735e-004	6.6245e-004
0.05	3.2657e-003	1.6526e-003	1.1062e-003	8.3136e-004	6.6593e-004
0.09	3.5460e-003	1.7987e-003	1.2051e-003	9.0609e-004	7.2594e-004
	Example 2				
0.03	2.1226e-003	1.0639e-003	7.0985e-004	5.3259e-004	4.2617e-004
0.05	2.1099e-003	1.0574e-003	7.0543e-004	5.2928e-004	4.2351e-004
0.09	2.0816e-003	1.0426e-003	6.9547e-004	5.2178e-004	4.1750e-004
	Example 3				
0.03	2.4582e-003	1.2196e-003	8.1096e-004	6.0742e-004	4.8554e-004
0.05	2.5127e-003	1.2472e-003	8.2948e-004	6.2134e-004	4.9669e-004
0.09	2.6198e-003	1.3016e-003	8.6589e-004	6.4872e-004	5.1863e-004
	Example 4				
0.03	1.8682e-002	9.0640e-003	5.9795e-003	4.4608e-003	3.5572e-003
0.05	1.4987e-002	7.2328e-003	4.7631e-003	3.5505e-003	2.8299e-003
0.09	2.1346e-002	1.0306e-002	6.7863e-003	5.0577e-003	4.0307e-003

Table 2: The maximum absolute error of the examples for different values of ε for $\delta = 0.03$

N	100	200	300	400	500
ε	Example 1				
2^{-1}	9.2363e-004	4.6407e-004	3.0991e-004	2.3263e-004	1.8619e-004
2^{-2}	1.6390e-003	8.2516e-004	5.5141e-004	4.1404e-004	3.3146e-004
2^{-3}	2.7044e-003	1.3653e-003	9.1315e-004	6.8602e-004	5.4937e-004
2^{-4}	4.1751e-003	2.1168e-003	1.4178e-003	1.0658e-003	8.5380e-004
2^{-5}	6.2518e-003	3.1866e-003	2.1382e-003	1.6088e-003	1.2895e-003
	Example 2				
2^{-1}	5.0597e-004	2.5321e-004	1.6886e-004	1.2667e-004	1.0134e-0 04
2^{-2}	9.6556e-004	4.8357e-004	3.2250e-004	2.4194e-004	1.9357e-004
2^{-3}	1.7698e-003	8.8657e-004	5.9149e-004	4.4377e-004	3.5508e-004
2^{-4}	3.0307e-003	1.5201e-003	1.0145e-003	7.6132e-004	6.0926e-004
2^{-5}	4.7379e-003	2.3810e-003	1.5901e-003	1.1937e-003	9.5544e-004

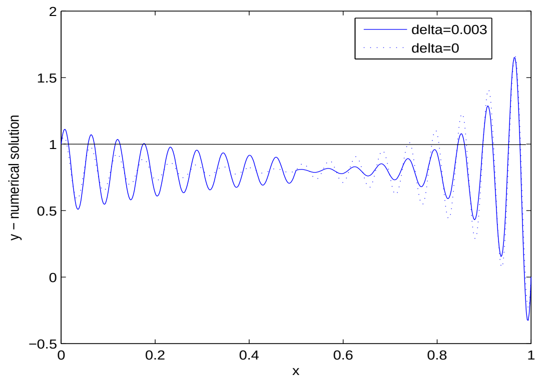

Fig. 5: The numerical solution of Ex.3 with $\varepsilon = 0.01$

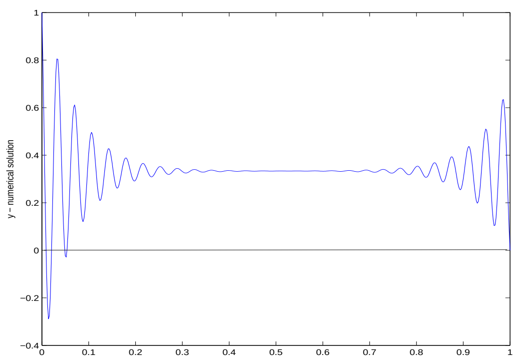

Fig. 6: The numerical solution of Ex. 4 with $\varepsilon = 0.01, \delta = 0$

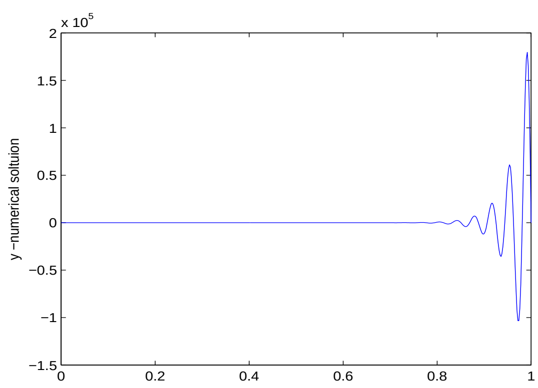

Fig. 7: The numerical solution of Ex. 4 with $\varepsilon = 0.01, \delta = 0.3$

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G. B. S. L. Soujanya received the PhD degree in Mathematics for numerical solution of singular perturbation problems from National Institute of Technology, Warangal, India. She is working as Assistant Professor in Mathematics, Kakatiya University, Warangal, India. Her research interests are in the areas of numerical methods for singular perturbation and singularly perturbed differential difference equations. She has published ten research articles in reputed international journals of mathematical sciences.

Y. N. Reddy is Professor of Mathematics at National Institute of Technology, Warangal, India. His research interests are in the area of numerical analysis, numerical methods for singular perturbation, singularly perturbed differential difference equations. He guided 9 research scholars under his supervision. He published 104 research articles in various international journals.